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Some interesting properties of the Smarandache function

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Abstract The main purpose of this paper is using the elementary method to study the property of the Smarandache function, and give an interesting result.
Keywords Smarandache function; Additive property; Greatest prime divisor.

§1. Introduction and results

Let n be an positive integer, the famous Smarandache function $S(n)$ is defined as following:

$$S(n) = \min\{m : m \in N, n|m!\}.$$

About this function and many other Smarandache type function, many scholars have studied its properties, see [1], [2], [3] and [4]. Let $p(n)$ denotes the greatest prime divisor of n, it is clear that $S(n) \geq p(n)$. In fact, $S(n) = p(n)$ for almost all n, as noted by Erdös [5]. This means that the number of $n \leq x$ for which $S(n) \neq p(n)$, denoted by $N(x)$, is $o(x)$. It is easily to show that $S(p) = p$ and $S(n) < n$ except for the case $n = 4, n = p$. So there have a closely relationship between $S(n)$ and $\pi(x)$:

$$\pi(x) = -1 + \sum_{n=2}^{[x]} \left[\frac{S(n)}{n}\right],$$

where $\pi(x)$ denotes the number of primes up to x, and $[x]$ denotes the greatest integer less than or equal to x. For two integer m and n, can you say $S(mn) = S(m) + S(n)$ is true or false? It is difficult to say. For some m an n, it is true, but for some other numbers it is false.

About this problem, J.Sandor [7] proved an very important conclusion. That is, for any positive integer k and any positive integers $m_1, m_2, \cdots, m_k$, we have the inequality

$$S\left(\prod_{i=1}^{k} m_i\right) \leq \sum_{i=1}^{k} S(m_i).$$

This paper as a note of [7], we shall prove the following two conclusions:

Theorem 1. *For any integer $k \geq 2$ and positive integers $m_1, m_2, \cdots, m_k$, we have the inequality*

$$S\left(\prod_{i=1}^{k} m_i\right) \leq \prod_{i=1}^{k} S(m_i).$$

Theorem 2. *For any integer $k \geq 2$, we can find infinite group numbers $m_1, m_2, \cdots, m_k$ such that:*

$$S\left(\prod_{i=1}^{k} m_i\right) = \sum_{i=1}^{k} S(m_i).$$

§2. Proof of the theorems

In this section, we will complete the proof of the Theorems. First we prove a special case of Theorem 1. That is, for any positive integers m and n, we have

$$S(m)S(n) \geq S(mn).$$

If $m = 1$ (or $n = 1$), then it is clear that $S(m)S(n) \geq S(mn)$. Now we suppose $m \geq 2$ and $n \geq 2$, so that $S(m) \geq 2$, $S(n) \geq 2$, $mn \geq m + n$ and $S(m)S(n) \geq S(m) + S(n)$. Note that $m|S(m)!$, $n|S(n)!$, we have $mn|S(m)!S(n)!|((S(m) + S(n))!$. Because $S(m)S(n) \geq S(m) + S(n)$, we have $(S(m) + S(n))!|(S(m)S(n))!$. That is, $mn|S(m)!S(n)!|(S(m) + S(n))!|(S(m)S(n))!$. From the definition of $S(n)$ we may immediately deduce that

$$S(mn) \leq S(m)S(n).$$

Now the theorem 1 follows from $S(mn) \leq S(m)S(n)$ and the mathematical induction.

Proof of Theorem 2. For any integer n and prime p, if $p^{\alpha} \| n!$, then we have

$$\alpha = \sum_{j=1}^{\infty} \left[\frac{n}{p^j}\right].$$

Let n_i are positive integers such that $n_i \neq n_j$, if $i \neq j$, where $1 \leq i, j \leq k$, $k \geq 2$ is any positive integer. Since

$$\sum_{r=1}^{\infty} \left[\frac{p^{n_i}}{p^r}\right] = p^{n_i - 1} + p^{n_i - 2} + \cdots + 1 = \frac{p^{n_i} - 1}{p - 1}.$$

For convenient, we let $u_i = \frac{p^{n_i}-1}{p-1}$. So we have

$$S(p^{u_i}) = p^{n_i}, \qquad i = 1, 2, \cdots, k. \tag{1}$$

In general, we also have

$$\sum_{r=1}^{\infty} \left[\frac{\sum_{i=1}^{k} p^{n_i}}{p^r}\right] = \sum_{i=1}^{k} \frac{p^{n_i} - 1}{p - 1} = \sum_{i=1}^{k} u_i.$$

So

$$S\left(p^{u_1 + u_2 + \cdots + u_k}\right) = \sum_{i=1}^{k} p^{n_i}. \tag{2}$$

Combining (1) and (2) we may immediately obtain

$$S\left(\prod_{i=1}^{k} p^{u_i}\right) = \sum_{i=1}^{k} S(p^{u_i}).$$

Let $m_i = p^{u_i}$, noting that there are infinity primes p and n_i, we can easily get Theorem 2.

This completes the proof of the theorems.

References

[1] C.Ashbacher, Some Properties of the Smarandache-Kurepa and Smarandache-Wagstaff Functions. Mathematics and Informatics Quarterly, **7**(1997), 114-116.

[2] A.Begay, Smarandache Ceil Functions Bulletin of Pure and Applied Sciences, **16**(1997), 227-229.

[3] Mark Farris and Patrick Mitchell, Bounding the Smarandache function Smarandache Notions Journal,**13**(2002), 37-42.

[4] Kevin Ford, The normal behavior of the Smarandache function, Smarandache Notions Journal, **10**(1999), 81-86.

[5] P.Erdös, Problem 6674 Amer. Math. Monthly, **98**(1991), 965.

[6] Pan Chengdong and Pan Chengbiao, Element of the analytic number theory, Science Press, Beijing, (1991).

[7] J.Sandor, On a inequality for the Smarandache function, Smarandache Notions Journal, **10**(1999), 125-127.