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Absolutely Harmonious Labeling of Graphs

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Abstract: Absolutely harmonious labeling f is an injection from the vertex set of a graph G with q edges to the set $\{0, 1, 2, ..., q-1\}$, if each edge uv is assigned $f(u)+f(v)$ then the resulting edge labels can be arranged as $a_0, a_1, a_2, ..., a_{q-1}$ where $a_i = q - i$ *or* $q + i$, $0 \leq i \leq q-1$. However, when G is a tree one of the vertex labels may be assigned to exactly two vertices. A graph which admits absolutely harmonious labeling is called *absolutely harmonious graph*. In this paper, we obtain necessary conditions for a graph to be absolutely harmonious and study absolutely harmonious behavior of certain classes of graphs.

Key Words: Graph labeling, Smarandachely k-labeling, harmonious labeling, absolutely harmonious labeling.

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§1. Introduction

A vertex labeling of a graph G is an assignment f of labels to the vertices of G that induces a label for each edge xy depending on the vertex labels. For an integer $k \geq 1$, a *Smarandachely k-labeling* of a graph G is a bijective mapping $f : V \rightarrow \{1, 2, \cdots, k|V(G)| + |E(G)|\}$ with an additional condition that $|f(u) - f(v)| \geq k$ for $\forall uv \in E$. particularly, if $k = 1$, i.e., such a Smarandachely 1-labeling is the usually labeling of graph. Among them, labelings such as those of *graceful labeling, harmonious labeling* and *mean labeling* are some of the interesting vertex labelings found in the dynamic survey of graph labeling by Gallian [2]. Harmonious labeling is one of the fundamental labelings introduced by Graham and Sloane [3] in 1980 in connection with their study on error correcting code. *Harmonious labeling* f is an injection from the vertex set of a graph G with q edges to the set $\{0, 1, 2, ..., q-1\}$, if each edge uv is assigned $f(u) + f(v)(mod\ q)$ then the resulting edge labels are distinct. However, when G is a tree one of the vertex labels may be assigned to exactly two vertices. Subsequently a few variations of harmonious labeling, namely, *strongly c-harmonious labeling* [1], *sequential labeling* [5], *elegant labeling* [1] and *felicitous labeling* [4] were introduced. The later three labelings were introduced to avoid such exceptions for the trees given in harmonious labeling. A strongly

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1-harmonious graph is also known as strongly harmonious graph.

It is interesting to note that if a graph G with q edges is harmonious then the resulting edge labels can be arranged as $a_0, a_1, a_2, \cdots, a_{q-1}$ where $a_i = i$ *or* $q+i$, $0 \le i \le q-1$. That is for each i, $0 \le i \le q-1$ there is a distinct edge with label either i or $q+i$. An another interesting and natural variation of edge label could be $q-i$ or $q+i$. This prompts to define a new variation of harmonious labeling called *absolutely harmonious labeling.*

Definition 1.1 *An absolutely harmonious labeling f is an injection from the vertex set of a graph G with q edges to the set $\{0,1,2,...,q-1\}$, if each edge uv is assigned $f(u)+f(v)$ then the resulting edge labels can be arranged as $a_0, a_1, a_2, ..., a_{q-1}$ where $a_i = q-i$ or $q+i$, $0 \le i \le q-1$. However, when G is a tree one of the vertex labels may be assigned to exactly two vertices. A graph which admits absolutely harmonious labeling is called absolutely harmonious graph.*

The result of Graham and Sloane [3] states that $C_n, n \cong 1(mod\ 4)$ is harmonious, but we show that $C_n, n \cong 1(mod\ 4)$ is not an absolutely harmonious graph. On the other hand, we show that C_4 is an absolutely harmonious graph, but it is not harmonious. We observe that a strongly harmonious graph is an absolutely harmonious graph.

To initiate the investigation on absolutely harmonious graphs, we obtain necessary conditions for a graph to be an absolutely harmonious graph and prove the following results:

1. Path $P_n, n \ge 3$, a class of banana trees, and $P_n \odot K_m^c$ are absolutely harmonious graphs.

2. Ladders, $C_n \odot K_m^c$, Triangular snakes, Quadrilateral snakes, and mK_4 are absolutely harmonious graphs.

3. Complete graph K_n is absolutely harmonious if and only if $n = 3$ or 4.

4. Cycle $C_n, n \cong 1$ *or* $2\ (mod\ 4)$, $C_m \times C_n$ where m and n are odd, $mK_3, m \ge 2$ are not absolutely harmonious graphs.

