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Mediate Dominating Graph of a Graph

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Abstract: Let S be the set of minimal dominating sets of graph G and $U,\ W \subset S$ with $U \bigcup W = S$ and $U \bigcap W = \emptyset$. A *Smarandachely mediate-(U, W) dominating graph* $D_m^S(G)$ of a graph G is a graph with $V(D_m^S(G)) = V' = V \bigcup U$ and two vertices $u, v \in V'$ are adjacent if they are not adjacent in G or $v = D$ is a minimal dominating set containing u. particularly, if $U = S$ and $W = \emptyset$, i.e., a Smarandachely mediate-$(S, \emptyset)$ dominating graph $D_m^S(G)$ is called the *mediate dominating graph* $D_m(G)$ of a graph G. In this paper, some necessary and sufficient conditions are given for $D_m(G)$ to be connected, Eulerian, complete graph, tree and cycle respectively. It is also shown that a given graph G is a mediate dominating graph $D_m(G)$ of some graph. One related open problem is explored. Finally, some bounds on domination number of $D_m(G)$ are obtained in terms of vertices and edges of G.

Key Words: Connectedness, connectivity, Eulerian, hamiltonian, dominating set, Smarandachely mediate-(U, W) dominating graph.

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§1. Introduction

The graphs considered here are finite and simple. Let $G = (V, E)$ be a graph and let the vertices and edges of a graph G be called the elements of G. The undefined terminology and notations can be found in [2]. The connectivity(edge connectivity) of a graph G, denoted by $\kappa(G)(\lambda(G))$, is defined to be the largest integer k for which G is k-connected(k-edge connected). For a vertex v of G, the eccentricity $ecc_G(v)$ of v is the largest distance between v and all the other vertices of G, i.e., $ecc_G(v) = max\{d_G(u, v)/u \in V(G)\}$. The diameter $diam(G)$ of G is the $\max\{ecc_G(v)/v \in V(G)\}$. The chromatic number $\chi(G)$ of a graph G is the minimum number of independent subsets that partition the vertex set of G. Any such minimum partition is called a chromatic partition of $V(G)$.

Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two graphs. We call G and H to be isomorphic, and we write $G \cong H$, if there exists a bijection $\theta : V(G) \longrightarrow V(H)$ with $xy \in E(G)$ if and only if $\theta(x)\theta(y) \in E(H)$ for all $x, y \in V(G)$.

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Let $G = (V, E)$ be a graph. A set $D \subseteq V$ is a dominating set of G if every vertex in $V - D$ is adjacent to some vertex in D. A dominating set D of G is minimal if for any vertex $v \in D$, $D - v$ is not a dominating set of G. The domination number $\gamma(G)$ of G is the minimum cardinality of a minimal dominating set of G. The upper domination number $\Gamma(G)$ of G is the maximum cardinality of a minimal dominating set of G. For details on $\gamma(G)$, refer [1].

The maximum number of classes of a domatic partition of G is called the domatic number of G and is denoted by $d(G)$. The vertex independence number $\beta_0(G)$ is the maximum cardinality among the independent set of vertices of G.

Our aim in this paper is to introduce a new graph valued function in the field of domination theory in graphs.

Definition 1.1 *Let S be the set of minimal dominating sets of graph G and U, $W \subset S$ with $U \bigcup W = S$ and $U \bigcap W = \emptyset$. A Smarandachely mediate-(U, W) dominating graph $D_m^S(G)$ of a graph G is a graph with $V(D_m^S(G)) = V' = V \bigcup U$ and two vertices u, $v \in V'$ are adjacent if they are not adjacent in G or $v = D$ is a minimal dominating set containing u. particularly, if $U = S$ and $W = \emptyset$, i.e., a Smarandachely mediate-$(S, \emptyset)$ dominating graph $D_m^S(G)$ is called the mediate dominating graph $D_m(G)$ of a graph G.*

In Fig.1, a graph G and its mediate dominating graph $D_m(G)$ are shown.

