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Smarandache Idempotents in finite ring Z_n and in Group Ring Z_nG

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Abstract In this paper we analyze and study the Smarandache idempotents (S-idempotents) in the ring Z_n and in the group ring Z_nG of a finite group G over the finite ring Z_n. We have shown the existance of Smarandache idempotents (S-idempotents) in the ring Z_n when $n = 2^m p$ (or $3p$), where p is a prime > 2 (or p a prime > 3). Also we have shown the existance of Smarandache idempotents (S-idempotents) in the group ring Z_2G and Z_2S_n where $n = 2^m p$ (p a prime of the form $2^m t + 1$).

§1. Introduction

This paper has 4 sections. In section 1, we just give the basic definition of S-idempotents in rings. In section 2, we prove the existence of S-idempotents in the ring Z_n where $n = 2^m p, m \in N$ and p is an odd prime. We also prove the existence of S-idempotents for the ring Z_n where n is of the form $n = 3p$, p is a prime greater than 3. In section 3, we prove the existence of S-idempotents in group rings Z_2G of cyclic group G over Z_2 where order of G is n, $n = 2^m p$ (p a prime of the form $2^m t + 1$). We also prove the existence of S-idempotents for the group ring Z_2S_n where $n = 2^m p$ (p a prime of the form $2^m t + 1$). In the final section, we propose some interesting number theoretic problems based on our study.

Here we just recollect the definition of Smarandache idempotents (S-idempotent) and some basic results to make this paper a self contained one.

Definition 1.1[5]. *Let R be a ring. An element $x \in R \quad 0$ is said to be a Smarandache idempotent (S-idempotent) of R if $x^2 = x$ and there exist $a \in R \quad x, 0$ such that*

$$i. \quad a^2 = x$$

$$ii. \quad xa = x \quad or \quad ax = a.$$

Example 1.1. Let $Z_1 0 = \{0, 1, 2, \ldots, 9\}$ be the ring of integers modulo 10. Here

$$6^2 \equiv 6(\text{mod}10), \quad 4^2 \equiv 6(\text{mod}10)$$

and

$$6 \cdot 4 \equiv 4(\text{mod}10).$$

So 6 is a S-idempotent in Z_{10}.

Example 1.2. Take $Z_{12} = \{0, 1, 2, \ldots, 11\}$ the ring of integers modulo 12. Here

$$4^2 \equiv 4(\mathrm{mod} 12), \quad 8^2 \equiv 4(\mathrm{mod} 12)$$

and

$$4 \cdot 8 \equiv 8(\mathrm{mod} 12).$$

So 4 is a S-idempotent in Z_{12}.

Example 1.3. In $Z_{30} = \{0, 1, 2, \ldots, 29\}$ the ring of integers modulo 30, 25 is a S-idempotent. As

$$25^2 \equiv 25(\mathrm{mod} 30), \quad 5^2 \equiv 25(\mathrm{mod} 30)$$

and

$$25 \cdot 5 \equiv 5(\mathrm{mod} 30).$$

So 25 is a S-idempotent in Z_{30}.

Theorem 1.1 [5]. *Let R be a ring. If $x \in R$ is a S-idempotent then it is an idempotent in R.*

Proof. From the very definition of S-idempotents.

§2. S-idempotents in the finite ring Z_n

In this section, we find conditions for Z_n to have S-idempotents and prove that when n is of the form $2^m p$, p a prime ¿2 or $n = 3p$ (p a prime ¿3) has S-idempotents. We also explicitly find all the S-idempotents.

Theorem 2.1. $Z_p = \{0, 1, 2, \ldots, p-1\}$, *the prime field of characteristic p, where p is a prime has no non-trivial S-idempotents.*

Proof. Straightforward, as every S-idempotents are idempotents and Z_p has no non-trivial idempotents.

