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Mean value of F. Smarandache LCM function

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Abstract For any positive integer n, the famous Smarandache function $S(n)$ defined as the smallest positive integer m such that $n \mid m!$. That is, $S(n) = \min\{m : n \mid m!, n \in N\}$. The Smarandache LCM function $SL(n)$ the smallest positive integer k such that $n \mid [1, 2, \cdots, k]$, where $[1, 2, \cdots, k]$ denotes the least common multiple of $1, 2, \cdots, k$. The main purpose of this paper is using the elementary methods to study the mean value properties of $(SL(n) - S(n))^2$, and give a sharper asymptotic formula for it.

Keywords Smarandache function, Smarandache LCM function, mean value, asymptotic formula.

§1. Introduction and result

For any positive integer n, the famous F.Smarandache function $S(n)$ defined as the smallest positive integer m such that $n \mid m!$. That is, $S(n) = \min\{m : n \mid m!, n \in N\}$. For example, the first few values of $S(n)$ are $S(1) = 1$, $S(2) = 2$, $S(3) = 3$, $S(4) = 4$, $S(5) = 5$, $S(6) = 3$, $S(7) = 7$, $S(8) = 4$, $S(9) = 6$, $S(10) = 5$, $S(11) = 11$, $S(12) = 4$, $\cdots\cdots$. The F.Smarandache LCM function $SL(n)$ defined as the smallest positive integer k such that $n \mid [1, 2, \cdots, k]$, where $[1, 2, \cdots, k]$ denotes the least common multiple of $1, 2, \cdots, k$. The first few values of $SL(n)$ are $SL(1) = 1$, $SL(2) = 2$, $SL(3) = 3$, $SL(4) = 4$, $SL(5) = 5$, $SL(6) = 3$, $SL(7) = 7$, $SL(8) = 8$, $SL(9) = 9$, $SL(10) = 5$, $SL(11) = 11$, $SL(12) = 4$, $\cdots\cdots$.

About the elementary properties of $S(n)$ and $SL(n)$, many authors had studied them, and obtained some interesting results, see reference [2], [3], [4] and [5]. For example, Murthy [2] proved that if n be a prime, then $SL(n) = S(n)$. Simultaneously, Murthy [2] also proposed the following problem:

$$SL(n) = S(n), \quad S(n) \neq n \ ? \tag{1}$$

Le Maohua [3] completely solved this problem, and proved the following conclusion:

Every positive integer n satisfying (1) can be expressed as

$$n = 12 \ \texttt{ or } \ n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} p,$$

where $p_1, p_2, \cdots, p_r, p$ are distinct primes, and $\alpha_1, \alpha_2, \cdots, \alpha_r$ are positive integers satisfying $p > p_i^{\alpha_i}$, $i = 1, 2, \cdots, r$.

Dr. Xu Zhefeng [4] studied the value distribution problem of $S(n)$, and proved the following conclusion:

Let $P(n)$ denotes the largest prime factor of n, then for any real number $x > 1$, we have the asymptotic formula

$$\sum_{n\leq x}(S(n)-P(n))^2=\frac{2\zeta\left(\frac{3}{2}\right)x^{\frac{3}{2}}}{3\ln x}+O\left(\frac{x^{\frac{3}{2}}}{\ln^2 x}\right),$$

where $\zeta(s)$ denotes the Riemann zeta-function.

Lv Zhongtian [5] proved that for any fixed positive integer k any real number $x > 1$, we have the asymptotic formula fixed positive integer k

$$\sum_{n\leq x}SL(n)=\frac{\pi^2}{12}\cdot\frac{x^2}{\ln x}+\sum_{i=2}^{k}\frac{c_i\cdot x^2}{\ln^i x}+O\left(\frac{x^2}{\ln^{k+1}x}\right),$$

where c_i $(i = 2,\ 3, \cdots, k)$ are computable constants.

The main purpose of this paper is using the elementary methods to study the mean value properties of $[SL(n) - S(n)]^2$, and give an interesting mean value formula for it. That is, we shall prove the following conclusion:

Theorem. Let k be a fixed positive integer. Then for any real number $x > 2$, we have the asymptotic formula

$$\sum_{n\leq x}[SL(n)-S(n)]^2=\frac{2}{3}\cdot\zeta\left(\frac{3}{2}\right)\cdot x^{\frac{3}{2}}\cdot\sum_{i=1}^{k}\frac{c_i}{\ln^i x}+O\left(\frac{x^{\frac{3}{2}}}{\ln^{k+1}x}\right),$$

where $\zeta(s)$ be the Riemann zeta-function, c_i $(i = 1,\ 2, \cdots,\ k)$ are computable constants.