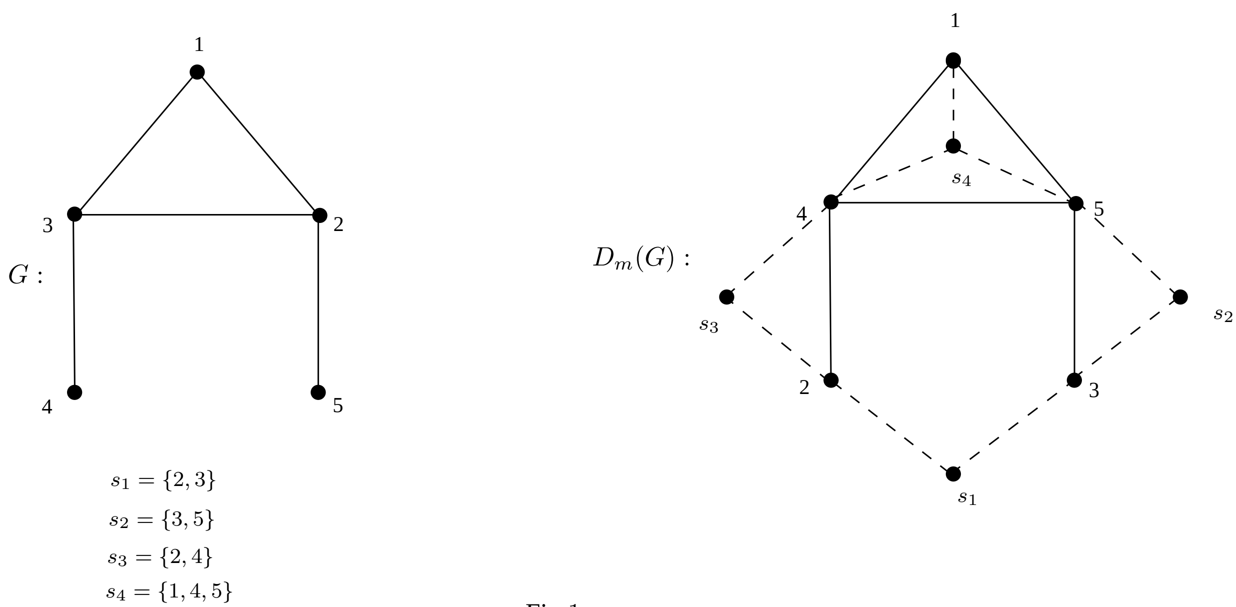

Fig.1

Observations 1.2 The following results are easily observed.:

(1) For any graph G, $\overline{G}$ is an induced subgraph of $D_m(G)$.

(2) Let $S = \{s_1, s_2, \cdots, s_n\}$ be the set of all minimal dominating sets of G, then each s_i; $1 \leq i \leq n$ will be independent in $D_m(G)$.

(3) If $G = K_p$, then $D_m(G) = pK_2$. (4) If $G = \overline{K}_p$, then $D_m(G) = K_{p+1}$.

§2. Results

When defining any class of graphs, it is desirable to know the number of vertices and edges. It is hard to determine for mediate dominating graph. So we obtain a bounds for $D_m(G)$ to determine the number of vertices and edges in $D_m(G)$.

Theorem 2.1 *For any graph G, $p + d(G) \leq p' \leq \dfrac{p(p+1)}{2}$, where $d(G)$ is the domatic number of G and p' denotes the number of vertices of $D_m(G)$. Further the lower bound is attained if and only if $G = \overline{K}_p$ and the upper bound is attained if and only if G is a $(p-2)$ regular graph.*

Proof The lower bound follows from the fact that every graph has at least $d(G)$ number of minimal dominating sets of G and the upper bound follows from the fact that every vertex is in at most $(p-1)$ minimal dominating sets of G.

Suppose the lower bound is attained. Then every vertex is in exactly one minimal dominating set of G and hence, every minimal dominating set is independent. Further, for any two minimal dominating sets D and D', every vertex in D is adjacent to every vertex in D'.