§2. Necessary Conditions

Theorem 2.1 *If G is an absolutely harmonious graph, then there exists a partition (V_1, V_2) of the vertex set $V(G)$, such that the number of edges connecting the vertices of V_1 to the vertices of V_2 is exactly $\left\lceil \frac{q}{2} \right\rceil$.*

Proof If G is an absolutely harmonious graph,then the vertices can be partitioned into two sets V_1 and V_2 having respectively even and odd vertex labels. Observe that among the q edges $\frac{q}{2}$ edges or $\left\lceil \frac{q}{2} \right\rceil$ edges are labeled with odd numbers, according as q is even or q is odd. For an edge to have odd label, one end vertex must be odd labeled and the other end vertex must be even labeled. Thus, the number of edges connecting the vertices of V_1 to the vertices of V_2 is exactly $\left\lceil \frac{q}{2} \right\rceil$. □

Remark 2.2 A simple and straight forward application of Theorem 2.1 identifies the non absolutely harmonious graphs. For example, complete graph K_n has $\frac{n(n-1)}{2}$ edges. If we assign

m vertices to the part V_1, there will be $m(n-m)$ edges connecting the vertices of V_1 to the vertices of V_2. If K_n has an absolutely harmonious labeling, then there is a choice of m for which $m(n-m) = \left\lceil \frac{n^2-n}{4} \right\rceil$. Such a choice of m does not exist for $n = 5, 7, 8.10, \ldots$.

A graph is called *even graph* if degree of each vertex is even.

Theorem 2.3 *If an even graph G is absolutely harmonious then $q \cong 0$ or 3 (mod 4).*

Proof Let G be an even graph with $q \cong 1$ *or* 2 (*mod* 4) and $d(v)$ denotes the degree of the vertex v in G. Suppose f be an absolutely harmonious labeling of G. Then the resulting edge labels can be arranged as $a_0, a_1, a_2, \ldots, a_{q-1}$ where $a_i = q - i$ *or* $q + i$, $0 \le i \le q-1$. In other words, for each i, the edge label a_i is $(q-i) + 2ib_i$, $0 \le i \le q-1$ where $b_i \in \{0, 1\}$. Evidently

$$\sum_{v \in V(G)} d(v)f(v) - 2\sum_{k=0}^{q-1} kb_k = \binom{q+1}{2}.$$

As $d(v)$ is even for each v and $q \cong 1$ *or* 2 (*mod* 4),

$$\sum_{v \in V(G)} d(v)f(v) - 2\sum_{k=0}^{q-1} kb_k \cong 0 \ (mod\ 2)$$

but $\binom{q+1}{2} \cong 1 \ (mod\ 2)$. This contradiction proves the theorem. □

Corollary 2.4 *A cycle C_n is not an absolutely harmonious graph if $n \cong 1$ or 2 (mod 4).*

Corollary 2.5 *A grid $C_m \times C_n$ is not an absolutely harmonious graph if m and n are odd.*

Theorem 2.6 *If f is an absolutely harmonious labeling of the cycle C_n , then edges of C_n can be partitioned into two sub sets E_1, E_2 such that*

$$\sum_{uv \in E_1} |f(u) + f(v) - n| = \frac{n(n+1)}{4} \quad and \quad \sum_{uv \in E_2} |f(u) + f(v) - n| = \frac{n(n-3)}{4}.$$

Proof Let $v_1v_2v_3 \ldots v_nv_1$ be the cycle C_n, where $e_i = v_{i-1}v_i, 2 \le i \le n$ and $e_1 = v_nv_1$. Define $E_1 = \{uv \in \ E/\, f(u) + f(v) - n \ is\ non\ negative\}$ and $E_2 = \{uv \in \ E/\, f(u) + f(v) - n\ is\ negative\}$. Since f is an absolutely harmonious labeling of the cycle C_n,

$$\sum_{uv \in E} |f(u) + f(v) - n| = \frac{n(n-1)}{2}.$$

In other words,

$$\sum_{uv \in E_1} |f(u) + f(v) - n| + \sum_{uv \in E_2} |f(u) + f(v) - n| = \frac{n(n-1)}{2}. \qquad (1)$$

Since $\sum_{uv \in E}(f(u) + f(v) - n) = -n$, we have

$$\sum_{uv \in E_1} |f(u) + f(v) - n| - \sum_{uv \in E_2} |f(u) + f(v) - n| = -n. \qquad (2)$$

Solving equations (1) and (2), we get the desired result. □

Remark 2.7 If $n \cong 1$ *or* 2 $(mod\ 4)$ then both $\dfrac{n(n+1)}{4}$ and $\dfrac{n(n-3)}{4}$ cannot be integers. Thus the cycle C_n is not an absolutely harmonious graph if $n \cong 1$ *or* 2 $(mod\ 4)$.

Remark 2.8 Observe that the conditions stated in Theorem 2.1, Theorem 2.3, and Theorem 2.6 are necessary but not sufficient. Note that C_8 satisfies all the conditions stated in Theorems 2.1, 2.3, and 2.6 but it is not an absolutely harmonious graph. For, checking each of the $\dfrac{8!}{2}$ possibilities reveals the desired result about C_8.