Theorem 2.2: *The ring Z_{2p}, where p is an odd prime has S-idempotents.*

Proof. Here p is an odd prime, so p must be of the form $2m + 1$ i.e $p = 2m + 1$. Take

$$x = p + 1 \qquad \textit{and} \qquad a = p - 1.$$

Here

$$\begin{aligned} p^2 = (2m+1)^2 &= 4m^2 + 4m + 1 \\ &= 2m(2m+1) + 2m + 1 \\ &= 2pm + p \\ &\equiv p(\mathrm{mod} 2p). \end{aligned}$$

So

$$p^2 \equiv p(\mathrm{mod} 2p).$$

Again

$$\begin{aligned} x^2 = (p+1)^2 &\equiv p^2 + 1(\text{mod}2p) \\ &\equiv p + 1(\text{mod}2p). \end{aligned}$$

Therefore

$$x^2 = x.$$

Also

$$a^2 = (p-1)^2 \equiv p + 1(\text{mod}2p),$$

therefore

$$a^2 = x.$$

And

$$\begin{aligned} xa &= (p+1)(p-1) \\ &= p^2 - 1 \\ &\equiv p - 1(\text{mod}2p) \end{aligned}$$

therefore

$$xa = a.$$

So $x = p + 1$ is a S-idempotent in Z_{2p}.

Example 2.1. Take $Z_6 = Z_{2\cdot 3} = \{0, 1, 2, 3, 4, 5\}$ the ring of integers modulo 6. Then $x = 3 + 1 = 4$ is a S-idempotent. As

$$x^2 = 4^2 \equiv 4(\text{mod}6),$$

take $a = 2$, then $a^2 = 2^2 \equiv 4(\text{mod}6)$.
Therefore

$$a^2 = x,$$

and

$$xa = 4 \cdot 2 \equiv 2(\text{mod}6)$$

i.e

$$xa = a.$$

Theorem 2.3. *The ring Z_{2^2p}, p a prime > 2 and is of the form $4m + 1$ or $4m + 3$ has (at least) two S-idempotents.*

Proof. Here p is of the form $4m + 1$ or $4m + 3$.

If $p = 4m + 1$, then $p^2 \equiv p(\text{mod}2^2p)$. As

$$\begin{aligned} p^2 &= (4m+1)^2 \\ &= 16m^2 + 8m + 1 \\ &= 4m(4m+1) + 4m + 1 \\ &= 4pm + p \\ &\equiv p(\text{mod}2^2p), \end{aligned}$$

therefore

$$p^2 \equiv p(\bmod 2^2 p).$$

Now, take $x = 3p + 1$ and $a = p - 1$ then

$$\begin{aligned} x^2 = (3p+1)^2 &= 9p^2 + 6p + 1 \\ &\equiv 9p + 6p + 1(\bmod 2^2 p) \\ &\equiv 3p + 1(\bmod 2^2 p) \end{aligned}$$

therefore

$$a^2 = x.$$

And

$$\begin{aligned} xa &= (3p+1)(p-1) \\ &= 3p^2 - 3p + p - 1 \\ &\equiv p - 1(\bmod 2^2 p) \end{aligned}$$

therefore

$$xa = a.$$

So x is an S-idempotent.

Similarly, we can prove that $y = p$, (here take $a = 3p$) is another S-idempotent. These are the only two S-idempotents in $Z_{2^2 p}$ when $p = 4m + 1$. If $p = 4m + 3$, then $p^2 \equiv 3p(\bmod 2^2 p)$.

As above, we can show that $x = p + 1, (a = 3p - 1)$ and $y = 3p, (a = p)$ are the two S-idempotents. So we are getting a nice pattern here for S-idempotents in $Z_{2^2 p}$:

I. If $p = 4m + 1$, then $x = 3p + 1$, $(a = p - 1)$ and $y = p$, $(a = 3p)$ are the two S-idempotents.

II. If $p = 4m+3$, $x = p+1$, $(a = 3p-1)$ and $y = 3p$, $(a = p)$ are the two S-idempotents.

Example 2.2. Take $Z_{2^2 \cdot 5} = \{0, 1, \ldots, 19\}$, here $5 = 4 \cdot 1 + 1$. So $x = 3 \cdot 5 + 1 = 16$, $(a = 5 - 1 = 4)$ is an S-idempotent. As $16^2 \equiv 16(\bmod 20), 4^2 \equiv 16(\bmod 20)$ and $16 \cdot 4 \equiv 4(\bmod 20)$. Also $y = 5, (a = 3 \cdot 5 = 15)$ is another S-idempotent. As $5^2 \equiv 5(\bmod 20), 15^2 \equiv 5(\bmod 20)$ and $5 \cdot 15 \equiv 15(\bmod 20)$.