§2. Proof of the theorem

In this section, we shall complete the proof of our theorem directly. In fact for any positive integer $n > 1$, let $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_s^{\alpha_s}$ be the factorization of n into prime powers, then from [2] we know that

$$S(n)=\max\{S(p_1^{\alpha_1}),\ S(p_2^{\alpha_2}),\ \cdots,\ S(p_s^{\alpha_s})\}\equiv S(p^{\alpha}) \tag{2}$$

and

$$SL(n)=\max\{p_1^{\alpha_1},\ p_2^{\alpha_2},\ \cdots,\ p_s^{\alpha_s}\}. \tag{3}$$

Now we consider the summation

$$\sum_{n\leq x}[SL(n)-S(n)]^2=\sum_{n\in A}[SL(n)-S(n)]^2+\sum_{n\in B}[SL(n)-S(n)]^2, \tag{4}$$

where A and B denote two subsets of all positive integer in the interval $[1,\ x]$. A denotes the set involving all integers $n \in [1,\ x]$ such that $SL(n) = p^2$ for some prime p; B denotes the set involving all integers $n \in [1,\ x]$ such that $SL(n) = p^{\alpha}$ for some prime p with $\alpha = 1$ or $\alpha \geq 3$. If

$n \in A$ and $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$, then $n = p^2 m$ with $p \dagger m$, and all $p_i^{\alpha_i} \le p^2$, $i = 1,\ 2,\ \cdots s$. Note that $S(p_i^{\alpha_i}) \le \alpha_i p_i$ and $\alpha_i \le \ln n$, from the definition of $SL(n)$ and $S(n)$ we have

$$\sum_{n \in A} [SL(n) - S(n)]^2 = \sum_{\substack{mp^2 \le x \\ SL(m) < p^2}} \left[p^2 - S(mp^2)\right]^2$$

$$= \sum_{\substack{mp^2 \le x \\ SL(m) < p^2}} \left(p^4 - 2p^2 S(mp^2) + S^2(mp^2)\right)$$

$$= \sum_{\substack{mp^2 \le x \\ SL(m) < p^2}} p^4 + O\left(\sum_{mp^2 \le x} p^2 \cdot p \cdot \ln x\right) + O\left(\sum_{mp^2 \le x} p^2 \cdot \ln^2 x\right)$$

$$= \sum_{m \le \sqrt{x}} \sum_{m < p^2 \le \frac{x}{m}} p^4 + O\left(\sum_{p^2 \le \sqrt{x}} \sum_{p^2 < m \le \frac{x}{p^2}} p^4\right) + O\left(x^2\right)$$

$$= \sum_{m \le \sqrt{x}} \sum_{p \le \sqrt{\frac{x}{m}}} p^4 + O\left(\sum_{m \le \sqrt{x}} \sum_{p^2 \le m} p^4\right) + O\left(x^2\right)$$

$$= \sum_{m \le \sqrt{x}} \sum_{p \le \sqrt{\frac{x}{m}}} p^4 + O\left(x^2\right). \qquad (5)$$

By the Abel's summation formula (See Theorem 4.2 of [6]) and the Prime Theorem (See Theorem 3.2 of [7]):

$$\pi(x) = \sum_{i=1}^{k} \frac{a_i \cdot x}{\ln^i x} + O\left(\frac{x}{\ln^{k+1} x}\right),$$

where a_i $(i = 1,\ 2, \cdots, k)$ are computable constants and $a_1 = 1$.

We have

$$\sum_{p \le \sqrt{\frac{x}{m}}} p^4 = \frac{x^2}{m^2} \cdot \pi\left(\sqrt{\frac{x}{m}}\right) - \int_{\frac{3}{2}}^{\sqrt{\frac{x}{m}}} 4y^3 \cdot \pi(y) dy$$

$$= \frac{x^2}{m^2} \cdot \pi\left(\sqrt{\frac{x}{m}}\right) - \int_{\frac{3}{2}}^{\sqrt{\frac{x}{m}}} 4y^3 \left[\sum_{i=1}^{k} \frac{a_i \cdot y}{\ln^i y} + O\left(\frac{y}{\ln^{k+1} y}\right)\right] dy$$

$$= \frac{1}{5} \cdot \frac{x^{\frac{5}{2}}}{m^{\frac{5}{2}}} \cdot \sum_{i=1}^{k} \frac{b_i}{\ln^i \sqrt{\frac{x}{m}}} + O\left(\frac{x^{\frac{5}{2}}}{m^{\frac{5}{2}} \cdot \ln^{k+1} x}\right), \qquad (6)$$

where we have used the estimate $m \le \sqrt{x}$, and all b_i are computable constants with $b_1 = 1$.

