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A note on the Smarandache inversion sequence

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Abstract In a recent paper, Muneer [1] introduced the Smarandache inversion sequence. In this paper, we study some properties of the Smarandache inversion sequence. Moreover, we find the necessary and sufficient condition such that $[SI(n)]^2 + [SI(n+1)]^2$ is a perfect square.

Keywords Smarandache reverse sequence, Smarandache inversion, perfect square.

§1. Introduction

The Smarandache reverse sequence is (see, for example, Ashbacher [2])

$$1, 21, 321, 4321, 54321, \cdots,$$

and in general, the $n-th$ term of the sequence is

$$S(n) = n(n-1)\cdots 321.$$

In connection with the Smarandache reverse sequence, Muneer [1] introduced the concept of the Smarandache inversion sequence, $SI(n)$, defined as follows :

Definition 1.1. The value of the Smarandache inversion of (positive) integers in a number is the number of order relations of the form $i > j$ (where i and j are digits of the positive integers of the number under consideration), with $SI(0) = 0$, $SI(1) = 0$.

More specifically, for the Smarandache reverse sequence number

$$S(n) = n(n-1)\cdots 321,$$

the following order relations hold :

$$(A-1) n > n-1 > \cdots > 3 > 2 > 1,$$
$$(A-2) n-1 > n-2 > \cdots > 3 > 2 > 1,$$
$$\cdots$$
$$(A-(n-1)) 2 > 1.$$

Note that, the number of order relations in $(A-1)$ is $n-1$, that in $(A-2)$ is $n-2$, and so on, and finally, the number of order relation in $(A-(n-1))$ is 1. We thus have the following result :

Lemma 1.1. $SI(n) = \frac{n(n-1)}{2}$ for any integer $n \geq 1$.

Proof. $SI(n) = (n-1) + (n-2) + \cdots + 1 = \frac{n(n-1)}{2}$.

Lemma 1.2. For any integer $n \geq 1$, $\sum_{i=1}^{n} SI(1) = \frac{n(n^2-1)}{6}$.

Proof. Using Lemma 1.1,

$$\begin{aligned}\sum_{i=1}^{n} SI(1) &= \sum_{i=1}^{n} \frac{i(i-1)}{2} = \frac{1}{2}\left(\sum_{i=1}^{n} i^2 - \sum_{i=1}^{n} i\right) \\ &= \frac{1}{2}\left[\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2}\right] = \frac{n(n^2-1)}{6}.\end{aligned}$$

Muneer [1] also derived the following results.

Lemma 1.3. $SI(n+1) + SI(n) = n^2$ for any integer $n \geq 1$.

Lemma 1.4. $SI(n+1) - SI(n) = n$ for any integer $n \geq 1$.

Proof. Since

$$SI(n+1) = \frac{n(n+1)}{2} = \frac{n(n-1)}{2} + n = SI(n) + n,$$

we get the desired result.

Lemma 1.5. $[SI(n+1)]^2 - [SI(n)]^2 = n^3$ for any integer $n \geq 1$.

Proof. Using Lemma 1.3 and Lemma 1.4,

$$[SI(n+1)]^2 - [SI(n)]^2 = [SI(n+1) + SI(n)][SI(n+1) - SI(n)] = (n^2)(n) = n^3.$$

Lemma 1.6. $SI(n+1)SI(n-1) + SI(n) = \left(\frac{n(n-1)}{2}\right)^2$ for any integer $n \geq 1$.

We also have the following recurrence relation.

Lemma 1.7. $SI(n+1) - SI(n-1) = 2n - 1$ for any integer $n \geq 1$.

Proof. Using Lemma 1.4,

$$\begin{aligned}SI(n+1) - SI(n-1) &= [SI(n+1)SI(n)] + [SI(n) - SI(n-1)] \\ &= n + (n-1) = 2n - 1.\end{aligned}$$

Muneer [1] also considered the equation

$$[SI(n)]^2 + [SI(n+1)]^2 = k^2 \tag{1}$$

for some integers $n \geq 1$, $k \geq 1$, and found two solutions, namely, $n = 7$ and $n = 8$.