§3. Absolutely Harmonious Graphs

Theorem 3.1 *The path P_{n+1}, where $n \geq 2$ is an absolutely harmonious graph.*

Proof Let $P_{n+1} : v_1v_2...v_{n+1}$ be a path, $r = \left\lceil \dfrac{n}{2} \right\rceil$, $s = \begin{cases} \lceil \frac{r}{2} \rceil + 1 & \text{if } n \cong 0\ (mod\ 4) \\ \lceil \frac{r}{2} \rceil & \text{otherwise} \end{cases}$,

$t = \begin{cases} s-1 & \text{if } n \cong 0 \text{ or } 1\ (mod\ 4) \\ s & \text{if } n \cong 2 \text{ or } 3\ (mod\ 4) \end{cases}$, $T_1 = n$, $T_2 = \begin{cases} 2t+2 & \text{if } n \cong 0 \text{ or } 1\ (mod\ 4) \\ 2t+1 & \text{if } n \cong 2 \text{ or } 3\ (mod\ 4) \end{cases}$ and $T_3 = \begin{cases} -1 & \text{if } n \cong 0 \text{ or } 1\ (mod\ 4) \\ -2 & \text{if } n \cong 2 \text{ or } 3\ (mod\ 4) \end{cases}$.

Then $r+s+t = n+1$. Define $f : V(P_{n+1}) \to \{0, 1, 2, 3, \cdots, n-1\}$ by:

$f(v_i) = T_1 - i$ *if* $1 \leq i \leq r$, $f(v_{r+i}) = T_2 - 2i$ *if* $1 \leq i \leq s$ *and* $f(v_{r+s+i}) = T_3 + 2i$ *if* $1 \leq i \leq t$.

Evidently f is an absolutely harmonious labeling of P_{n+1}. For example, an absolutely harmonious labeling of P_{12} is shown in Fig.3.1. □

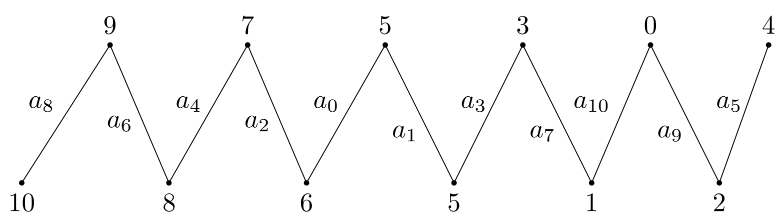

Fig.3.1

The tree obtained by joining a new vertex v to one pendant vertex of each of the k disjoint stars $K_{1,n_1}, K_{1,n_2}, K_{1,n_3}, ..., K_{1,n_k}$ is called a *banana tree*. The class of all such trees is denoted by $BT(n_1, n_2, n_3, ..., n_k)$.

Theorem 3.2 *The banana tree $BT(n, n, n, ..., n)$ is absolutely harmonious.*

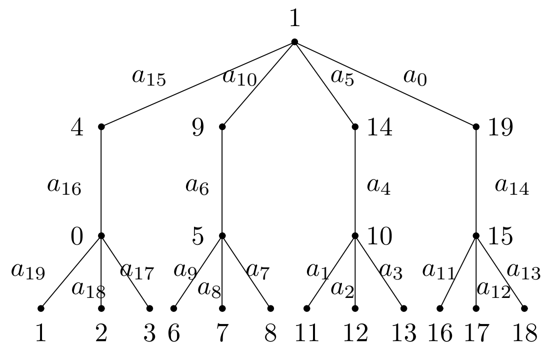

Fig.3.2

Proof Let $V(BT(n,n,n,\cdots,n)=\{v\}\cup\{v_j,v_{jr}:1\leq j\leq k \ and \ 1\leq r\leq n\}$ where $d(v_j)=n$ and $E(BT(n,n,n,...,n)=\{vv_{jn}:1\leq j\leq k\}\cup\{v_jv_{jr}:1\leq j\leq k,1\leq r\leq n\}$. Clearly $BT(n,n,\cdots,n)$ has order $(n+1)k+1$ and size $(n+1)k$. Define

$$f:V(BT(n,n,\cdots,n)\to\{1,2,3,...,(n+1)k-1\}$$

as follows:

$$f(v)=1,\ f(v_j)=(n+1)(j-1):1\leq j\leq k,\ f(v_{jr})=(n+1)(j-1)+r:1\leq r\leq n.$$

It can be easily verified that f is an absolutely harmonious labeling of $BT(n,n,n,...,n)$. For example an absolutely harmonious labeling of $BT(4,4,4,4)$is shown in Fig.3.2. □

The *corona* $G_1\odot G_2$ of two graphs $G_1(p_1,q_1)$ and $G_2(p_2,q_2)$ is defined as the graph obtained by taking one copy of G_1 and p_1 copies of G_2 and then joining the i^{th} vertex of G_1 to all the vertices in the i^{th} copy of G_2.

