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On the property of the Smarandache-Riemann zeta sequence

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Abstract In this paper, some elementary methods are used to study the property of the Smarandache-Riemann zeta sequence and obtain a general result.

Keywords Riemann zeta function, Smarandache-Riemann zeta sequence, positive integer.

§1. Introduction and result

For any complex number s, let

$$\zeta(s) = \sum_{k=1}^{\infty} k^{-s}$$

be the Riemann zeta function. For any positive integer n, let T_n be a positive real number such that

$$\zeta(2n) = \frac{\pi^{2n}}{T_n}, \tag{1}$$

where π is ratio of the circumference of a circle to its diameter. Then the sequence $T = \{T_n\}_{n=1}^{\infty}$ is called the Smarandache-Riemann zeta sequence. About the elementary properties of the Smarandache-Riemann zeta sequence, some scholars have studied it, and got some useful results. For example, in [2], Murthy believed that T_n is a sequence of integers. Meanwhile, he proposed the following conjecture:

Conjecture. No two terms of T_n are relatively prime.

In [3], Le Maohua proved some interesting results. That is, if

$$ord(2, (2n)!) < 2n - 2,$$

where $ord(2, (2n)!)$ denotes the order of prime 2 in $(2n)!$, then T_n is not an integer, and finally he defies Murthy's conjecture.

In reference [4], Li Jie proved that for any positive integer $n \geq 1$, we have the identity

$$ord(2, (2n)!) = \alpha_2(2n) \equiv \sum_{i=1}^{+\infty} \left[\frac{2n}{2^i}\right] = 2n - a(2n, 2),$$

where $[x]$ denotes the greatest integer not exceeding x.

So if $2n - a(2n, 2) < 2n - 2$, or $a(2n, 2) \geq 3$, then T_n is not an integer.

In fact, there exist infinite positive integers n such that $a(2n, 2) \geq 3$, and T_n is not an integer. From this, we know that Murthy's conjecture is not correct, because there exist infinite positive integers n such that T_n is not an integer.

In this paper, we use the elementary methods to study another property of the Smarandache-Riemann zeta sequence, and give a general result for it. That is, we shall prove the following conclusion:

Theorem. If T_n are positive integers, then 3 divides T_n, more generally, if $n = 2k$, then 5 divides T_n; If $n = 3k$, then 7 divides T_n, where $k \neq 0$ is an integer.

So from this Theorem we may immediately get the following

Corollary. For any positive integers m and $n (m \neq n)$, if T_m and T_n are integers, then

$$(T_m, T_n) \geq 3, \quad (T_{2m}, T_{2n}) \geq 15, \quad (T_{3m}, T_{3n}) \geq 21.$$

§2. Proof of the theorem

In this section, we shall complete the proof of our theorem. First we need two simple Lemmas which we state as follows:

Lemma 1. If n is a positive integer, then we have

$$\zeta(2n) = (-1)^{n+1} \frac{(2\pi)^{2n} B_{2n}}{2(2n)!}, \tag{2}$$

where B_{2n} is the Bernoulli number.

Proof. See reference [1].

Lemma 2. For any positive integer n, we have

$$B_{2n} = I_n - \sum_{p-1|2n} \frac{1}{p}, \tag{3}$$

where I_n is an integer and the sum is over all primes p such that $p - 1$ divides $2n$.

Proof. See reference [3].

Lemma 3. For any positive integer n, we have

$$T_n = \frac{(2n)! b_n}{2^{2n-1} a_n}, \tag{4}$$

where a_n and b_n are coprime positive integers satisfying $2||b_n, 3|b_n, n \geq 1$.

Proof. It is a fact that

$$\zeta(2n) = (-1)^{n-1} \frac{2^{2n-1}\pi^{2n}}{(2n)!} \cdot B_{2n}, \quad n \geq 1, \tag{5}$$

where

$$B_{2n} = (-1)^{n-1} \frac{a_n}{b_n}, \quad n \geq 1. \tag{6}$$

Using (1), (5) and (6), we get (4).

Now we use above Lemmas to complete the proof of our theorem.

For any positive integer n, from (4) we can directly obtain that if T_n is an integer, then 3 divides T_n, since $(a_n, b_n) = 1$.

From (1), (2) and (3) we have the following equality

$$\zeta(2n) = \frac{\pi^{2n}}{T_n} \quad = \quad (-1)^{n+1}\frac{(2\pi)^{2n}B_{2n}}{2(2n)!} = (-1)^{n+1}\frac{(2\pi)^{2n}}{2(2n)!} \cdot \left(I_n - \sum_{p-1|2n}\frac{1}{p}\right),$$

Let

$$\prod_{p-1|2n} p = p_1p_2\cdots p_s,$$

where $p_i (1 \le i \le s)$ is a prime number, and $p_1 < p_2 \cdots < p_s$.

Then from the above, we have

$$\begin{aligned} T_n &= \frac{(-1)^{n+1}\cdot\pi^{2n}}{\frac{(2\pi)^{2n}}{2(2n)!}\cdot\left(I_n - \sum\limits_{p-1|2n}\frac{1}{p}\right)} = \frac{(-1)^{n+1}\cdot(2n)!}{2^{2n-1}\cdot\left(I_n - \sum\limits_{p-1|2n}\frac{1}{p}\right)} \qquad (7) \\ &= \frac{(-1)^{n+1}\cdot(2n)!\cdot\prod\limits_{p-1|2n}p}{2^{2n-1}\cdot\left(I_n\cdot\prod\limits_{p-1|2n}p - \prod\limits_{p-1|2n}p\cdot\sum\limits_{p-1|2n}\frac{1}{p}\right)} \\ &= \frac{(-1)^{n+1}\cdot(2n)!\cdot p_1p_2\cdots p_s}{2^{2n-1}\cdot\left(I_n\cdot p_1p_2\cdots p_s - p_1p_2\cdots p_s\cdot\left(\frac{1}{p_1}+\frac{1}{p_2}+\cdots\frac{1}{p_s}\right)\right)} \\ &= \frac{(-1)^{n+1}\cdot(2n)!\cdot p_1p_2\cdots p_s}{2^{2n-1}\cdot(I_n\cdot p_1p_2\cdots p_s - p_2p_3\cdots p_s - p_1p_3\cdots p_s - \cdots - p_1p_2\cdots p_{s-1})} \end{aligned}$$

Then we find that if $p_i | p_1p_2\cdots p_s, \quad 1 \le i \le s$, but

$$p_i \dagger (I_n\cdot p_1p_2\cdots p_s - p_2p_3\cdots p_s - p_1p_3\cdots p_s - \cdots - p_1p_2\cdots p_{s-1}).$$

So we can easily deduce that if T_n are integers, when $n = 2k$, 5 can divide T_n; While $n = 3k$, then 7 can divide T_n.

This completes the proof of Theorem.

References

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