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On the generalized constructive set

Qianli Yang

Department of Mathematics, Weinan Teacher's College
Weinan, Shaanxi, P.R.China

Abstract In this paper, we use the elementary methods to study the properties of the constructive set S, and obtain some interesting properties for it.

Keywords Generalized constructive set, summation, recurrence equation, characteristic equation.

§1. Introduction and Results

The generalized constructive set S is defined as: numbers formed by digits $d_1, d_2, \cdots, d_m$ only, all d_i being different each other, $1 \leq m \leq 9$. That is to say,

(1) $d_1,\ d_2,\ \cdots, d_m$ belongs to S;

(2) If $a,\ b$ belong to S, then $\overline{ab}$ belongs to S too;

(3) Only elements obtained by rules (1) and (2) applied a finite number of times belongs to S.

For example, the constructive set (of digits 1, 2) is: $1, 2, 11, 12, 21, 22, 111, 112, 121, 122, 211, 212, 221, 222, 1111, 1112, 1121 \cdots$. And the constructive set (of digits 1, 2, 3) is: $1, 2, 3, 11, 12, 13, 21, 22, 23, 31, 32, 33, 111, 112, 113, 121, 122, 123, 131, 132, 133, 211, 212, 213, 221, 222, 223, 231, 232, 233, 311, 312, 313, 321, 322, 323, 331, 332, 333, 1111, \cdots$. In problem 6, 7 and 8 of reference [1], Professor F.Smarandache asked us to study the properties of this sequence. In [2], Gou Su had studied the convergent properties of the series

$$\sum_{n=1}^{+\infty} \frac{1}{a_n^{\alpha}},$$

and proved that the series is convergent if $\alpha > \log m$, and divergent if $\alpha \leq \log m$, where $\{a_n\}$ denotes the sequence of the constructive set S, formed by digits $d_1, d_2, \cdots, d_m$ only, all d_i being different each other, $1 \leq m \leq 9$.

In this paper, we shall use the elementary methods to study the summation $\sum\limits_{k=1}^{n} S_k$ and $\sum\limits_{k=1}^{n} T_k$, where S_k denotes the summation of all k digits numbers in S, T_k denotes the summation of each digits of all k digits numbers in S.

That is, we shall prove the following

Theorem 1. For the generalized constructive set S of digits $d_1,\ d_2,\cdots,d_m\ (1\leq m\leq 9)$, we have

$$\sum_{k=1}^{n} S_k=\frac{d_1+d_2+\cdots+d_m}{9}\left(10\times\frac{(10m)^n-1}{10m-1}-\frac{m^n-1}{m-1}\right),$$

where S_k denotes the summation of all k digits numbers in S.

Taking $m=2,\ d_1=1$ and $d_2=2$ in Theorem 1, we may immediately get

Corollary 1. For the generalized constructive set S of digits 1 and 2, we have

$$\sum_{k=1}^{n} S_k=\frac{1}{3}\left(10\times\frac{20^n-1}{19}-2^n+1\right).$$

Taking $m=3,\ d_1=1,\ d_2=2$ and $d_3=3$ in Theorem 1, we may immediately get the following:

Corollary 2. For the generalized constructive set S of digits $1,2$ and 3, we have

$$\sum_{k=1}^{n} S_k=\frac{2}{3}\left(10\times\frac{30^n-1}{29}-\frac{3^n}{2}+\frac{1}{2}\right).$$

Theorem 2. For the generalized constructive set S of digits $d_1,\ d_2,\cdots,d_m\ (1\leq m\leq 9)$, we have

$$\sum_{k=1}^{n} T_k=(d_1+d_2+\cdots+d_m)\cdot\frac{nm^{n+1}-(n+1)m^n+1}{(m-1)^2},$$

where T_k denotes the summation of each digits of all k digits numbers in S.