Theorem 3.3 *The corona $P_n\odot K_m^C$ is absolutely harmonious.*

Proof Let $V(P_n\odot K_m^C)=\{u_i:1\leq i\leq n\}\cup\{u_{ij}:1\leq i\leq n,1\leq j\leq m\}$ and $E(P_n\odot K_m^C)=\{u_iu_{i+1}:1\leq i\leq n-1\}\cup\{u_iu_{ij}:1\leq i\leq n,1\leq j\leq m\}$. We observe that $P_n\odot K_m^C$ has order $(m+1)n$ and size $(m+1)n-1$. Define $f:V(P_n\odot K_m^C)\longrightarrow\{0,1,2,...,mn+n-2\}$ as follows:

$$f(u_i)=\begin{cases}0 & if\ i=1,\\ (m+1)(i-1) & \text{if } i=\lceil\frac{n}{2}\rceil\\ (m+1)(i-1)-1 & \text{otherwise,}\end{cases}\qquad f(u_{im})=\begin{cases}(m+1)i & \text{if } 1\leq i\leq\lceil\frac{n}{2}\rceil-2,\\ (m+1)i-1 & \text{if } i=\lceil\frac{n}{2}\rceil-1,\\ (m+1)i-2 & \lceil\frac{n}{2}\rceil\leq i\leq n,\end{cases}$$

and for $1\leq j\leq m-1$,

$$f(u_{ij})=\begin{cases}(m+1)(i-1)+j & \text{if } 1\leq i\leq\lceil\frac{n}{2}\rceil-1,\\ (m+1)(i-1)+j-1 & \text{if } \lceil\frac{n}{2}\rceil\leq i\leq n.\end{cases}$$

It can be easily verified that f is an absolutely harmonious labeling of $P_n\odot K_m^C$. For example an absolutely harmonious labeling of $P_5\odot K_3^C$ is shown in Fig. 3.3. □

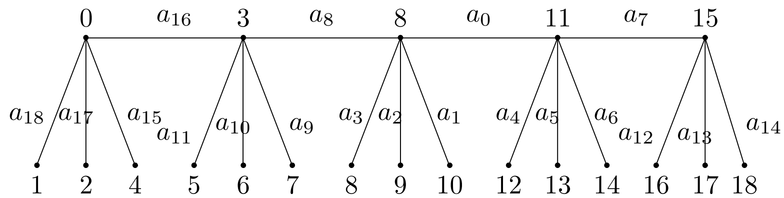

Fig.3.3

Theorem 3.4 *The corona $C_n \odot K_m^C$ is absolutely harmonious.*

Proof Let $V(C_n \odot K_m^C) = \{u_i : 1 \leq i \leq n\} \cup \{u_{ij} : 1 \leq i \leq n, 1 \leq j \leq m\}$ and $E(C_n \odot K_m^C) = \{u_i u_{i+1} : 1 \leq i \leq n-1\} \cup \{u_n u_1\} \cup \{u_i u_{ij} : 1 \leq i \leq n, 1 \leq j \leq m\}$. We observe that $C_n \odot K_m^C$ has order $(m+1)n$ and size $(m+1)n$. Define $f : V(C_n \odot K_m^C) \longrightarrow \{0, 1, 2, ..., mn + n - 1\}$ as follows:

$$f(u_i) = \begin{cases} 0 & \text{if } i = 1, \\ (m+1)(i-1) - 1 & \text{if } 2 \leq i \leq \frac{n-1}{2}, \\ (m+1)(i-1) & \text{otherwise,} \end{cases}, \quad f(u_{im}) = \begin{cases} (m+1)i & \text{if } 1 \leq i \leq \frac{n-3}{2}, \\ (m+1)i - 1 & \text{otherwise} \end{cases}$$

and for $1 \leq j \leq m-1$

$$f(u_{ij}) = \begin{cases} (m+1)(i-1) + j & \text{if } 1 \leq i \leq \lceil \frac{n}{2} \rceil - 1, \\ (m+1)(i-1) + j - 1 & \text{if } \lceil \frac{n}{2} \rceil \leq i \leq n. \end{cases}$$

It can be easily verified that f is an absolutely harmonious labeling of $C_n \odot K_m^C$. For example an absolutely harmonious labeling of $C_5 \odot K_3^C$ is shown in Figure 3.4. □

