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On the Smarandache power function and Euler totient function

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Abstract For any positive integer n, let $SP(n)$ denotes the Smarandache power function, and $\phi(n)$ is the Euler totient function. The main purpose of this paper is using the elementary method to study the solutions of the equation $SP(n^k) = \phi(n)$, and give its all positive integer solutions for $k =$ 1, 2, 3.

Keywords Smarandache power function, Euler totient function, equation, positive integer solutions.

§1. Introduction and Results

For any positive integer n, the famous Smarandache power function $SP(n)$ is defined as the smallest positive integer m such that m^m is divisible by n. That is,

$$SP(n) = \min\{m : n \mid m^m, m \in N, \prod_{p|m} p = \prod_{p|n} p\}$$

where N denotes the set of all positive integers. For example, the first few values of $SP(n)$ are: $SP(1) = 1$, $SP(2) = 2$, $SP(3) = 3$, $SP(4) = 2$, $SP(5) = 5$, $SP(6) = 6$, $SP(7) = 7$, $SP(8) = 4$, $SP(9) = 3$, $SP(10) = 10$, $SP(11) = 11$, $SP(12) = 6$, $SP(13) = 13$, $SP(14) = 14$, $SP(15) = 15$, $SP(16) = 4$, $SP(17) = 17$, $SP(18) = 6$, $SP(19) = 19$, $SP(20) = 10$, $\cdots$. In reference [1], Professor F.Smarandache asked us to study the properties of $SP(n)$. From the definition of $SP(n)$ we can easily get the following conclusions: if $n = p^\alpha$, then

$$SP(n) = \begin{cases} p, & 1 \le \alpha \le p; \\ p^2, & p+1 \le \alpha \le 2p^2; \\ p^3, & 2p^2+1 \le \alpha \le 3p^3; \\ \cdots & \cdots \\ p^\alpha, & (\alpha-1)p^{\alpha-1}+1 \le \alpha \le \alpha p^\alpha. \end{cases}$$

Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ denotes the factorization of n into prime powers. If $\alpha_i \le p_i$ for all $\alpha_i (i = 1, 2, \cdots, r)$, then we have $SP(n) = U(n)$, where $U(n) = \prod_{p|n} p$, $\prod_{p|n}$ denotes the product over all different prime divisors of n. It is clear that $SP(n)$ is not a multiplicative function. For example, $SP(3) = 3$, $SP(8) = 4$, $SP(24) = 6 \neq SP(3) \times SP(8)$. But for most n we have $SP(n) = U(n)$.

About other properties of $SP(n)$, many scholars had studied it, and obtained some interesting results. For example, In reference [2], Dr.Zhefeng Xu studied the mean value properties of $SP(n)$, and obtain some sharper asymptotic formulas, one of them as follows: for any real number $x \geq 1$,

$$\sum_{n\leq x} SP(n) = \frac{1}{2}x^2 \prod_p \left(1 - \frac{1}{p(p+1)}\right) + O\left(x^{\frac{2}{3}+\epsilon}\right),$$

where $\prod_p$ denotes the product over all prime numbers, ϵ is any given positive number. Huanqin Zhou [3] studied the convergent properties of an infinite series involving $SP(n)$, and gave some interesting identities. That is, she proved that for any complex number s with $Re(s) > 1$,

$$\sum_{n=1}^{\infty} \frac{(-1)^{\mu(n)}}{(SP(n^k))^s} = \begin{cases} \frac{2^s+1}{2^s-1}\frac{1}{\zeta(s)}, & k = 1, 2; \\ \frac{2^s+1}{2^s-1}\frac{1}{\zeta(s)} - \frac{2^s-1}{4^s}, & k = 3; \\ \frac{2^s+1}{2^s-1}\frac{1}{\zeta(s)} - \frac{2^s-1}{4^s} + \frac{3^s-1}{9^s}, & k = 4, 5. \end{cases}$$

If $n \geq 1$, the Euler function $\phi(n)$ is defined as the number of all positive integers not exceeding n, which are relatively prime to n. It is clear that $\phi(n)$ is a multiplicative function. In this paper, we shall use the elementary method to study the solutions of the equation $SP(n^k) = \phi(n)$, and give its all solutions for $k = 1, 2, 3$. That is, we shall prove the following:

Theorem 1. The equation $SP(n) = \phi(n)$ have only 4 positive integer solutions, namely, $n = 1, 4, 8, 18$.

Theorem 2. The equation $SP(n^2) = \phi(n)$ have only 3 positive integer solutions, namely, $n = 1, 8, 18$.

Theorem 3. The equation $SP(n^3) = \phi(n)$ have only 3 positive intrger solutions, namely, $n = 1, 16, 18$.

Generally, for any given positive integer number $k \geq 4$, we conjecture that the equation $SP(n^k) = \phi(n)$ has only finite positive integer solutions. This is an open problem.