In this note, we derive a necessary and sufficient condition such that (1) is satisfied. This is given in the next section.

§2. Main Results

We consider the equation

$$[SI(n)]^2 + [SI(n+1)]^2 = k^2 \tag{2}$$

for some integers $n \geq 1$, $k \geq 1$. By definition,

$$[SI(n)]^2 + [SI(n+1)]^2 = \left(\frac{n(n-1)}{2}\right)^2 + \left(\frac{n(n+1)}{2}\right)^2 = \frac{1}{2}n^2(n^2+1)$$

We thus arrive at the following result.

Lemma 2.1. The equation (2) has a solution (for n and k) if and only if $\frac{1}{2}(n^2+1)$ is a perfect square.

Lemma 2.2. The Diophantine equation

$$\frac{1}{2}(n^2+1) = k^2 \tag{3}$$

has a solution (for n and k) if and only if there is an integer $m \geq 1$ such that $m^2 + (m+1)^2$ is a perfect square, and in that case, $n = 2m+1$, $k^2 = m^2 + (m+1)^2$.

Proof. We consider the equation (3) in the equivalent form

$$n^2 + 1 = 2k^2, \tag{4}$$

which shows that n must be odd; so let

$$n = 2m + 1. \tag{5}$$

for some integer $m \geq 1$. Then, from (4),

$$(2m+1)^2 + 1 = 2k^2,$$

that is, $(4m^2 + 4m + 1) + 1 = 2k^2$, that is, $m^2 + (m+1)^2 = k^2$.

Searching for all consecutive integers upto 1500, we found only four pairs of consecutive integers whose sums of squares are perfect squares. These are

$$(1)32 + 42 = 52, \tag{6}$$

$$(2)202 + 212 = 292, \tag{7}$$

$$(3)1192 + 1202 = 1692, \tag{8}$$

$$(4)6962 + 6972 = 9852. \tag{9}$$

The first two give respectively the solutions

(a) $[SI(7)]^2 + [SI(8)]^2 = 35^2$,

(b) $[SI(41)]^2 + [SI(42)]^2 = 1189^2$,

which were found by Muneer [1], while the other two give respectively the solutions

(c) $[SI(239)]^2 + [SI(240)]^2 = 40391^2$,

(d) $[SI(1393)]^2 + [SI(1394)]^2 = 1372105^2$.

The following lemma, giving the general solution of the Diophantine equation $x^2+y^2 = z^2$, is a well-known result (see, for example, Hardy and Wright [3]).

Lemma 2.3. The most general (integer) solution of the Diophantine equation $x^2+y^2 = z^2$ is

$$x = 2ab, \quad y = a^2 - b^2, \quad z = a^2 + b^2, \tag{10}$$

where $x > 0$, $y > 0$, $z > 0$ are integers with $(x, y) = 1$ and x is even, and a and b are of opposite parity with $(a, b) = 1$.

Lemma 2.4. The problem of solving the Diophantine equation

$$m^2 + (m+1)^2 = k^2, \tag{11}$$

is equivalent to the problem of solving the Diophantine equations

$$x^2 - 2y^2 = 1.$$

Proof. By Lemma 2.3, the general solution of the Diophantine equation

$$(m+1)^2 + m^2 = k^2$$

has one of the following two forms :

(a) $m = 2ab$, $m + 1 = a^2 - b^2$, $k = a^2 + b^2$ for some integers $a, b \geq 1$ with $(a, b) = 1$;

(b) $m = a^2 - b^2$, $m + 1 = 2ab$, $k = a^2 + b^2$ for some integers $a, b \geq 1$ with $(a, b) = 1$.

In case (a),

$$1 = (m+1) - m = (a^2 - b^2)^2 - 2ab = (a-b)^2 - 2ab^2,$$

which leads to the Diophantine equation $x^2 - 2y^2 = 1$.

In case (b),

$$-1 = m - (m+1) = (a^2 - b^2)^2 - 2ab = (a-b)^2 - 2ab^2,$$

leading to the Diophantine equation $x^2 - 2y^2 = -1$.

The general solutions of the Diophantine equations $x^2 - 2y^2 = \pm 1$ are given in the following lemma (see, for example, Hardy and Wright [3]).