Example 2.3. In the ring $Z_{2^2 \cdot 7} = \{0, 1, \ldots, 27\}$, here $7 = 4 \cdot 1 + 3, x = 7 + 1 = 8, (a = 3 \cdot 7 - 1 = 20)$ is an S-idempotent. As $8^2 \equiv 8(\bmod 28), 20^2 \equiv 8(\bmod 28)$ and $8 \cdot 20 \equiv 20(\bmod 28)$. Also $y = 3 \cdot 7 = 21, (a = 7)$ is another S-idempotent. As $21^2 \equiv 21(\bmod 28), 7^2 \equiv 21(\bmod 28)$ and $21 \cdot 7 \equiv 7(\bmod 28)$.

Theorem 2.4. *The ring $Z_{2^3 p}$, p a prime > 2 has (at least) two S-idempotents of $\phi(2^3)$ types (where $\phi(n)$ is the number of integer less than n and relatively prime to n).*

Proof. As p is prime > 2. So p is one of the $8m + 1, 8m + 3, 8m + 5, 8m + 7$. Now we will get the following two S-idempotents for each $\phi(2^3) = 4$ types of prime p.

I. If $p = 8m + 1$, then $x = 7p + 1, (a = p - 1)$ and $y = p$, $(a = 7p)$ are S-idempotents.

II. If $p = 8m + 3$, then $x = 5p + 1, (a = 3p - 1)$ and $y = 3p$, $(a = 5p)$ are S-idempotents.

III. If $p = 8m + 5$, then $x = 3p + 1, (a = 5p - 1)$ and $y = 5p$, $(a = 3p)$ are S-idempotents.

IV. If $p = 8m + 7$, then $x = p + 1, (a = 7p - 1)$ and $y = 7p$, $(a = p)$ are S-idempotents.

Example 2.4. In the ring $Z_{2^3 \cdot 3} = \{0, 1, \ldots, 23\}$, here $3 = 8 \cdot 0 + 3$. So $x = 5 \cdot 3 + 1 = 16, (a = 3 \cdot 3 - 1 = 8)$ is an S-idempotent. As $16^2 \equiv 16(\text{mod}24), 8^2 \equiv 16(\text{mod}24)$ and $16 \cdot 8 \equiv 8(\text{mod}24)$. Also $y = 3 \cdot 3 = 9, (a = 5 \cdot 3 = 15)$ is another S-idempotent. As $9^2 \equiv 9(\text{mod}24), 15^2 \equiv 9(\text{mod}24)$ and $9 \cdot 15 \equiv 15(\text{mod}24)$.

Example 2.5. Take $Z_{2^3 \cdot 13} = Z_{104} = \{0, 1, \ldots, 103\}$, here $13 = 8 \cdot 1 + 5$. So $x = 3 \cdot 13 + 1 = 40, (a = 5 \cdot 13 - 1 = 64)$ is an S-idempotent. As $40^2 \equiv 40(\text{mod}104), 64^2 \equiv 40(\text{mod}104)$ and $40 \cdot 64 \equiv 64(\text{mod}104)$. Also $y = 5 \cdot 13 = 65, (a = 3 \cdot 13 = 39)$ is another S-idempotent. As $65^2 \equiv 65(\text{mod}104), 39^2 \equiv 65(\text{mod}104)$ and $65 \cdot 39 \equiv 39(\text{mod}104)$.

Theorem 2.5. *The ring $Z_{2^4 p}$, p a prime > 2 has (at least) two S-idempotents for each of $\phi(2^4)$ types of prime p.*

Proof. As above, we can list the S-idempotents for all $\phi(2^4) = 8$ types of prime p.