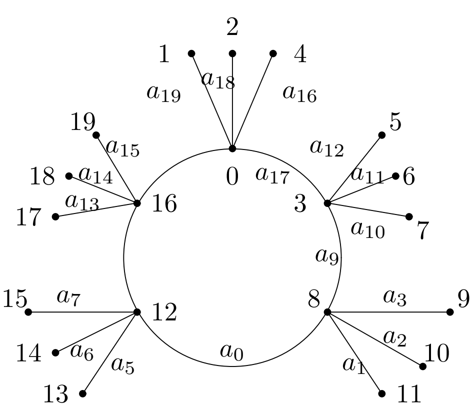

Fig.3.4

Theorem 3.5 *The ladder $P_n \times P_2$, where $n \geq 2$ is an absolutely harmonious graph.*

Proof Let $V(P_n \times P_2) = \{u_1, u_2, u_3, ..., u_n\} \cup \{v_1, v_2, v_3, ..., v_n\}$ and $E(P_n \times P_2) = \{u_i u_{i+1} : 1 \leq i \leq n-1\} \cup \{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{u_i v_i : 1 \leq i \leq n\}$. We note that $P_n \times P_2$ has order

$2n$ and size $3n-2$.

Case 1. $n \equiv 0(mod\ 4)$.

Define $f : V(P_n \times P_2) \longrightarrow \{0, 1, 2, ..., 3n-3\}$ by

$$f(u_i) = \begin{cases} 3i-2 & \text{if } i \text{ is odd,} \\ 3i-2 & \text{if } i \text{ is even and } 2 \le i \le \frac{n-4}{2}, \\ 3i-1 & \text{if } i \text{ is even and } i = \frac{n}{2}, \\ 3i-3 & \text{if } i \text{ is even and } \frac{n+4}{2} \le i \le n, \end{cases}$$

$$f(v_1) = 0, \quad f(v_{\frac{n+2}{2}}) = \frac{3n-6}{2}, \quad f(v_{i+1}) = f(u_i) + 1 \text{ if } 1 \le i \le n-1 \text{ and } i \ne \frac{n}{2}.$$

Case 2. $n \equiv 1(mod\ 4)$.

Define $f : V(P_n \times P_2) \longrightarrow \{0, 1, 2, ..., 3n-3\}$ by

$$f(u_i) = \begin{cases} 3i-2 & \text{if } i \text{ is odd and } 1 \le i \le \frac{n-3}{2}, \\ 3i-1 & \text{if } i = \frac{n+1}{2}, \\ 3i-3 & \text{if } i \text{ is odd and } \frac{n+5}{2} \le i \le n, \\ 3i-2 & \text{if } i \text{ is even,} \end{cases}$$

$$f(v_1) = 0, \quad f(v_{\frac{n+3}{2}}) = \tfrac{3n-3}{2}, \quad f(v_{i+1}) = f(u_i) + 1 \text{ if } 1 \le i \le n-1 \text{ and } i \ne \tfrac{n+1}{2}.$$

Case 3. $n \equiv 2(mod\ 4)$.

Define $f : V(P_n \times P_2) \longrightarrow \{0, 1, 2, ..., 3n-3\}$ by

$$f(u_i) = \begin{cases} 3i-2 & \text{if } i \text{ is odd,} \\ 3i-2 & \text{if } i \text{ is even and } 2 \le i \le \frac{n-2}{2}, \\ 3i-3 & \text{if } i \text{ is even and } \frac{n+2}{2} \le i \le n, \end{cases}$$

$$f(v_1) = 0, \quad f(v_{i+1}) = f(u_i) + 1 \text{ if } 1 \le i \le n-1.$$

Case 4. $n \equiv 3(mod\ 4)$.

Define $f : V(P_n \times P_2) \longrightarrow \{0, 1, 2, ..., 3n-3\}$ by

$$f(u_i) = \begin{cases} 3i-2 & \text{if } i \text{ is odd and } 1 \le i \le \frac{n-1}{2}, \\ 3i-3 & \text{if } i \text{ is odd and } \frac{n+3}{2} \le i \le n, \\ 3i-2 & \text{if } i \text{ is even.} \end{cases}$$

$$f(v_1) = 0, \quad f(v_{i+1}) = f(u_i) + 1 \text{ if } 1 \le i \le n-1.$$

In all four cases, it can be easily verified that f is an absolutely harmonious labeling of $P_n \times P_2$. For example, an absolutely harmonious labeling of $P_9 \times P_2$ is shown in Fig.3.5. $\square$

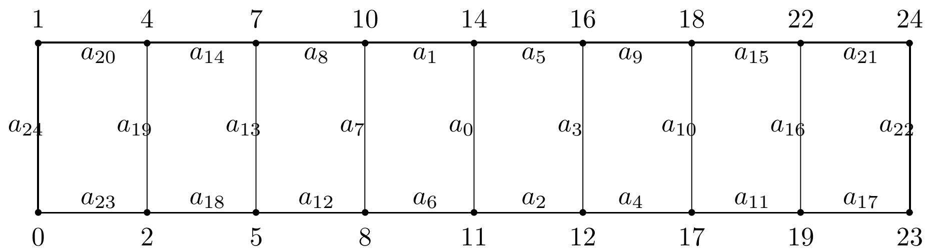

Fig.3.5

A K_n-*snake* has been defined as a connected graph in which all blocks are isomorphic to K_n and the block-cut point graph is a path. A K_3-snake is called *triangular snake.*