Lemma 2.5. All solutions of the Diophantine equation

$$x^2 - 2y^2 = 1$$

are given by

$$x + \sqrt{2}y = (1 + \sqrt{2})^{2n}, \tag{12}$$

$n \geq 0$ is an integer; and all solutions of the Diophantine equation

$$x^2 - 2y^2 = -1,$$

are given by

$$x + \sqrt{2}y = (1 + \sqrt{2})^{2n+1}, \tag{13}$$

$n \geq 0$ is an integer.

Remark 2.1. Lemma 2.5 shows that the Diophantine equation $m^2 + (m+1)^2 = k^2$ has infinite number of solutions. The first four solutions of the Diophantine equation (11) are given in (6 - 9). It may be mentioned here that the first and third solutions can be obtained from

(12) corresponding to $n = 1$ and $n = 2$ respectively, while the second and the fourth solutions can be obtained from (13) corresponding to $n = 0$ and $n = 1$ respectively. The fifth solution may be obtained from (12) with $n = 3$ as follows :

$$x + \sqrt{2}y = (1 + \sqrt{2})^6 = 99 + 70\sqrt{2} \Rightarrow x = 99, \quad y = 70.$$

Therefore,

$$a - b = 99, \quad b = 70 \Rightarrow a = 169, b = 70,$$

and finally,

$$m = 2ab = 23660, \quad m + 1 = a^2 - b^2 = 23661.$$

Corresponding to this, we get the following solution to (2) :

$$[SI(47321)]^2 + [SI(47322)]^2 = 1583407981^2.$$

§3. Some Observations

In [1], Muneer has found three relations connecting four consecutive Smarandache inversion functions. These are as follows :

(1) $SI(6) + SI(7) + SI(8) + SI(9) = 10^2$,

(2) $SI(40) + SI(41) + SI(42) + SI(43) = 58^2$,

(3) $SI(238) + SI(239) + SI(240) + SI(241) = 338^2$.

Searching for more such relations upto $n = 1500$, we got a fourth one :

(4) $SI(1392) + SI(1393) + SI(1394) + SI(1395) = 1970^2$.

Since

$$SI(n-1) + SI(n) + SI(n+1) + SI(n+2) = (n-1)^2 + (n+1)^2,$$

the problem of finding four consecutive Smarandache inversion functions whose sum is a perfect square reduces to the problem of solving the Diophantine equation

$$m^2 + (m+2)^2 = k^2.$$

In this respect, we have the following result.

Lemma 3.1. If m_0, $m_0 + 1$ and $k_0 = \sqrt{m_0^2 + (m_0+1)^2}$ is a solution of the Diophantine equation

$$m^2 + (m+1)^2 = k^2, \tag{14}$$

then $2m_0$, $2(m_0 + 1)$ and $l_0 = 2\sqrt{m_0^2 + (m_0+1)^2}$ is a solution of the Diophantine equation

$$m^2 + (m+2)^2 = l^2, \tag{15}$$

and conversely.

Proof. First, let m_0, $m_0 + 1$ and $k_0 = \sqrt{m_0^2 + (m_0+1)^2}$ be a solution of (14), so that

$$m_0^2 + (m_0+1)^2 = k_0^2, \tag{16}$$

Multiplying throughout of (1) by 4, we get

$$(2m_0)^2 + [2(m_0+1)]^2 = (2k_0)^2,$$

so that $2m_0$, $2(m_0+1)$ and $l_0 = 2k_0$ is a solution of (15).

Conversely, let m_0, m_0+2 and $l_0 = \sqrt{m_0^2 + (m_0+2)^2}$ be a solution of (15). Note that, m_0 and m_0+2 are of the same parity. Now, both m_0 and m_0+2 cannot be odd, for otherwise,

$$m_0 \equiv 1(mod2), \quad m_0+2 \equiv 1(mod2) \;\Rightarrow\; l_0^2(mod4),$$

which is impossible. Thus, both m_0 and m_0+2 must be even. It, therefore, follows that $\frac{m_0}{2}$, $\frac{m_0}{2}+1$ and $k_0 = \frac{l_0}{2}$ is a solution of (14).

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