§2. Proof of the theorems

In this section, we shall complete the proof of the theorems. First we prove Theorem 1. It is easy to versify that $n = 1$ is one solution of the equation $SP(n) = \phi(n)$. In order to obtain the other positive integer solution, we discuss in the following cases:

1. $n > 1$ is an odd number.

At this time, from the definition of the Smarandache power function $SP(n)$ we know that $SP(n)$ is an odd number, but $\phi(n)$ is an even number, hence $SP(n) \neq \phi(n)$.

2. n is an even number.

(1) $n = 2^\alpha$, $\alpha \geq 1$. It is easy to versify that $n = 2$ is not a solution of the equation $SP(n) = \phi(n)$ and $n = 4, 8$ are the solutions of the equation $SP(n) = \phi(n)$. If $\alpha \geq 4$, $(\alpha-2)2^{\alpha-2} \geq \alpha$, so $2^\alpha \mid (2^{\alpha-2})^{2^{\alpha-2}}$, namely $n \mid (\frac{\phi(n)}{2})^{\frac{\phi(n)}{2}}$, which implies $SP(n) \leq \frac{\phi(n)}{2} < \phi(n)$.

(2) $n = 2^{\alpha}p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$, where p_i is an odd prime, $p_1 < p_2 < \cdots < p_k$, $\alpha_i \geq 1$, $i = 1, 2, \cdots, k$, $\alpha \geq 2$, $k \geq 1$. At this time,

$$\phi(n) = 2^{\alpha-1}p_1^{\alpha_1-1}p_2^{\alpha_2-1}\cdots p_k^{\alpha_k-1}(p_1-1)(p_2-1)\cdots(p_k-1).$$

If $n \dagger (\phi(n))^{\phi(n)}$, then from the definition of the Smarandache power function $SP(n)$ we know that $SP(n) \neq \phi(n)$.

If $n \mid (\phi(n))^{\phi(n)}$, then from the form of $\phi(n)$, we can imply $\alpha_k \geq 2$.

(i) for 2^{α}. $\alpha \geq 2$, so

$$(\alpha-1)\frac{\phi(n)}{2} \geq (\alpha-1)2^{\alpha-1}p_k^{\alpha_k-1}\frac{p_k-1}{2} \geq (\alpha-1)\cdot 2\cdot 3 \geq 6(\alpha-1) \geq 3\alpha > \alpha,$$

which implies $2^{\alpha} \mid (2^{(\alpha-1)})^{\frac{\phi(n)}{2}}$. Hence $2^{\alpha} \mid (\frac{\phi(n)}{2})^{\frac{\phi(n)}{2}}$.

(ii) for $p_i^{\alpha_i} \mid n$. If $\alpha_i = 1$, associating

$$\frac{\phi(n)}{2} \geq 2^{\alpha-1}p_k^{\alpha_k-1}\frac{p_k-1}{2} \geq 2\cdot 3 = 6 > 1$$

with $p_i \mid (\phi(n))^{\phi(n)}$ which implies $p_i \mid \frac{\phi(n)}{2}$, we can deduce that $p_i \mid (\frac{\phi(n)}{2})^{\frac{\phi(n)}{2}}$. If $\alpha_i \geq 2$,

$$(\alpha_i-1)\frac{\phi(n)}{2} \geq (\alpha_i-1)2^{\alpha-1}p_i^{\alpha_i-1}\frac{p_i-1}{2} \geq (\alpha_i-1)\cdot 2\cdot 3 \geq 6(\alpha_i-1) \geq 3\alpha_i > \alpha_i,$$

which implies $p_i^{\alpha_i} \mid (p_i^{(\alpha_i-1)})^{\frac{\phi(n)}{2}}$. Hence $p_i^{\alpha_i} \mid (\frac{\phi(n)}{2})^{\frac{\phi(n)}{2}}$. Consequently, $\forall p_i^{\alpha_i} \mid n$, $p_i^{\alpha_i} \mid (\frac{\phi(n)}{2})^{\frac{\phi(n)}{2}}$.

Combining (i) and (ii), we can deduce that if $n \mid (\phi(n))^{\phi(n)}$, then $n \mid (\frac{\phi(n)}{2})^{\frac{\phi(n)}{2}}$. Hence $SP(n) \leq \frac{\phi(n)}{2} < \phi(n)$.

(3) $n = 2p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$, where p_i is an odd prime, $p_1 < p_2 < \cdots < p_k$, $\alpha_i \geq 1$, $i = 1, 2, \cdots, k$, $k \geq 1$. At this time,

$$\phi(n) = p_1^{\alpha_1-1}p_2^{\alpha_2-1}\cdots p_k^{\alpha_k-1}(p_1-1)(p_2-1)\cdots(p_k-1).$$

If $n \dagger (\phi(n))^{\phi(n)}$, then from the definition of the Smarandache power function $SP(n)$ we know that $SP(n) \neq \phi(n)$.

If $n \mid (\phi(n))^{\phi(n)}$, then from the form of $\phi(n)$, we can imply $\alpha_k \geq 2$.

