ON THE GENERALIZATION OF THE FLOOR OF THE SQUARE ROOT SEQUENCE

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Abstract The floor of the square root sequence is the natural sequence, where each number is repeated $2n+1$ times. In this paper, we use analytic method to study the mean value properties of its generalization, and give an interesting asymptotic formula.

Keywords: the floor of the square root sequence; mean value; asymptotic formula.

§1. Introduction

The floor of the square roots of the natural numbers are:
0,1,1,1,2,2,2,2,2,3,3,3,3,3,3,3,4,4,4,4,4,4,4,4,4, 5,5,5,5,5,5,5,5,5,5,5,6,6, 6,6,6,6,6,6,6,6,6,6,6, 7,7,7,7,7,7,7,7,7,7,7,7,7,7,7,8, $\cdots$

This sequence is the natural sequence, where each number is repeated $2n+1$ times. In reference [1], Professor F.Smarandache asked us to study the properties of this sequence. We denote the generalization of the sequence as $b(n)$, in which each number is repeated $kn+1$ times, and express it as $b(n) = [n^{1/k}]$. In reference [2], He Xiaolin and Guo Jinbao studied the mean value of $d(b(n))$, and obtain an asymptotic formula for it. In this paper, as a generalization of [2], we use analytic method to study the mean value properties of $\sigma_\alpha(b(n))$, and give a general asymptotic formula for $\sigma_\alpha(b(n))$. That is, we shall prove the following :

Theorem. For any real number $x > 1$ and integer $n \geq 1$, we have

$$\sum_{n\leq x}\sigma_\alpha(b(n)) = \begin{cases} \frac{k\zeta(\alpha+1)}{\alpha+k}x^{\frac{\alpha+k}{k}} + O(x^{\frac{\beta+k-1}{k}}), & \text{if } \alpha > 0; \\ \frac{1}{k}x\log x + O(x), & \text{if } \alpha = 0; \\ \zeta(2)x + 0(x^{\frac{k+\varepsilon-1}{k}}), & \text{if } \alpha = -1; \\ \zeta(1-\alpha)x + O(x^{\frac{\delta+k-1}{k}}), & \text{if } \alpha < 0 \text{ and } \alpha \neq -1 \end{cases}$$

where $\sigma_\alpha(n) = \sum\limits_{d|n} d^\alpha$ be the divisor function, $\zeta(n)$ be the Riemann Zeta function, $\beta = \max\{1, \alpha\}, \delta = \max\{0, 1+\alpha\}$ and $\varepsilon > 0$ be an arbitrary real number.

§2. Proof of the Theorem

In this section, we shall complete the proof of the theorem. Firstly, we need following:

Lemma 1. Let $\alpha > 0$ be a fixed real number. Then for $x > 1$, we have

$$\sum_{n \le x} \sigma_\alpha(n) = \frac{\zeta(\alpha+1)}{\alpha+1} x^{\alpha+1} + O(x^\beta),$$

and

$$\sum_{n \le x} \sigma_{-\alpha}(n) = \begin{cases} \zeta(\alpha+1)x + O(x^\delta), & \text{if } \alpha \ne 1; \\ \zeta(2)x + O(\log x), & \text{if } \alpha = 1. \end{cases}$$

where $\beta = \max\{1, \alpha\}$, $\delta = \max\{0, 1-\alpha\}$, $\zeta(n)$ denotes the Riemann zeta-function.

Proof See reference [3].

Lemma 2. Let n be a positive integer, and $b(n) = [n^{1/k}]$, $d(n)$ be the divisor function, then

$$\sum_{n \le x} \sigma_0(b(n)) = \sum_{n \le x} d\left([n^{1/k}]\right) = \frac{1}{k} x \log x + O(x).$$

Proof. See reference [2].

Now we use the above Lemmas to complete the proof of Theorem. We separate α into three cases respectively.

Case 1 , when $\alpha > 0$, we have

$$\begin{aligned} & \sum_{n \le x} \sigma_\alpha(b(n)) = \sum_{n \le x} \sigma_\alpha([n^{1/k}]) \\ = & \sum_{1^k \le n < 2^k} \sigma_\alpha\left([n^{1/k}]\right) + \sum_{2^k \le n < 3^k} \sigma_\alpha\left([n^{1/k}]\right) + \cdots \\ & + \sum_{N^k \le n < (N+1)^k} \sigma_\alpha\left([n^{1/k}]\right) + O(N^\beta) \\ = & \sum_{j \le N} \left((j+1)^k - j^k\right) \sigma_\alpha(j) + O(N^\beta). \end{aligned}$$