Suppose the upper bound is attained. Then each vertex is in exactly $(p-1)$ minimal dominating sets hence G is $(p-2)$ regular.

Conversely, we first consider the converse part of the equality of the lower bound. If $G = \overline{K}_p$, then $d(\overline{K}_p) = 1$ and there exist exactly one minimal dominating set $S(G)$. Therefore by the definition of $D_m(G)$, $V(D_m(G)) = p + |S(G)| = p + 1 = p + d(G)$.

Now, we consider the converse part of the equality of the upper bound. Suppose G is a $(p-2)$ regular graph. Then G has $\dfrac{p(p-1)}{2}$ minimal dominating sets of G. Therefore by the definition of $D_m(G)$, $V(D_m(G)) = p + |S(G)| = p + \dfrac{p(p-1)}{2} = \dfrac{p(p+1)}{2}$. □

Theorem 2.2 *For any graph G, $p \leq q' \leq \dfrac{p(p+1)}{2}$, where q' denotes the number of edges of $D_m(G)$. Further, the lower bound is attained if and only if $G = K_p$ and the upper bound is attained if and only if $G = \overline{K}_p$.*

Proof First we consider the lower bound. Suppose the lower bound is attained. Then $p = q'$, it follows that $\overline{G}$ contains no edges in $D_m(G)$. Therefore by observation 3, $G = K_p$; $p \geq 2$. Conversely, if $G = K_p$; $p \geq 2$ the $D_m(G) = pK_2$. Therefore $p = q'$.

Now consider the upper bound. Suppose the upper bound is attained. Then $q' = \dfrac{p(p+1)}{2}$. Therefore $\delta(D_m(G)) = \Delta(D_m(G)) = p - 1$. Hence $D_m(G) = K_{p+1}$. By observation 4, $G = \overline{K}_p$.

Conversely, if $G = \overline{K}_p$, then $D_m(G) = K_{p+1}$, since K_{p+1} has $\frac{p(p+1)}{2}$ edges. Therefore $q' = \dfrac{p(p+1)}{2}$. □

In the next theorem, we prove the necessary and sufficient condition for $D_m(G)$ to be connected.

Theorem 2.3 *For any (p,q) graph G, the mediate dominating graph $D_m(G)$ is connected if and only if $\Delta(G) < p - 1$.*

Proof Let $\Delta(G) < p - 1$. We consider the following cases.

Case 1 Let u and v be any two adjacent vertices in G. Suppose there is no minimal dominating set containing both u and v. Then there exist another vertex w in V which is not adjacent to both u and v. Let D and D' be any two maximal independent sets containing u, w and v, w respectively. Since every maximal independent set is a minimal dominating set, hence u and v are connected by a path $uDwD'v$. Thus $D_m(G)$ is connected.

Case 2 Let u and v be any two nonadjacent vertices in G. Then by observation 1, $\overline{G}$ is an induced subgraph of $D_m(G)$. Clearly u and v are connected in $D_m(G)$. Thus from the above two cases $D_m(G)$ is connected.

Conversely, suppose $D_m(G)$ is connected. On the contrary assume that $\Delta(G) = p - 1$. Let u be any vertex of degree $p - 1$. Then u is a minimal dominating set of G and $V - u$ also contains a minimal dominating set of G. It follows that $D_m(G)$ has two components, a contradiction. □

Theorem 2.4 *For any graph G, $D_m(G)$ is either connected or has at least one component which is K_2.*

Proof We consider the following cases:

Case 1 If $\Delta(G) < p - 1$, then by Theorem 2.1, $D_m(G)$ is connected.

Case 2 If $\delta(G) = \Delta(G) = p - 1$, then G is K_p. By Observation 3, $D_m(K_p) = pK_2$.

Case 3 If $\delta(G) < \Delta(G) = p - 1$.