I. If $p = 16m + 1$, then $x = 15p + 1, (a = p - 1)$ and $y = p$, $(a = 15p)$ are S-idempotents.

II. If $p = 16m + 3$, then $x = 13p + 1, (a = 3p - 1)$ and $y = 3p$, $(a = 13p)$ are S-idempotents.

III. If $p = 16m + 5$, then $x = 11p + 1, (a = 5p - 1)$ and $y = 5p$, $(a = 11p)$ are S-idempotents.

IV. If $p = 16m + 7$, then $x = 9p + 1, (a = 7p - 1)$ and $y = 7p$, $(a = 9p)$ are S-idempotents.

V. If $p = 16m + 9$, then $x = 7p + 1, (a = 9p - 1)$ and $y = 9p$, $(a = 7p)$ are S-idempotents.

VI. If $p = 16m + 11$, then $x = 5p + 1, (a = 11p - 1)$ and $y = 11p$, $(a = 5p)$ are S-idempotents.

VII. If $p = 16m + 13$, then $x = 3p + 1, (a = 13p - 1)$ and $y = 13p$, $(a = 13p)$ are S-idempotents.

VIII. If $p = 16m + 15$, then $x = p + 1, (a = 15p - 1)$ and $y = 15p$, $(a = p)$ are S-idempotents.

Example 2.6. In the ring $Z_{2^4 \cdot 17} = Z_{272} = \{0, 1, \ldots, 271\}$, here $17 = 16 \cdot 1 + 1$. So $x = 15 \cdot 17 + 1 = 256, (a = 17 - 1 = 16)$ is an S-idempotent. As $256^2 \equiv 256(\text{mod}272), 16^2 \equiv 256(\text{mod}272)$ and $256 \cdot 16 \equiv 16(\text{mod}272)$. Also $y = 17, (a = 15 \cdot 17 = 255)$ is another S-idempotent. As $17^2 \equiv 17(\text{mod}272), 255^2 \equiv 17(\text{mod}272)$ and $17 \cdot 255 \equiv 255(\text{mod}272)$.

We can generalize the above result as followings:

Theorem 2.6. *The ring $Z_{2^n p}$, p a prime > 2 has (at least) two S-idempotents for each of $\phi(2^n)$ types of prime p.*

Proof. Here p is one of the $\phi(2^n)$ form:

$$2^n m_1 + 1, \quad 2^n m_2 + 3, \quad \ldots \quad 2^n m_{\phi(2^n)} + (2^n - 1).$$

We can find the two S-idempotents for each p as above. We are showing here for the prime $p = 2^n m_1 + 1$ only. If

$$p = 2^n m_1 + 1,$$

then

$$x = (2^n - 1)p + 1, \quad (a = p - 1)$$

and

$$y = p, \quad (a = (2^n - 1)p)$$

are S-idempotents.

Similarly we can find S-idempotents for each of the $\phi(2^n)$ form of prime p.

Theorem 2.7. *The ring Z_{3p}, p a prime > 3 has (at least) two S-idempotents of $\phi(3)$ types.*

Proof. Here p can be one of the form $3m+1$ or $3m+2$. We can apply the Theorem 2.6 for Z_{3p} also.

I. If $p = 3m+1$, then $x = 2p+1, (a = p-1)$ and $y = p$, $(a = 2p)$ are S-idempotents.

II. If $p = 3m+2$, then $x = p+1, (a = 2p-1)$ and $y = 2p$, $(a = p)$ are S-idempotents.

Example 2.7. In the ring $Z_{3\cdot 5} = Z_{15} = \{0, 1, \ldots, 14\}$, here $5 = 3 \cdot 1 + 2$. So $x = 5+1 = 6, (a = 2\cdot 5 - 1 = 9)$ is an S-idempotent. As $6^2 \equiv 6(\text{mod}15), 9^2 \equiv 6(\text{mod}15)$ and $6 \cdot 9 \equiv 9(\text{mod}15)$. Also $y = 2 \cdot 5 = 10, (a = 5)$ is another S-idempotent. As $10^2 \equiv 10(\text{mod}15), 5^2 \equiv 10(\text{mod}15)$ and $10 \cdot 5 \equiv 5(\text{mod}15)$.

Remark: The above result is not true for the ring Z_{3^2p}, p prime > 3. As, for $p = 9m+5; x = 4p+1, (a = 5p-1)$ should be an S-idempotent from the above result. But we see it is not the case in general; for take the ring $Z_{3^2\cdot 23} = Z_{207} = \{0, 1, \ldots, 206\}$. Here $p = 9\cdot 2+5$. Now take

$$x = 4 \cdot 23 + 1 = 93 \quad \textit{and} \quad a = 5 \cdot 23 - 1 = 114.$$

But

$$x^2 \not\equiv x(\text{mod}207).$$

So x is not even an idempotent. So $x = 4p+1$ is not an S-idempotent of Z_{3^2p}.