Theorem 3.6 *A triangular snake with n blocks is absolutely harmonious if and only if $n \cong$ 0 or 1 $(mod\ 4)$.*

Proof The necessity follows from Theorem 2.3.Let G_n be a triangular snake with n blocks on p vertices and q edges. Then $p = 2n - 1$ and $q = 3n$. Let $V(G_n) = \{u_i : 1 \leq i \leq n+1\} \cup \{v_i : 1 \leq i \leq n\}$ and $E(G_n) = \{u_iu_{i+1}, u_iv_i, u_{i+1}v_i : 1 \leq i \leq n\}$.

Case 1. $n \equiv 0\ (mod\ 4)$.

Let $m = \frac{n}{4}$. Define $f : V(G_n) \longrightarrow \{0, 1, 2, ..., 3n - 1\}$ as follows:

$$f(u_i) = \begin{cases} 0 & \text{if } i = 1, \\ 2i - 2 & \text{if } 2 \leq i \leq 3m \text{ and } i \equiv 0 \text{ or } 2\ (mod\ 3), \\ 2i - 1 & \text{if } 2 \leq i \leq 3m \text{ and } i \equiv 1\ (mod\ 3), \\ 6i - 3n - 7 & \text{otherwise,} \end{cases}$$

$$f(v_i) = \begin{cases} 1 & \text{if } i = 1, \\ 2i - 1 & \text{if } 2 \leq i \leq 3m - 1 \text{ and } i \equiv \ \text{ or } 2\ (mod\ 3), \\ 2i - 2 & \text{if } 2 \leq i \leq 3m - 1 \text{ and } i \equiv 1\ (mod\ 3), \\ 6m + 1 & \text{if } i = 3m, \\ 6i - 3n - 3 & \text{otherwise.} \end{cases}$$

Case 2. $n \equiv 1\ (mod\ 4)$.

Let $m = \frac{n-1}{4}$. Define $f : V(G_n) \longrightarrow \{0, 1, 2, ..., 3n - 1\}$ as follows:

$$f(u_i) = \begin{cases} 0 & \text{if } i = 1, \\ 2i - 2 & \text{if } 2 \le i \le 3m+2 \text{ and } i \equiv 0 \; or \; 2 \; (mod\ 3), \\ 2i - 1 & \text{if } 2 \le i \le 3m+2 \text{ and } i \equiv 1 \; (mod\ 3), \\ 6i - 3n - 7 & \text{otherwise,} \end{cases}$$

$$f(v_i) = \begin{cases} 1 & \text{if } i = 1, \\ 2i - 1 & \text{if } 2 \le i \le 3m+1 \text{ and } i \equiv 0 \; or \; 2 \; (mod\ 3) \\ 2i - 2 & \text{if } 2 \le i \le 3m+1 \text{ and } i \equiv 1 \; (mod\ 3) \\ 6i - 3n - 3 & \text{otherwise.} \end{cases}$$

In both cases, it can be easily verified that f is an absolutely harmonious labeling of the triangular snake G_n. For example, an absolutely harmonious labeling of a triangular snake with five blocks is shown in Fig.3.6. □

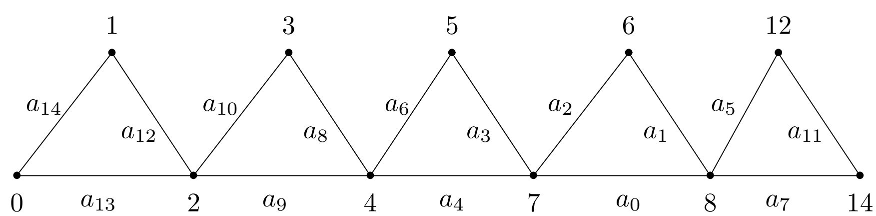

Fig.3.6

Theorem 3.7 *K_4-snakes are absolutely harmonious.*

Proof Let G_n be a K_4-snake with n blocks on p vertices and q edges. Then $p = 3n+1$ and $q = 6n$. Let $V(G_n) = \{u_i, v_i, w_i : 1 \le i \le n\} \cup \{v_{n+1}\}$ and $E(G_n) = \{u_iv_i, u_iw_i, v_iw_i : 1 \le i \le n\} \cup \{u_iv_{i+1}, v_iv_{i+1}, w_iv_{i+1} : 1 \le i \le n\}$ Define $f : V(G_n) \longrightarrow \{0, 1, 2, ..., 6n-1\}$ as follows:

$$f(u_i) = 3i - 3, f(v_i) = 3i - 2, f(w_i) = 3i - 1$$

where $1 \le i \le n$, and $f(v_{n+1}) = 3n+1$. It can be easily verified that f is an absolutely harmonious labeling of G_n and hence K_4-snakes are absolutely harmonious. For example, an absolutely harmonious labeling of a K_4-snake with five blocks is shown in Fig.3.7. □

