THE FIRST CONSTANT OF SMARANDACHE

by

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In this note we prove that the series $\sum_{n=2}^{\infty} \frac{1}{S(n)!}$ is convergent to a real number $s \in (0.717, 1.253)$ that we call the first constant constant of Smarandache.

It appears as an open problem, in [1], the study of the nature of the series $\sum_{n=2}^{\infty} \frac{1}{S(n)!}$. We can write it as it follows :

$$\frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{3!} + \cdots = \frac{1}{2!} + \frac{2}{3!} + \frac{4}{4!} + \frac{8}{5!} + \frac{14}{6!} + \cdots =$$

$= \sum_{n=2}^{\infty} \frac{a(n)}{n!}$, where $a(n)$ is the number of the equation $S(x) = n$, $n \in \mathbf{N}$, $n \geq 2$ solutions.

It results from the equality $S(x) = n$ that x is a divisor of $n!$, so $a(n)$ is smaller than $d(n!)$.

So, $\quad a(n) < d(n!). \qquad (1)$

Lemma 1. We have the inequality :

$d(n) \leq n - 2$, for each $n \in \mathbf{N}$, $n \geq 7$. $\qquad (2)$

Proof. Be $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ with $p_1, p_2, \ldots, p_k$ prime numbers, and $a_i \geq 1$ for each $i \in \{1, 2, \ldots, k\}$. We consider the function $f : [1, \infty) \to \mathbf{R}$, $f(x) = a^x - x - 2$, $a \geq 2$, fixed. It is derivable on $[1, \infty)$ and $f'(x) = a^x \ln a - 1$. Because $a \geq 2$, and $x \geq 1$ it results that $a^x \geq 2$, so $a^x \ln a \geq 2 \ln a = \ln a^2 \geq \ln 4 > \ln e = 1$, i.e., $f'(x) > 0$ for each $x \in [1, \infty)$ and $a \geq 2$, fixed. But $f(1) = a - 3$. It results that for $a \geq 3$ we have $f(x) \geq 0$, that means $a^x \geq x + 2$.

Particularly, for $a = p_i$, $i \in \{1, 2, \ldots, k\}$, we obtain $p_i^{a_i} \geq a_i + 2$ for each $p_i \geq 3$.

If $n = 2^s$, $s \in \mathbf{N}^*$, then $d(n) = s + 1 < 2^s - 2 = n - 2$ for $s \geq 3$.

So we can assume $k \geq 2$, i.e. $p_2 \geq 3$. It results the inequalities :

$$p_1^{a_1} \geq a_1 + 1$$

$$p_2^{a_2} \geq a_2 + 2$$

....

$$p_k^{a_k} \geq a_k + 2,$$

equivalent with

$$p_1^{a_1} \geq a_1 + 1, \; p_2^{a_2} - 1 \geq a_2 + 1, \; \ldots, \; p_k^{a_k} - 1 \geq a_k + 1. \tag{3}$$

Multiplying, member with member, the inequalities (3) we obtain :

$$p_1^{a_1}(p_2^{a_2} - 1)\cdots(p_k^{a_k} - 1) \geq (a_1 + 1)(a_2 + 1)\cdots(a_k + 1) = d(n). \tag{4}$$

Considering the obvious inequality :

$$n - 2 \geq p_1^{a_1}(p_2^{a_2} - 1)\cdots(p_k^{a_k} - 1) \tag{5}$$

and using (4) it results that :

$n - 2 \geq d(n)$ for each $n \geq 7$.

Lemma 2. $d(n!) < (n - 2)!$ for each $n \in \mathbf{N}, n \geq 7$. (6)

Proof. We ration trough induction after n. So, for $n = 7$,

$$d(7!) = d(2^4 \cdot 3^2 \cdot 5 \cdot 7) = 60 < 120 = 5!.$$

We assume that $d(n!) < (n - 2)!$.

$$d((n+1)!) = d(n!(n + 1)) \leq d(n!) \cdot d(n + 1) < (n - 2)!\, d(n + 1) < (n-2)!\,(n - 1) = (n - 1)!,$$

because in accordance with Lemma 1, $d(n + 1) < n - 1$.

Proposition. The series $\sum_{n=2}^{\infty} \frac{1}{S(n)!}$ is convergent to a number $s \in (0.717, 1.253)$, that we call the first constant constant of Smarandache.

Proof. From Lemma 2 it results that $a(n) < (n-2)!$, so $\frac{a(n)}{n!} < \frac{1}{n(n-1)}$ for every $n \in \mathbf{N}$, $n \geq 7$ and $\sum_{n=2}^{\infty} \frac{1}{S(n)!} = \sum_{n=2}^{6} \frac{a(n)}{n!} + \sum_{n=7}^{\infty} \frac{1}{(n-1)}$.

Therefore $\sum_{n=2}^{\infty} \frac{1}{S(n)!} < \frac{1}{2!} + \frac{2}{3!} + \frac{4}{4!} + \frac{8}{5!} + \frac{14}{6!} + \sum_{n=7}^{\infty} \frac{1}{n^2 - n}$. (7)

Because $\sum_{n=2}^{\infty} \frac{1}{n^2 - n} = 1$ we have : it exists the number $s > 0$, that we call the Smarandache constant, $s = \sum_{n=2}^{\infty} \frac{1}{S(n)!}$.

From (7) we obtain :

$$\sum_{n=2}^{\infty} \frac{1}{S(n)!} < \frac{391}{360} + 1 - \frac{1}{2^2 - 2} - \frac{1}{3^2 - 3} - \frac{1}{4^2 - 4} +$$

$$+ \frac{1}{5^2 - 5} + \frac{1}{6^2 - 6} = \frac{751}{360} - \frac{5}{6} = \frac{451}{360} < 1,253.$$

But, because $S(n) \leq n$ for every $n \in \mathbf{N}^*$, it results :

$$\sum_{n=2}^{\infty} \frac{1}{S(n)!} \geq \sum_{n=2}^{\infty} \frac{1}{n!} = e - 2.$$

Consequently, for this first constant we obtain the framing $e - 2 < s < 1,253$, i.e., $0,717 < s < 1,253$.

REFERENCES

[1] **I. Cojocaru, S. Cojocaru** : *On some series involving the Smarandache Function* (to appear).

[2] **F. Smarandache** : *A Function in the Number Theory* (An. Univ. Timisoara, Ser. St. Mat., vol. XVIII, fasc. 1 (1980), 79 - 88.

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