Taking $m=2,\ d_1=1$ and $d_2=2$ in Theorem 2, we may immediately get the following:

Corollary 3. For the the generalized constructive set S of digits 1 and 2, we have

$$\sum_{k=1}^{n} T_k=3n\cdot 2^{n+1}-3(n+1)2^n+3.$$

Taking $m=3,\ d_1=1,\ d_2=2$ and $d_3=3$ in Theorem 2, we may immediately get

Corollary 4. For the the generalized constructive set S of digits $1,2$ and 3, we have

$$\sum_{k=1}^{n} T_k=\frac{3}{2}n\cdot 3^{n+1}-\frac{3}{2}(n+1)3^n+\frac{3}{2}.$$

§2. Proof of the theorems

In this section, we shall complete the proof of the theorems. First we prove Theorem 1. Let S_k denotes the summation of all k digits numbers in the generalized constructive set S. Note that for $k=1,2,3,\cdots$, there are m^k numbers of k digits in S. So we have

$$S_k=10^{k-1}m^{k-1}(d_1+d_2+\cdots+d_m)+mS_{k-1}. \tag{1}$$

Meanwhile, we have

$$S_{k-1}=10^{k-2}m^{k-2}(d_1+d_2+\cdots+d_m)+mS_{k-2}. \tag{2}$$

Combining (1) and (2), we can get the following recurrence equation

$$S_k - 11mS_{k-1} + 10m^2S_{k-2} = 0.$$

Its characteristic equation

$$x^2 - 11mx + 10m^2 = 0$$

have two different real solutions

$$x = m,\ 10m.$$

So we let

$$S_k = A \cdot m^k + B \cdot (10m)^k.$$

Note that

$$S_0 = 0, \quad S_1 = d_1 + d_2 + \cdots + d_m,$$

we can get

$$A = -\frac{d_1 + d_2 + \cdots + d_m}{9m}, \quad B = \frac{d_1 + d_2 + \cdots + d_m}{9m}.$$

So

$$S_k = \frac{d_1 + d_2 + \cdots + d_m}{9m}\left((10m)^k - m^k\right).$$

Then

$$\sum_{k=1}^{n} S_k = \frac{d_1 + d_2 + \cdots + d_m}{9}\left(10 \times \frac{(10m)^n - 1}{10m - 1} - \frac{m^n - 1}{m - 1}\right).$$

This completes the proof of Theorem 1.

Now we come to prove Theorem 2. Let T_k is denotes the summation of each digits of all k digits numbers in the generalized constructive set S.

Similarly, note that for $k = 1, 2, 3, \cdots$, there are m^k numbers of k digits in S, so we have

$$T_k \ = \ m^{k-1}(d_1 + d_2 + \cdots + d_m) + mT_{k-1} \tag{3}$$

Meanwhile, we have

$$T_{k-1} = m^{k-2}(d_1 + d_2 + \cdots + d_m) + mT_{k-2} \tag{4}$$

Combining (3) and (4), we can get the following recurrence equation

$$T_k - 2mT_{k-1} + m^2T_{k-2} = 0,$$

its characteristic equation

$$x^2 - 2mx + m^2 = 0$$

have two solutions

$$x_1 = x_2 = m.$$

So we let

$$T_k = A \cdot m^k + k \cdot B \cdot m^k.$$

Note that

$$T_0 = 0, \quad T_1 = d_1 + d_2 + \cdots + d_m.$$

We may immediately deduce that

$$A = 0, \quad B = \frac{d_1 + d_2 + \cdots + d_m}{m}.$$

So

$$T_k = (d_1 + d_2 + \cdots + d_m) \cdot km^{k-1}.$$

Then

$$\begin{aligned} \sum_{k=1}^{n} T_k &= (d_1 + d_2 + \cdots + d_m) \sum_{k=1}^{n} k \cdot m^{k-1} \\ &= (d_1 + d_2 + \cdots + d_m) \left(\sum_{k=1}^{n} m^k \right)' \\ &= (d_1 + d_2 + \cdots + d_m) \cdot \frac{nm^{n+1} - (n+1)m^n + 1}{(m-1)^2}. \end{aligned}$$

This completes the proof of Theorem 2.

References

[1] F. Smarandache, Only Problems, Not Solutions, Chicago, Xiquan Publishing House, 1993.

[2] Gou Su, On the generalized constructive set, Research on Smarandache problems in number theory, Hexis, 2005, 53-55.