Note that $\sum_{m=1}^{\infty} \frac{1}{m^{\frac{5}{2}}} = \zeta\left(\frac{5}{2}\right)$, and $\sum_{m=1}^{\infty} \frac{\ln^i m}{m^{\frac{5}{2}}}$ is convergent for all $i = 1,\ 2,\ 3,\ \cdots,\ k$. So from (5) and (6) we have

$$\begin{aligned}
&\sum_{n\in A}[SL(n)-S(n)]^2\\
=&\sum_{m\le\sqrt{x}}\left[\frac{1}{5}\cdot\frac{x^{\frac{5}{2}}}{m^{\frac{3}{2}}}\cdot\sum_{i=1}^{k}\frac{b_i}{\ln^i\sqrt{\frac{x}{m}}}+O\left(\frac{x^{\frac{5}{2}}}{m^{\frac{5}{2}}\cdot\ln^{k+1}x}\right)\right]+O\left(x^2\right)\\
=&\frac{2}{5}\cdot\zeta\left(\frac{5}{2}\right)\cdot x^{\frac{5}{2}}\cdot\sum_{i=1}^{k}\frac{c_i}{\ln^i x}+O\left(\frac{x^{\frac{5}{2}}}{\ln^{k+1}x}\right),
\end{aligned}\tag{7}$$

where c_i $(i=1,\ 2,\ 3,\cdots,k)$ are computable constants and $c_1=1$.

Now we estimate the summation in set B. For any positive integer $n\in B$. If $n\in B$ and $SL(n)=p$, then we also have $S(n)=p$. So $[SL(n)-S(n)]^2=[p-p]^2=0$. If $SL(n)=p^\alpha$ with $\alpha\ge 3$, then $[SL(n)-S(n)]^2=[p^\alpha-S(n)]^2\le p^{2\alpha}+\alpha^2p^2$ and $\alpha\le\ln n$. So we have

$$\sum_{n\in B}[SL(n)-S(n)]^2\ll\sum_{\substack{np^\alpha\le x\\ \alpha\ge 3}}\left(p^{2\alpha}+p^2\ln^2 n\right)\ll x^2.\tag{8}$$

Combining (4), (7) and (8) we may immediately deduce the asymptotic formula

$$\sum_{n\le x}[SL(n)-S(n)]^2=\frac{2}{5}\cdot\zeta\left(\frac{5}{2}\right)\cdot x^{\frac{5}{2}}\cdot\sum_{i=1}^{k}\frac{c_i}{\ln^i x}+O\left(\frac{x^{\frac{5}{2}}}{\ln^{k+1}x}\right),$$

where c_i $(i=1,\ 2,\ 3,\cdots,k)$ are computable constants and $c_1=1$.

This completes the proof of Theorem.

References

[1] F. Smarandache, Only Problems, Not Solutions, Chicago, Xiquan Publishing House, 1993.

[2] A. Murthy, Some notions on least common multiples, Smarandache Notions Journal, **12**(2001), 307-309.

[3] Le Maohua, An equation concerning the Smarandache LCM function, Smarandache Notions Journal, **14**(2004), 186-188.

[4] Xu Zhefeng, The value distribution property of the Smarandache function, Acta Mathematica Sinica (Chinese Series), **49**(2006), No.5, 1009-1012.

[5] Lv Zhongtian, On the F.Smarandache LCM function and its mean value. Scientia Magna, **3**(2007), No.1, 22-25.

[6] Tom M. Apostol, Introduction to Analytic Number Theory. New York, Springer-Verlag, 1976

[7] Pan Cengdong and Pan Chengbiao, The elementary proof of the prime theorem, Shanghai Science and Technology Press, Shanghai, 1988.

[8] G.H.Hardy and E.M.Wright, An introduction to the theory of numbers, Oxford University Press, 1981.