§3. S-idempotents in the group rings Z_2G

Here we prove the existance of Smarandache idempotents for the group rings Z_{3^2p} of the cyclic group G of order $2^n p$ where p is a prime of the form $2^n t + 1$.

Example 3.2. Let $G = \{g/g^{52} = 1\}$ be the cyclic group of order $2^2 \cdot 13$. Consider the group ring Z_2G of the group G over Z_2. Take

$$x = 1 + g^4 + g^8 + g^{12} + \ldots + g^{44} + g^{48}$$

and

$$a = 1 + g^2 + g^4 + \ldots + g^{22} + g^{24}$$

then

$$x^2 = x, \quad \textit{and} \quad a^2 = x$$

also

$$x \cdot a = x.$$

So $x = 1 + g^4 + g^8 + g^{12} + \ldots + g^{44} + g^{48}$ is a S-idempotent in Z_2G.

Theorem 3.1. *Let Z_2G be the group ring of the finite cyclic group G of order 2^2p, where p is a prime of the form 2^2m+1, then the group ring Z_2G has non-trivial S-idempotents.*

Proof. Here G is a cyclic group of order 2^2p, where p of the form 2^2m+1.

Take

$$x = 1 + g^4 + g^8 + \ldots + g^{16m}$$

and

$$a = 1 + g^2 + g^4 + \ldots + g^{8m}$$

then

$$\begin{aligned} x^2 &= (1+g^4+g^8+\ldots+g^{16m})^2 \\ &= 1+g^4+g^8+\ldots+g^{16m} \\ &= x. \end{aligned}$$

And

$$\begin{aligned} a^2 &= (1+g^2+g^4+\ldots+g^{8m})^2 \\ &= 1+(g^2)^2+(g^4)^2+\ldots+(g^{8m})^2 \\ &= x. \end{aligned}$$

Also

$$\begin{aligned} x\cdot a &= (1+g^4+g^8+\ldots+g^{16m})(1+g^2+g^4+\ldots+g^{8m}) \\ &= 1+g^4+g^8+\ldots+g^{16m} \\ &= x. \end{aligned}$$

So $x = 1+g^4+g^8+\ldots+g^{16m}$ is a S-idempotent in Z_2G.

Example 3.3. Let $G = \{g/g^{136} = 1\}$ be the cyclic group of order $2^3 \cdot 17$. Consider the group ring Z_2G of the group G over Z_2.

Take

$$x = 1+g^8+g^{16}+\ldots+g^{128}$$

and

$$a = 1+g^4+g^8+\ldots+g^{64}$$

then

$$\begin{aligned} x^2 &= (1+g^8+g^{16}+\ldots+g^{128})^2 \\ &= 1+g^8+g^{16}+\ldots+g^{128} \\ &= x. \end{aligned}$$

And

$$\begin{aligned} a^2 &= (1+g^4+g^8+\ldots+g^{64})^2 \\ &= 1+(g^4)^2+(g^8)^2+\ldots+(g^{64})^2 \\ &= x. \end{aligned}$$

Also

$$\begin{aligned} x\cdot a &= (1+g^8+g^{16}+\ldots+g^{128})(1+g^4+g^8+\ldots+g^{64}) \\ &= 1+g^8+g^{64}+\ldots+g^{128} \\ &= x. \end{aligned}$$

So $x = 1+g^8+g^{16}+\ldots+g^{128}$ is a S-idempotent in Z_2G.

Theorem 3.2. *Let Z_2G be the group ring of a finite cyclic group G of order 2^3p, where p is a prime of the form 2^3m+1, then the group ring Z_2G has non-trivial S-idempotents.*

Proof. Here G is a cyclic group of order 2^3p, where p of the form 2^3m+1.

