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On the integer part of the M-th root and the largest M-th power not exceeding N

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Abstract The main purpose of this paper is using the elementary methods to study the properties of the integer part of the m-th root and the largest m-th power not exceeding n, and give some interesting identities involving these numbers.

Keywords The integer part of the m-th root, the largest m-th power not exceeding n, Dirichlet series, identities.

§1. Introduction and Results

Let m be a fixed positive integer. For any positive integer n, we define the arithmetical function $a_m(n)$ as the integer part of the m-th root of n. That is, $a_m(n) = [n^{\frac{1}{m}}]$, where $[x]$ denotes the greatest integer not exceeding to x. For example, $a_2(1) = 1$, $a_2(2) = 1$, $a_2(3) = 1$, $a_2(4) = 2$, $a_2(5) = 2$, $a_2(6) = 2$, $a_2(7) = 2$, $a_2(8) = 2$, $a_2(9) = 3$, $a_2(10) = 3$, $\cdots$. In [1], Professor F. Smarandache asked us to study the properties of the sequences $\{a_k(n)\}$. About this problem, Z. H. Li [2] studied its mean value properties, and given an interesting asymptotic formula:

$$\sum_{\substack{n\le x\\ n\in\mathcal{A}_k}} a_m(n) = \frac{1}{\zeta(k)}\frac{m}{m+1}x^{\frac{m+1}{m}} + O(x),$$

where $\mathcal{A}_k$ denotes the set of all k-th power free numbers, $\zeta(k)$ is the Riemann zeta-function. X. L. He and J. B. Guo [3] also studied the mean value properties of $\sum_{n\le x} a(n)$, and proved that

$$\sum_{n\le x} a(n) = \sum_{n\le x}[x^{\frac{1}{k}}] = \frac{k}{k+1}x^{\frac{k+1}{k}} + O(x).$$

Let n be a positive integer. It is clear that there exists one and only one integer k such that

$$k^m \le n < (k+1)^m.$$

Now we define $b_m(n) = k^m$. That is, $b_m(n)$ is the largest m-th power not exceeding n. If $m = 2$, then $b_2(1) = 1$, $b_2(2) = 1$, $b_2(3) = 1$, $b_2(4) = 4$, $b_2(5) = 4$, $b_2(6) = 4$, $b_2(7) = 4$, $b_2(8) = 4$, $b_2(9) = 9$, $b_2(10) = 9$, $\cdots$. In problem 40 and 41 of [1], Professor F. Smarandache asked us to study the properties of the sequences $\{b_2(n)\}$ and $\{b_3(n)\}$. For these problems, some people had studied them, and obtained many results. For example, W. P. Zhang [4] gave an useful asymptotic formula:

$$\sum_{n\le x} d(u(n)) = \frac{2}{9\pi^4}Ax\ln^3 x + Bx\ln^2 x + Cx\ln x + Dx + O\left(x^{\frac{5}{6}+\varepsilon}\right),$$

where $u(n)$ denotes the largest cube part not exceeding n, $A=\prod_p(1-\frac{1}{(p+1)^2})$, B, C and D are constants, ε denotes any fixed positive number.
And in [5], J. F. Zheng made further studies for $\sum_{n\le x} d(b_m(n))$, and proved that

$$\sum_{n\le x} d(b_m(n)) = \frac{1}{kk!}\left(\frac{6}{k\pi^2}\right)^{k-1} A_0 x\ln^k x + A_1 x\ln^{k-1} x + \cdots + A_{k-1}x\ln x + A_k x + O\left(x^{1-\frac{1}{2k}+\varepsilon}\right),$$

where A_0, A_1,$\cdots$ A_k are constants, especially when k equals to 2, $A_0 = 1$.

In this paper, we using the elementary methods to study the convergent properties of two Dirichlet serieses involving $a_m(n)$ and $b_m(n)$, and give some interesting identities. That is, we shall prove the following conclusions:

Theorem 1. Let m be a fixed positive integer. Then for any real number $s > 1$, the Dirichlet series $f(s) = \sum_{n=1}^{\infty} \frac{(-1)^n}{a_m^s(n)}$ is convergent and

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{a_m^s(n)} = \left(\frac{1}{2^{s-1}} - 1\right)\zeta(s),$$

where $\zeta(s)$ is the Riemann zeta-function.