(i) $k \geq 2$. We will prove that $n \mid (\frac{\phi(n)}{2})^{\frac{\phi(n)}{2}}$.

For one hand, obviously, $2 \mid (\frac{\phi(n)}{2})^{\frac{\phi(n)}{2}}$. For the other hand, $\forall p_i^{\alpha_i} \mid n$, if $\alpha_i = 1$, associating

$$\frac{\phi(n)}{2} \geq p_k^{\alpha_k-1}(p_i-1)\frac{p_k-1}{2} \geq 3\cdot 2 = 6 > 1$$

with $p_i \mid (\phi(n))^{\phi(n)}$ which implies $p_i \mid \frac{\phi(n)}{2}$, we can deduce that $p_i \mid (\frac{\phi(n)}{2})^{\frac{\phi(n)}{2}}$. If $\alpha_i \geq 2$,

$$(\alpha_i-1)\frac{\phi(n)}{2} \geq (\alpha_i-1)p_k^{\alpha_k-1}(p_1-1)\frac{p_k-1}{2} \geq (\alpha_i-1)\cdot 5\cdot 2\cdot 2 \geq 20(\alpha_i-1) \geq 10\alpha_i > \alpha_i,$$

which implies $p_i^{\alpha_i} \mid (p_i^{(\alpha_i-1)})^{\frac{\phi(n)}{2}}$. Hence $p_i^{\alpha_i} \mid (\frac{\phi(n)}{2})^{\frac{\phi(n)}{2}}$. Consequently, $n \mid (\frac{\phi(n)}{2})^{\frac{\phi(n)}{2}}$, which implies $SP(n) \leq \frac{\phi(n)}{2} < \phi(n)$.

(ii) $k = 1$. At this time, $n = 2p_1^{\alpha_1}$, $\alpha_1 \geq 2$, $\phi(n) = p_1^{\alpha_1-1}(p_1-1)$.

which implies $p_i^{\alpha_i} \mid (p_i^{(\alpha_i-1)})^{\frac{\phi(n)}{2}}$. Hence $p_i^{\alpha_i} \mid (\frac{\phi(n)}{2})^{\frac{\phi(n)}{2}}$. Consequently, $n \mid (\frac{\phi(n)}{2})^{\frac{\phi(n)}{2}}$, which implies $SP(n) \leq \dfrac{\phi(n)}{2} < \phi(n)$.

(ii) $k = 1$. At this time, $n = 2p_1^{\alpha_1}$, $\alpha_1 \geq 2$, $\phi(n) = p_1^{\alpha_1-1}(p_1 - 1)$.

$(ii)'$ $p_1 \geq 5$, because $\alpha_1 \geq 2$,

$$(\alpha_1 - 1)\frac{\phi(n)}{\frac{p_1-1}{2}} = (\alpha_1 - 1)p_1^{\alpha_1-1}2 \geq (\alpha_1 - 1) \cdot 5 \cdot 2 \geq 10(\alpha_1 - 1) \geq 5\alpha_1 > \alpha_1,$$

which implies $p_1^{\alpha_1} \mid (p_1^{(\alpha_1-1)})^{\frac{\phi(n)}{\frac{p_1-1}{2}}}$. Hence $p_1^{\alpha_1} \mid (\frac{\phi(n)}{\frac{p_1-1}{2}})^{\frac{\phi(n)}{\frac{p_1-1}{2}}}$. Obviously, $2 \mid (\frac{\phi(n)}{\frac{p_1-1}{2}})^{\frac{\phi(n)}{\frac{p_1-1}{2}}}$. Consequently, $n \mid (\frac{\phi(n)}{\frac{p_1-1}{2}})^{\frac{\phi(n)}{\frac{p_1-1}{2}}}$, which implies $SP(n) \leq \dfrac{\phi(n)}{\frac{p_1-1}{2}} < \phi(n)$.

$(ii)''$ $p_1 = 3$, namely $n = 2 \cdot 3^{\alpha_1}$.

$\alpha_1 = 1$, $\phi(n) = \phi(6) = 2$, $SP(n) = SP(6) = 6$, so $SP(n) \neq \phi(n)$.

$\alpha_1 = 2$, $\phi(n) = \phi(18) = 6$, $SP(n) = SP(18) = 6$, so $SP(n) = \phi(n)$.

$\alpha_1 \geq 3$, $(\frac{\phi(n)}{3})^{\frac{\phi(n)}{3}} = (2 \cdot 3^{\alpha_1-2})^{2\cdot 3^{\alpha_1-2}}$, so $n \mid (\frac{\phi(n)}{3})^{\frac{\phi(n)}{3}}$, which implies $SP(n) \leq \dfrac{\phi(n)}{3} < \phi(n)$.

Combining (1), (2) and (3), we know that if n is an even number, then $SP(n) = \phi(n)$ if and only if $n =$ 4, 8, 18.

Associating the cases 1 and 2, we complete the proof of Theorem 1.

Using the similar discussion, we can easily obtain the proofs of Theorem 2 and Theorem 3.

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