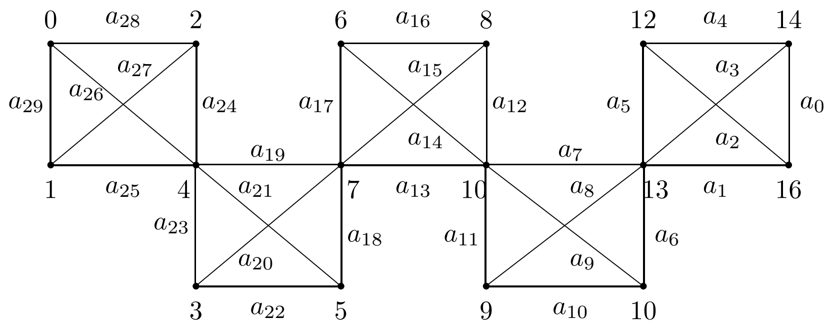

Fig.3.7

A *quadrilateral snake* is obtained from a path $u_1u_2...u_{n+1}$ by joining u_i, u_{i+1} to new vertices v_i, w_i respectively and joining v_i and w_i.

Theorem 3.8 *All quadrilateral snakes are absolutely harmonious.*

Proof Let G_n be a quadrilateral snake with $V(G_n) = \{u_i : 1 \leq i \leq n+1\} \cup \{v_i, w_i : 1 \leq i \leq n\}$ and $E(G_n) = \{u_iu_{i+1}, u_iv_i, u_{i+1}w_i, v_iw_i : 1 \leq i \leq n\}$. Then $p = 3n+1$ and $q = 4n$. Let

$$m = \begin{cases} \frac{n}{2} & if\ n \equiv 0\ (mod\ 2) \\ \frac{n-1}{2} & if\ n \equiv 1\ (mod\ 2) \end{cases}.$$

Define $f : V(G_n) \longrightarrow \{0, 1, 2, ...4n-1\}$ as follows:

$$f(u_i) = \begin{cases} 0 & \text{if } i = 1, \\ 4i-6 & \text{if } 2 \leq i \leq m+1, \\ 4i-7 & \text{if } m+2 \leq i \leq n+1 \end{cases}, \quad f(v_i) = \begin{cases} 4i-3 & \text{if } 1 \leq i \leq m, \\ 4i-2 & \text{if } m+1 \leq i \leq n, \end{cases}$$

$$f(w_i) = \begin{cases} 4i & \text{if } 1 \leq i \leq m, \\ 4i-1 & \text{if } m+1 \leq i \leq n. \end{cases}$$

It can be easily verified that f is an absolutely harmonious labeling of the quadrilateral snake G_n and hence quadrilateral snakes are absolutely harmonious. For example, an absolutely harmonious labeling of a quadrilateral snake with six blocks is shown in Fig.3.8. □

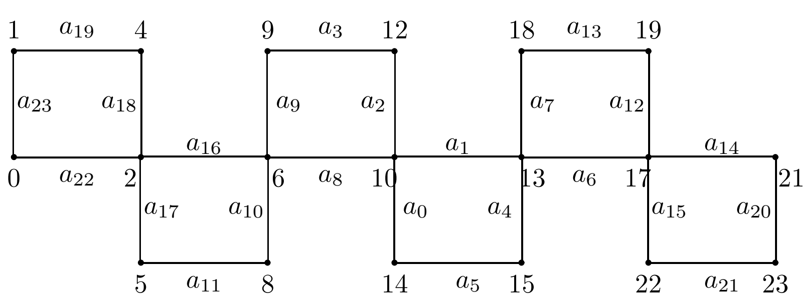

Fig.3.8

Theorem 3.9 *The disjoint union of m copies of the complete graph on four vertices, mK_4 is absolutely harmonious.*

Proof Let u_i^j where $1 \leq i \leq 4$ and $1 \leq j \leq m$ denotes the i^{th} vertex of the j^{th} copy of mK_4. We note that that mK_4 has order $4m$ and size $6m$. Define $f : V(mK_4) \longrightarrow \{0, 1, 2, ...6m-1\}$ as follows:$f(u_1^1) = 0, f(u_2^1) = 1, f(u_3^1) = 2, f(u_4^1) = 4, f(u_1^2) = q-3, f(u_2^2) = q-4, f(u_3^2) = q-5, f(u_4^2) = q-7, f(u_i^{j+2}) = f(u_i^j) + 6\ if\ j\ is\ odd, and\ f(u_i^{j+2}) = f(u_i^j) - 6\ if\ j\ is\ even,$ where $1 \leq i \leq 4$ and $1 \leq j \leq m-2$. Clearly f is an absolutely harmonious labeling. For example, an absolutely harmonious labeling of $5K_4$ is shown in Figure 11. *Box*

Observation 3.10 If f is an absolutely harmonious labeling of a graph G,which is not a tree, then

1. Each x in the set $\{0, 1, 2\}$ has inverse image.
2. Inverse images of 0 and 1 are adjacent in G.
3. Inverse images of 0 and 2 are adjacent in G.