Let $u_1, u_2, \cdots, u_i$ be the vertices of degree $p-1$ in G. Let $H = G - \{u_1, u_2, \cdots, u_i\}$. Then clearly $\Delta(H) < p-1$. By Theorem 2.1, $D_m(H)$ is connected. Since $D_m(G) = D_m(H) \cup (\{u_1\} + u_1) \cup (\{u_2\} + u_2) \cup \cdots \cup (\{u_n\} + u_n)$. Therefore it follows that at least one component of $D_m(G)$ is K_2. □

Corollary 1 *For any graph G, $D_m(G) = K_p \cup K_2$ if and only if $G = K_{1,p-1}$.*

Proof The proof follows from Observation 3 and Theorem 2.6. □

In the next theorem, we characterize the graphs G for which $D_m(G)$ is a tree.

Theorem 2.5 *The mediate dominating graph $D_m(G)$ of G is a tree if and only if $G = K_1$.*

Proof Let the mediate dominating graph $D_m(G)$ of G be a tree and $G \neq K_1$. Then by Theorem 2.3, $\Delta(G) < p - 1$. Hence $D_m(G)$ is connected. Now consider the following cases.

Case 1 Let G be a disconnected graph. If G is totally disconnected graph, then by the observation 4, $D_m(G) = K_{p+1}$, a contradiction.

Let us consider at least one component of G containing an edge uev. Then the smallest possible graph is $G = K_2 \cup K_1$. Therefore $D_m(G) = C_3 \cdot C_3$, a contradiction. Hence for any disconnected graph G of order at least two, $D_m(G)$ must contain a cycle of length at least three, a contradiction. Thus $G = K_1$.

Case 2 Let G be a connected graph with $\Delta(G) < p-1$. By Theorem 2.3, $D_m(G)$ is connected. For $D_m(G)$ to be connected and $\Delta(G) < p-1$, the order of the graph G must be greater than or equal to four. Then there exist at least two nonadjacent vertices u and v in G, which belong to at least one minimal dominating set D of G. Therefore $uvDu$ is a cycle in $D_m(G)$, a contradiction. Thus from above two cases we conclude that $G = K_1$.

Conversely, if $G = K_1$, then by the definition of $D_m(G)$, $D_m(G) = K_2$, which is a tree. □

In the next theorem we characterize the graphs G for which $D_m(G)$ is a cycle.

Theorem 2.6 *The mediate dominating graph $D_m(G)$ of G is a cycle if and only if $G = 2K_1$.*

Proof Let $D_m(G)$ be a cycle. Then by Theorem 2.3, $\Delta(G) < p-1$. Suppose $G \neq 2K_1$, then by Theorem 2.5, $D_m(G)$, $D_m(G)$ is either a tree or containing at least one vertex of degree greater than or equal to 3, a contradiction. Hence $G = 2K_1$.

Conversely, if $G = 2K_1$ then by observation, $D_m(G) = K_3$ or C_3 a cycle. □

Proposition 1 *The mediate dominating graph $D_m(G)$ of G is a complete graph if and only if $G = \overline{K}_p$.*

In the next theorem, we find the diameter of $D_m(G)$.

Theorem 2.7 *Let G be any graph with $\Delta(G) < p-1$, then $diam(D_m(G)) \leq 3$, where $diam(G)$ is the diameter of G.*

Proof Let G be any graph with $\Delta(G) < p-1$, then by Theorem 2.3, $D_m(G)$ is connected. Let $u, v \in V(D_m(G))$ be any two arbitrary vertices in $D_m(G)$. We consider the following cases.

Case 1 Suppose $u, v \in V(G)$, u and v are nonadjacent vertices in G, then $d_{D_m(G)}(u, v) = 1$. If u and v are adjacent in G, suppose there is no minimal dominating set containing both u and v. Then there exist another vertex w in $V(G)$, which is not adjacent to both u and v. Let D and D' be any two maximal independent sets containing u, w and v, w respectively. Since every maximal independent set is a minimal dominating set, hence u and v are connected in $D_m(G)$ by a path $uDwD'v$. Thus, $d_{D_m(G)}(u, v) \leq 3$.