Theorem 2. Let m be a fixed positive integer. Then for any real number $s > \frac{1}{m}$, the Dirichlet series $g_m(s) = \sum_{n=1}^{\infty} \frac{(-1)^n}{b_m^s(n)}$ is convergent and

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{b_m^s(n)} = \left(\frac{1}{2^{ms-1}} - 1\right)\zeta(ms).$$

From our Theorems, we may immediately deduce the following:

Corollary 1. Taking $s = 2$ or $s = 3$ in Theorem 1, then we have the identities

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{a_m^2(n)} = -\frac{\pi^2}{12} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{a_m^3(n)} = -\frac{3}{4}\zeta(3).$$

Corollary 2. Taking $m = 2$ and $s = 2$ or $m = 2$ and $s = 3$ in Theorem 2, then we have the identities

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{b_2^2(n)} = -\frac{7}{720}\pi^4 \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{b_2^3(n)} = -\frac{31}{30240}\pi^6.$$

Corollary 3. Taking $m = 3$ and $s = 2$ or $m = 3$ and $s = 3$ in Theorem 2, then we have the identities

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{b_3^2(n)} = -\frac{31}{30240}\pi^6 \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{b_3^3(n)} = -\frac{255}{256}\zeta(9).$$

Corollary 4. For any positive integer s and $m \geq 2$, we have

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{b_m^s(n)} = \sum_{n=1}^{\infty} \frac{(-1)^n}{b_s^m(n)}.$$

§2. Proof of the theorems

In this section, we shall complete the proof of our Theorems. For any positive integer n, let $a_m(n) = k$. It is clear that there are exactly $(k+1)^m - k^m$ integer n such that $a_m(n) = k$. So we may get

$$f(s) = \sum_{n=1}^{\infty} \frac{(-1)^n}{a_m^s(n)} = \sum_{k=1}^{\infty} \sum_{\substack{n=1 \\ a_m(n)=k}}^{\infty} \frac{(-1)^n}{k^s},$$

where if k be an odd number, then $\sum\limits_{\substack{n=1 \\ a_m(n)=k}}^{\infty} \frac{(-1)^n}{k^s} = \frac{-1}{k^s}$. And if k be an even number, then $\sum\limits_{\substack{n=1 \\ a_m(n)=k}}^{\infty} \frac{(-1)^n}{k^s} = \frac{1}{k^s}$. Combining the above two cases we have

$$\begin{aligned} f(s) &= \sum_{\substack{t=1 \\ k=2t}}^{\infty} \frac{1}{(2t)^s} + \sum_{\substack{t=1 \\ k=2t-1}}^{\infty} \frac{-1}{(2t-1)^s} \\ &= \sum_{t=1}^{\infty} \frac{1}{(2t)^s} - \left(\sum_{t=1}^{\infty} \frac{1}{t^s} - \sum_{t=1}^{\infty} \frac{1}{(2t)^s} \right) \\ &= \sum_{t=1}^{\infty} \frac{2}{2^s t^s} - \sum_{t=1}^{\infty} \frac{1}{t^s}. \end{aligned}$$

From the integral criterion, we know that $f(s)$ is convergent if $s > 1$. If $s > 1$, then $\zeta(s) = \sum\limits_{n=1}^{\infty} \frac{1}{n^s}$, so we have

$$f(s) = \sum_{n=1}^{\infty} \frac{(-1)^n}{a_m^s(n)} = \left(\frac{1}{2^{s-1}} - 1 \right) \zeta(s).$$

This completes the proof of Theorem 1.