Theorem 3.11 *The disjoint union of m copies of the complete graph on three vertices, mK_3 is absolutely harmonious if and only if $m = 1$.*

Proof Let u_i^j,where1 $\leq i \leq 3$ and $1 \leq j \leq m$ denote the i^{th} vertex of the j^{th} copy of mK_3. Assignments of the values $0, 1, 2$ to the vertices of K_3 gives the desired absolutely harmonious labeling of K_3. For $m \geq 2$, mK_3 has $3m$ vertices and $3m$ edges. If mK_3 is an absolutely harmonious graph, we can assign the numbers $\{0, 1, 2,\ 3m-1\}$ to the vertices of mK_3 in such a way that its edges receive each of the numbers a_0,a_1,...,a_{q-1} where $a_i = q-i\ or\ q+i$, $0 \leq i \leq q-1$. By Observation 3.10, we can assume, without loss of generality that $f(u_1^1) = 0, f(u_2^1) = 1, f(u_3^1) = 2$. Thus we get the edge labels a_{q-1}, a_{q-2} and a_{q-3}. In order to have an edge labeled a_{q-4}, we must have two adjacent vertices labeled $q-1$ and $q-3$. we can assume without loss of generality that $f(u_1^2) = q-1$ and $f(u_2^2) = q-3$. In order to have an edge labeled a_{q-5}, we must have $f(u_2^3) = q-4$. There is now no way to obtain an edge labeled a_{q-6}. This contradiction proves the theorem. □

Theorem 3.12 *A complete graph K_n is absolutely harmonious graph if and only if $n = 3$ or 4.*

Proof From the definition of absolutely harmonious labeling, it can be easily verified that K_1 and K_2 are not absolutely harmonious graphs. Assignments of the values $0, 1, 2$ and $0, 1, 2, 4$ respectively to the vertices of K_3 and K_4 give the desired absolutely harmonious labeling of them. For $n > 4$, the graph K_n has $q \geq 10$ edges. If K_n is an absolutely harmonious graph, we can assign a subset of the numbers $\{0, 1, 2,\ q-1\}$ to the vertices of K_n in such a way that the edges receive each of the numbers a_0,a_1,...,a_{q-1} where $a_i = q-i\ or\ q+i$, $0 \leq i \leq q-1$. By Observation 3.10, 0, 1, and 2 must be vertex labels. With vertices labeled $0, 1$, and 2, we have edges labeled a_{q-1}, a_{q-2} and a_{q-3}. To have an edge labeled a_{q-4} we must adjoin the vertex label 4. Had we adjoined the vertex label 3 to induce a_{q-4}, we would have two edges labeled a_{q-3}, namely, between 0 and 3, and between 1 and 2. Had we adjoined the vertex labels $q-1$

and $q-3$ to induce a_{q-4}, we would have three edges labeled a_1, namely, between $q-1$ and 0, between $q-1$ and 2, and between $q-3$ and 2. With vertices labeled $0, 1, 2,$and 4, we have edges labeled a_{q-1}, a_{q-2}, a_{q-3}, a_{q-4}, a_{q-5}, and a_{q-6}. Note that for K_4 with $q = 6$, this gives the absolutely harmonious labeling. To have an edge labeled a_{q-7}, we must adjoin the vertex label 7; all the other choices are ruled out. With vertices labeled $0, 1, 2, 4$ and 7, we have edges labeled a_{q-1}, a_{q-2}, a_{q-3}, a_{q-4}, a_{q-5}, a_{q-6}, a_{q-7}, a_{q-8}, a_{q-9}, and a_{q-11}. There is now no way to obtain an edge labeled a_{q-10}, because each of the ways to induce a_{q-10} using two numbers contains at least one number that can not be assigned as vertex label. We may easily verify that the following boxed numbers are not possible choices as vertex labels:

$$\begin{array}{cc}
0 & \boxed{10} \\
1 & \boxed{9} \\
2 & \boxed{8} \\
\boxed{3} & 7 \\
4 & \boxed{6} \\
\boxed{q-1} & \boxed{q-9} \\
\boxed{q-2} & \boxed{q-8} \\
\boxed{q-3} & \boxed{q-7} \\
\boxed{q-4} & \boxed{q-6}
\end{array}$$

This contradiction proves the theorem. □

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