Case 2 Suppose $u \in V$ and $v \notin V$. Then $v = D$ is a minimal dominating set of G. If $u \in D$, then $d_{D_m(G)}(u, v) = 1$. If $u \notin D$, then there exist a vertex $w \in D$ which is adjacent to both u and v. Hence $d_{D_m(G)}(u, v) = d(u, w) + d(w, v) = 2$.

Case 3 Suppose $u, v \in V$. Then $u = D$ and $v = D'$ are two minimal dominating sets of G. If D and D' are disjoint, then every vertex in $w \in D$ is adjacent to some vertex $x \in D'$ and vice versa. This implies that, $d_{D_m(G)}(u, v) = d(u, w) + d(w, x) + d(x, v) = 3$. If D and D' have a vertex in common, then $d_{D_m(G)}(u, v) = d(u, w) + d(w, v) = 2$. Thus from all these cases the result follows. □

In the next two results we prove the vertex and edge connectivity of $D_m(G)$.

Theorem 2.8 *For any graph G,*

$$\kappa(D_m(G)) = \min\{\min(deg_{D_m(G)} v_i)_{1\leq i\leq p}, \min_{1\leq j\leq n} |S_j|\},$$

where $S_j's$ are the minimal dominating sets of G

Proof Let G be a (p,q) graph. We consider the following cases:

Case 1 Let $x \in v_i$ for some i, having minimum degree among all $v_i's$ in $D_m(G)$. If the degree of x is less than any vertex in $D_m(G)$, then by deleting those vertices of $D_m(G)$ which are adjacent with x, results in a disconnected graph.

Case 2 Let $y \in S_j$ for some j, having minimum degree among all vertices of $S_j's$. If degree of y is less than any other vertices in $D_m(G)$, then by deleting those vertices which are adjacent with y, results in a disconnected graph.
Hence the result follows. □

Theorem 2.9 *For any graph G,*

$$\lambda(D_m(G)) = \min\{\min(deg_{D_m(G)} v_i)_{1\leq i\leq p}, \min_{1\leq j\leq n} |S_j|\},$$

where $S_j's$ are the minimal dominating sets of G

Proof The proof is on the same lines of the proof of Theorem 2.8. □

§3. Traversability in $D_m(G)$

The following will be useful in the proof of our results.

Theorem A([2]) *A graph G is Eulerian if and only if every vertex of G has even degree. Next, we prove the necessary and sufficient conditions for $D_m(G)$ to be Eulerian.*

Theorem 3.1 *For any graph G with $\Delta(G) < p-1$, D_m is Eulerian if and only if it satisfies the following conditions:*

(i) *Every minimal dominating set contains even number of vertices;*

(ii) *If $v \in V$ is a vertex of odd degree, then it is in odd number of minimal dominating sets, otherwise it is in even number of minimal dominating sets.*

Proof Suppose $\Delta(G) < p-1$. By Theorem 2.3, $D_m(G)$ is connected. If $D_m(G)$ is Eulerian. On the contrary, if condition (i) is not satisfied, then there exists a minimal dominating set containing odd number of vertices and hence $D_m(G)$ has a vertex of odd degree, therefore by Theorem A, $D_m(G)$ is Eulerian, a contradiction. Similarly we can prove (ii). Conversely, suppose the given conditions are satisfied. Then degree of each vertex in $D_m(G)$ is even. Therefore by Theorem A, $D_m(G)$ is Eulerian. □

Theorem 3.2 *Let G be any graph with $\Delta(G) < p-1$ and $\Gamma(G) = 2$. If every vertex is in exactly two minimal dominating sets of G, then $D_m(G)$ is Hamiltonian.*

Proof Let $\Delta(G) < p-1$. Then by Theorem 2.3, $D_m(G)$ is connected. Clearly $\gamma(G) = \Gamma(G)$ and if every vertex is in exactly two minimal dominating sets then there exist an induced two regular graph in $D_m(G)$. Hence $D_m(G)$ contains a hamiltonian cycle. Therefore $D_m(G)$ is hamiltonian. □

Next, we prove the chromatic number of $D_m(G)$.