Take

$$x = 1 + g^8 + g^{16} + \ldots + g^{8(p-1)}$$

and

$$a = 1 + g^4 + g^8 + \ldots + g^{4(p-1)}$$

then

$$\begin{aligned} x^2 &= (1 + g^8 + g^{16} + \ldots + g^{8(p-1)})^2 \\ &= 1 + g^8 + g^{16} + \ldots + g^{8(p-1)} \\ &= x. \end{aligned}$$

And

$$\begin{aligned} a^2 &= (1 + g^4 + g^8 + \ldots + g^{4(p-1)})^2 \\ &= 1 + (g^4)^2 + (g^8)^2 + \ldots + (g^{8(p-1)})^2 \\ &= x. \end{aligned}$$

Also

$$\begin{aligned} x \cdot a &= (1 + g^8 + g^{16} + \ldots + g^{8(p-1)})(1 + g^4 + g^8 + \ldots + g^{4(p-1)}) \\ &= 1 + g^8 + g^{16} + \ldots + g^{8(p-1)} \\ &= x. \end{aligned}$$

So $x = 1 + g^8 + g^{16} + \ldots + g^{8(p-1)}$ is a S-idempotent in Z_2G.

We can generalize the above two results as followings:

Theorem 3.3. *Let Z_2G be the group ring of a finite cyclic group G of order 2^np, where p is a prime of the form 2^nt+1, then the group ring Z_2G has non-trivial S-idempotents.*

Proof. Here G is a cyclic group of order 2^np, where p of the form 2^nt+1.

Take

$$x = 1 + g^{2^n} + g^{2^n \cdot 2} + \ldots + g^{2^n(p-1)}$$

and

$$a = 1 + g^{2^{n-1}} + g^{2^{n-1} \cdot 2} + \ldots + g^{2^{n-1} \cdot (p-1)}$$

then

$$\begin{aligned} x^2 &= (1 + g^{2^n} + g^{2^n \cdot 2} + \ldots + g^{2^n(p-1)})^2 \\ &= 1 + g^{2^n} + g^{2^n \cdot 2} + \ldots + g^{2^n(p-1)} \\ &= x. \end{aligned}$$

And

$$\begin{aligned} a^2 &= (1 + g^{2^{n-1}} + g^{2^{n-1} \cdot 2} + \ldots + g^{2^{n-1} \cdot (p-1)})^2 \\ &= 1 + (g^{2^{n-1}})^2 + (g^{2^{n-1} \cdot 2})^2 + \ldots + (g^{2^{n-1} \cdot (p-1)})^2 \\ &= x. \end{aligned}$$

Also

$$\begin{aligned} x \cdot a &= (1 + g^{2^n} + g^{2^n \cdot 2} + \ldots + g^{2^n(p-1)})(1 + g^{2^{n-1}} + g^{2^{n-1}\cdot 2} + \ldots + g^{2^{n-1}\cdot(p-1)}) \\ &= 1 + g^{2^n} + g^{2^n \cdot 2} + \ldots + g^{2^n(p-1)} \\ &= x. \end{aligned}$$

So $x = 1 + g^{2^n} + g^{2^n \cdot 2} + \ldots + g^{2^n(p-1)}$ is a S-idempotent in Z_2G.

Corollary 3.1. *Let Z_2S_n be the group ring of a symmetric group S_n where $n = 2^n p$, and p is a prime of the form $2^n t + 1$, then the group ring Z_2S_n has non-trivial S-idempotents.*

Proof. Here Z_2S_n is a group ring where $n = 2^n p$, and p of the form $2^n t+1$. Clearly Z_2S_n contains a finite cyclic group of order $2^n p$. Then by the Theorem 3.3, Z_2S_n has a non-trivial S-idempotent.

§4. Conclusions

Here we have mainly proved the existance of S-idempotents in certain types of group rings. But it is interesting to enumerate the number of S-idempotents for the group rings Z_2G and Z_2S_n in the Theorem 3.3 and Corollary 3.1. We feel that Z_2G can have only one S-idempotent but we are not in a position to give a proof for it. Also, the problem of finding S-idempotents in Z_pS_n (and Z_pG) where $(p, n) = 1$ (and $(p, |G|) = 1$) or $(p, n) = d \neq 1$ (and $(p, |G|) = d \neq 1$) are still interesting number theoretic problems.

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