Using the same method of proving Theorem 1 we have

$$\begin{aligned} g_m(s) &= \sum_{n=1}^{\infty} \frac{(-1)^n}{b_m^s(n)} \\ &= \sum_{k=1}^{\infty} \sum_{\substack{n=1 \\ b_m(n)=k^m}}^{\infty} \frac{(-1)^n}{k^{ms}} \\ &= \sum_{\substack{t=1 \\ k=2t}}^{\infty} \frac{1}{(2t)^{ms}} + \sum_{\substack{t=1 \\ k=2t-1}}^{\infty} \frac{-1}{(2t-1)^{ms}} \\ &= \sum_{t=1}^{\infty} \frac{2}{2^{ms}t^{ms}} - \sum_{t=1}^{\infty} \frac{1}{t^{ms}}. \end{aligned}$$

From the integral criterion, we know that $g_m(s)$ is also convergent if $s > \frac{1}{m}$. If $s > 1$, $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, so we may easily deduce

$$g_m(s) = \sum_{n=1}^{\infty} \frac{(-1)^n}{b_m^s(n)} = \left(\frac{1}{2^{ms-1}} - 1\right) \zeta(ms).$$

This completes the proof of Theorem 2.

From our two Theorems, and note that $\zeta(2) = \frac{\pi^2}{6}$, $\zeta(4) = \frac{\pi^4}{90}$, $\zeta(6) = \frac{\pi^6}{945}$ (see [6]), we may immediately deduce Corollary 1, 2, and 3. Then, Corollary 4 can also be obtained from Theorem 2.

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On the ten's complement factorial Smarandache function

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Sequence A110396 by Amarnath Murthy in the on-line encyclopedia of integer sequences [1] is defined as "the 10's complement factorial of n." Let $t(n)$ denote the difference between n and the next power of 10. This is the ten's complement of a number. E.g., $t(27) = 73$, because $100 - 27 = 73$. Hence the $10's$ complement factorial simply becomes

$$\begin{aligned} tcf(n) &= (10's \text{ complement } of\ n) * (10's \text{ complement } of\ n-1)\cdots \\ &\quad (10's \text{ complement } of\ 2) * (10's \text{ complement } of\ 1). \end{aligned}$$

How would the Smarandache function behave if this variation of the factorial function were used in place of the standard factorial function? The Smarandache function $S(n)$ is defined as the smallest integer m such that n evenly divides m factorial. Let $TS(n)$ be the smallest integer m such that n divides the ($10's$ complement factorial of m.)

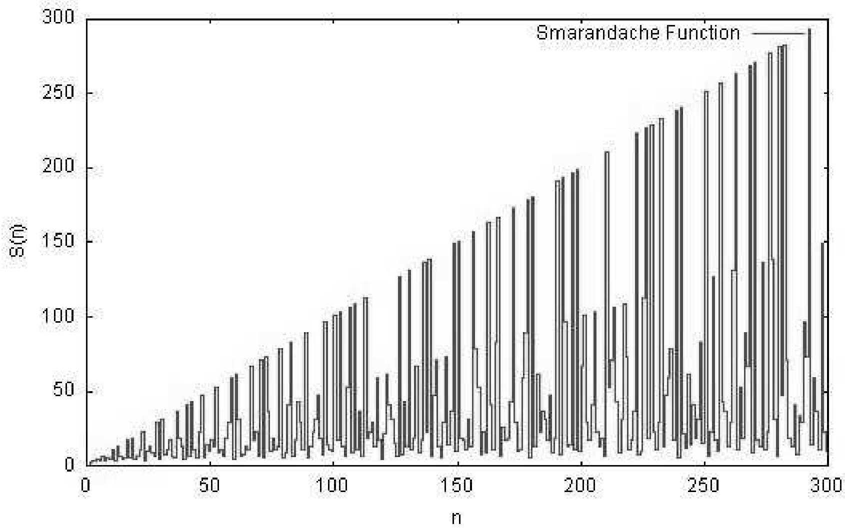
This new $TS(n)$ function produces the following sequence (which is A109631 in the OEIS [2]).