Let $A(n) = \sum\limits_{n \le x} \sigma_\alpha(j)$ and $f(j) = \sum\limits_{j \le N} \left((j+1)^k - j^k\right)$, applying Abel's identity and Lemma 1, we have

$$\begin{aligned} & \sum_{n \le x} \sigma_\alpha(b(n)) \\ = & A(N)f(N) - A(1)f(1) - \int_1^N A(t) f^{'}(t) dt + O(N^\beta) \\ = & \frac{k\zeta(\alpha+1)}{\alpha+1} N^{\alpha+k} - k(k-1) \int_1^N \frac{\zeta(\alpha+1)}{\alpha+k} t^{\alpha+k+1} dt \end{aligned}$$

$$
\begin{aligned}
& +O(N^{\beta+k-1}) \\
= & \frac{k\zeta(\alpha+1)}{\alpha+k}N^{\alpha+k}+O(N^{\beta+k-1}) \\
= & \frac{k\zeta(\alpha+1)}{\alpha+k}x^{\frac{\alpha+k}{k}}+O(x^{\frac{\beta+k-1}{k}}).
\end{aligned}
$$

Case 2, when $\alpha=-1$, we have

$$
\begin{aligned}
& \sum_{n\le x}\sigma_{-1}(b(n))=\sum_{n\le x}\sigma_{-1}([n^{1/k}]) \\
= & \sum_{1^k\le n<2^k}\sigma_{-1}\left([n^{1/k}]\right)+\sum_{2^k\le n<3^k}\sigma_{-1}\left([n^{1/k}]\right)+\cdots \\
& +\sum_{N^k\le n<(N+1)^k}\sigma_{-1}\left([n^{1/k}]\right)+O(N^{\varepsilon}) \\
= & \sum_{j\le N}\left((j+1)^k-j^k\right)\sigma_{-1}(j)+O(N^{\varepsilon}).
\end{aligned}
$$

Let $A(n)=\sum\limits_{j\le N}\sigma_{-1}(j)$ and $f(j)=\sum\limits_{j\le N}\left((j+1)^k-j^k\right)$, we have

$$
\begin{aligned}
& \sum_{n\le x}\sigma_{-1}(b(n))=\sum_{n\le x}\sigma_{-1}[n^{1/k}] \\
= & A(N)f(N)-A(1)f(1)-\int_1^N A(t)f'(t)dt+O(N^{\varepsilon}) \\
= & \zeta(2)N^k+O(N^{k+\varepsilon-1}) \\
= & \zeta(2)x+O(x^{\frac{k+\varepsilon-1}{k}}),
\end{aligned}
$$

where $\varepsilon>0$ be an arbitrary real number.

Case 3, when $\alpha<0$ and $\alpha\ne-1$, we have

$$
\begin{aligned}
& \sum_{n\le x}\sigma_{\alpha}(b(n))=\sum_{n\le x}\sigma_{\alpha}([n^{1/k}]) \\
= & \sum_{1^k\le n<2^k}\sigma_{\alpha}\left([n^{1/k}]\right)+\sum_{2^k\le n<3^k}\sigma_{\alpha}\left([n^{1/k}]\right)+\cdots \\
& +\sum_{N^k\le n<(N+1)^k}\sigma_{\alpha}\left([n^{1/k}]\right)+O(N^{\delta}) \\
= & \sum_{j\le N}\left((j+1)^k-j^k\right)\sigma_{\alpha}(j)+O(N^{\delta}).
\end{aligned}
$$

Let $A(n)=\sum\limits_{j\le N}\sigma_{\alpha}(j)$ and $f(j)=\sum\limits_{j\le N}\left((j+1)^k-j^k\right)$, we have

$$
\sum_{n\le x}\sigma_{\alpha}(b(n))=\sum_{n\le x}\sigma_{\alpha}([n^{1/k}])
$$

$$\begin{aligned}
&= A(N)f(N) - A(1)f(1) - \int_1^N A(t)f^{'}(t)dt + O(N^{\delta}) \\
&= k\zeta(1-\alpha)N^k + O(N^{k+\delta-1}) - (k-1)\zeta(1-\alpha)N^k \\
&= \zeta(1-\alpha)N^k + O(N^{k+\delta-1}) \\
&= \zeta(1-\alpha)x + O(x^{\frac{\delta+k-1}{k}}).
\end{aligned}$$

Combining the above result, we have

$$\sum_{n\le x}\sigma_\alpha(b(n)) = \begin{cases} \frac{k\zeta(\alpha+1)}{\alpha+k}x^{\frac{\alpha+k}{k}} + O(x^{\frac{\beta+k-1}{k}}), & \text{if } \alpha > 0; \\ \frac{1}{k}x\log x + O(x), & \text{if } \alpha = 0; \\ \zeta(2)x + 0(x^{\frac{k+\varepsilon-1}{k}}), & \text{if } \alpha = -1; \\ \zeta(1-\alpha)x + O(x^{\frac{\delta+k-1}{k}}), & \text{if } \alpha < 0 \text{ and } \alpha \ne -1 \end{cases}$$

This completes the proof of Theorem.

References

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