Theorem 3.3 *For any graph G,*

$$\chi(D_m(G)) = \begin{cases} \chi(\overline{G}) + 1 & \textit{if vertices of any minimal dominating sets colored by } \chi(\overline{G}) \textit{ colors} \\ \chi(\overline{G}) & \textit{otherwise} \end{cases}$$

Proof Let G be a graph with $\chi(\overline{G}) = k$ and D be the set of all minimal dominating sets of G. Since by the definition of $D_m(G)$, $\overline{G}$ is an induced subgraph of $D_m(G)$ and by Observation 2, D is an independent set. Therefore to color $D_m(G)$, either we can make use of the colors which are used to color $\overline{G}$ that is $\chi(D_m(G)) = k = \chi(\overline{G})$ or we should have to use one more new color. In particular, if the vertices of any minimal dominating set x of G are colored with $k-$colors, then we require one more new color to color x in $D_m(G)$. Hence in this case we require $k+1$ colors to color $D_m(G)$. Therefore $\chi(D_m(G)) = k+1$ This implies, $\chi(D_m(G)) = \chi(\overline{G}) + 1$. □

§4. Characterization of $D_m(G)$

Question. *Is it possible to determine the given graph G is a mediate dominating graph of some graph?*

A partial solution to the above problem is as follows.

Theorem 4.1 *If $G = K_p$; $p \geq 2$, then it is a mediate dominating graph of $\overline{K}_{p-1}$.*

Proof The proof follows from Theorem 2.2. □

Problem 4.1 *Give necessary and sufficient condition for a given graph G is a mediate dominating graph of some graph.*

§5. Domination in $D_m(G)$

We first calculate the domination number of $D_m(G)$ of some standard class of graphs.

Theorem 5.1 (i) *If $G = K_p$, then $\gamma(D_m(K_p)) = p$;*

(ii) *If $G = K_{1,p}$, then $\gamma(D_m(K_{1,p})) = 2$;*

(iii) *If $G = W_p$; $p \geq 4$ then $\gamma(D_m(W_p)) = \gamma(\overline{C}_{p-1}) + 1$;*

(iv) *If $G = P_p$; $p \geq 2$ then $\gamma(D_m(P_p)) = 2$;* □

Theorem 5.2 *Let G be any graph of order p and $S = \{s_1, s_2, \cdots, s_n\}$ be the set of all minimal dominating sets of G, then $\gamma(D_m(G)) \leq \gamma(\overline{G}) + |S|$.*

Proof Let $D = \{v_1, v_2, \cdots, v_i\}$; $1 \leq i \leq p$ be a minimum dominating set of $\overline{G}$. By the definition of $D_{m(G)}$, $\overline{G}$ is an induced subgraph of $D_m(G)$ and by Observation 2, each s_i; $1 \leq i \leq n$ is independent in $D_m(G)$. Hence $D' = D \cup S$ will form a dominating set in $D_m(G)$. Therefore $\gamma(D_m(G)) \leq |D'| = |D \cup S| = \gamma(\overline{G}) + |S|$. □

Theorem 5.3 *Let G be any connected graph with $\delta(G) = 1$, then $\gamma(D_m(G)) = 2$.*

Proof Let G be any connected graph with a minimum degree vertex u, such that $deg(u) = 1$. Let v be a vertex adjacent to u in G. Then $deg_{\overline{G}}(u) = p - 2$, and every minimal dominating set contains either u or v. Hence $D = \{u, v\}$ is a minimal dominating set of $D_m(G)$. Therefore, $\gamma(D_m(G)) = |D| = |\{u, v\}| = 2$. □

Corollary 2 *For any nontrivial tree T, $\gamma(D_m(T)) = 2$.*

Furthermore, we get a Nordhaus-Gaddum type result following.

Theorem 5.4 *Let G be any graph of order p, then*

(i) $\gamma(D_m(G)) + \gamma(D_m(\overline{G})) \leq p + 1$;
(ii) $\gamma(D_m(G)) \cdot \gamma(D_m(\overline{G})) \leq p$.

Further, equality holds if and only if $G = K_p$.

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