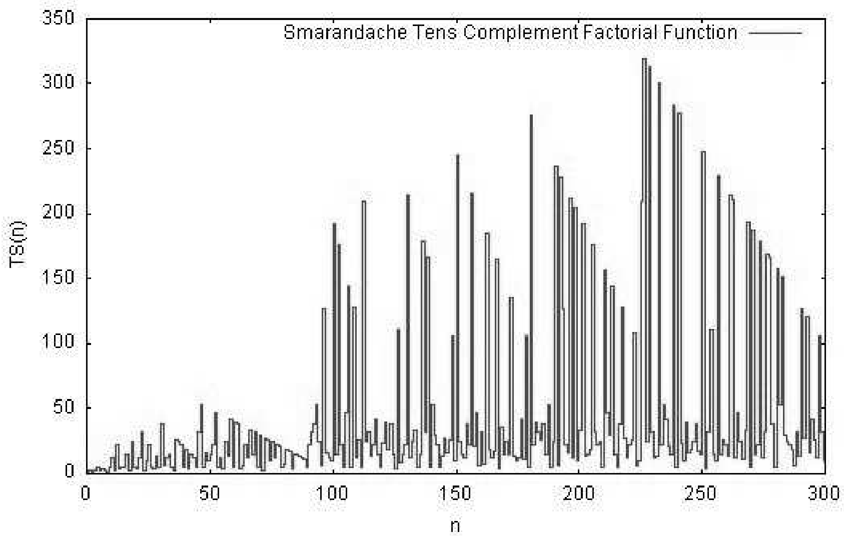
$n = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20 \cdots,$

$TS(n) = 1, 2, 1, 2, 5, 2, 3, 2, 1, 5, 12, 2, 22, 3, 5, 4, 15, 2, 24, 5 \cdots.$

For example, $TS(7) = 3$, because 7 divides $(10-3)*(10-2)*(10-1)$; and 7 does not divide (10's complement factorial of m) for $m < 3$.

Not surprisingly, the $TS(n)$ function differs significantly from the standard Smarandache function. Here are graphs displaying the behavior of each for the first 300 terms:





Four Problems Concerning the New $TS(n)$ Function

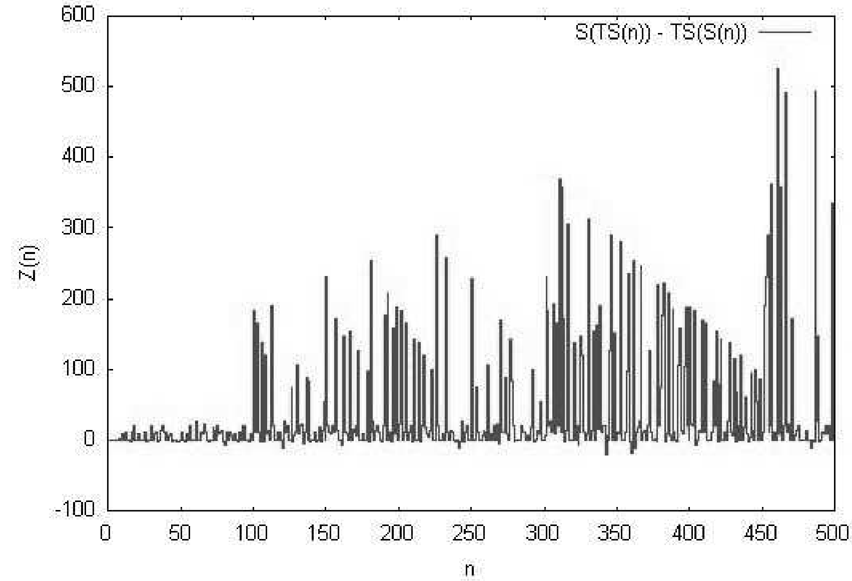
1. The Smarandache function and the ten's complement factorial Smarandache function have many values in common. Here are the initial solutions to $S(n) = TS(n)$:

1, 2, 5, 10, 15, 20, 25, 30, 40, 50, 60, 75, 100, 120, 125, 128, 150, 175, 200, 225, 250, 256, 300, 350, 375, 384, 400, 450, 500, 512, 525, 600, 625, 640, 675, 700, 750, 768, $\cdots$.

Why are most of the solutions multiples of 5 or 10? Are there infinitely many solutions?

2. After a computer search for all values of n from 1 to 1000, the only solution found for $TS(n) = TS(n+1)$ is 374. We conjecture there is at least one more solution. But are there infinitely many?

3. Let $Z(n) = TS(S(n)) - S(TS(n))$. Is $Z(n)$ positive infinitely often? Negative infinitely often? The $Z(n)$ sequence seems highly chaotic with most of its values positive. Here is a graph of the first 500 terms:



4. The first four solutions to $TS(n) + TS(n+1) = TS(n+2)$ are 128, 186, 954, and 1462. Are there infinitely many solutions?

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