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Super Fibonacci Graceful Labeling

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Abstract: A *Smarandache-Fibonacci Triple* is a sequence $S(n),\ n \geq 0$ such that $S(n) = S(n-1)+S(n-2)$, where $S(n)$ is the Smarandache function for integers $n \geq 0$. Certainly, it is a generalization of *Fibonacci sequence*. A *Fibonacci graceful labeling* and a *super Fibonacci graceful labeling* on graphs were introduced by Kathiresan and Amutha in 2006. Generally, let G be a (p,q)-graph and $\{S(n)|n \geq 0\}$ a Smarandache-Fibonacci Triple. An bijection $f\colon V(G) \rightarrow \{S(0), S(1), S(2), \ldots, S(q)\}$ is said to be a *super Smarandache-Fibonacci graceful graph* if the induced edge labeling $f^*(uv) = |f(u) - f(v)|$ is a bijection onto the set $\{S(1), S(2), \ldots, S(q)\}$. Particularly, if $S(n), n \geq 0$ is just the Fibonacci sequence $F_i,\ i \geq 0$, such a graph is called a *super Fibonacci graceful graph*. In this paper, we construct new types of graphs namely $F_n \oplus K^+_{1,m}$, $C_n \oplus P_m$, $K_{1,n} \oslash K_{1,2}$, $F_n \oplus P_m$ and $C_n \oplus K_{1,m}$ and we prove that these graphs are super Fibonacci graceful graphs.

Key Words: Smarandache-Fibonacci triple, graceful labeling, Fibonacci graceful labeling, super Smarandache-Fibonacci graceful graph, super Fibonacci graceful graph.

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§1. Introduction

By a graph, we mean a finite undirected graph without loops or multiple edges. A path of length n is denoted by P_{n+1}. A cycle of length n is denoted by C_n. G^+ is a graph obtained from the graph G by attaching pendant vertex to each vertex of G. Graph labelings, where the vertices are assigned certain values subject to some conditions, have often motivated by practical problems.

In the last five decades enormous work has been done on this subject [1]. The concept of graceful labeling was first introduced by Rosa [5] in 1967. A function f is a graceful labeling of a graph G with q edges if f is an injection from the vertices of G to the set $\{0, 1, 2, \ldots, q\}$ such that when each edge uv is assigned the label $|f(u) - f(v)|$, the resulting edge labels are distinct. The notion of Fibonacci graceful labeling and Super Fibonacci graceful labeling were introduced by Kathiresan and Amutha [3]. We call a function f, a *Fibonacci graceful labeling* of a graph G

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with q edges if f is an injection from the vertices of G to the set $\{0, 1, 2, \ldots, F_q\}$, where F_q is the q^{th} Fibonacci number of the Fibonacci series $F_1 = 1$,$F_2 = 2$,$F_3 = 3$, $F_4 = 5, \cdots$, such that each edge uv is assigned the labels $|f(u) - f(v)|$, the resulting edge labels are $F_1, F_2, \ldots, F_q$. An injective function $f : V(G) \to \{F_0, F_1, \ldots, F_q\}$, where F_q is the q^{th} Fibonacci number, is said to be a super Fibonacci graceful labeling if the induced edge labeling $|f(u) - f(v)|$ is a bijection onto the set $\{F_1, F_2, \ldots, F_q\}$. In the labeling problems the induced labelings must be distinct. So to introduce Fibonacci graceful labelings we assume $F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, \cdots$, as the sequence of Fibonacci numbers instead of $0, 1, 2, \ldots$, [3].

Generally, a *Smarandache-Fibonacci Triple* is a sequence $S(n)$, $n \geq 0$ such that $S(n) = S(n-1) + S(n-2)$, where $S(n)$ is the Smarandache function for integers $n \geq 0$ [2]. A (p, q)-graph G is a *super Smarandache-Fibonacci graceful graph* if there is an bijection $f : V(G) \to \{S(0), S(1), S(2), \ldots, S(q)\}$ such that the induced edge labeling $f^*(uv) = |f(u) - f(v)|$ is a bijection onto the set $\{S(1), S(2), \ldots, S(q)\}$. So a super Fibonacci graceful graph is a special type of Smarandache-Fibonacci graceful graph by definition.

In this paper, we prove that $F_n \oplus K_{1,m}^+$, $C_n \oplus P_m$, $K_{1,n} \oslash K_{1,2}$, $F_n \oplus P_m$ and $C_n \oplus K_{1,m}$ are super Fibonacci graceful graphs.

§2. Main Results

In this section, we show that some new types of graphs namely $F_n \oplus K_{1,m}^+$, $C_n \oplus P_m$, $K_{1,n} \oslash K_{1,2}$, $F_n \oplus P_m$ and $C_n \oplus K_{1,m}$ are super Fibonacci graceful graphs.

Definition 2.1([4]) *Let G be a (p, q) graph. An injective function $f : V(G) \to \{F_0, F_1, F_2, \ldots, F_q\}$, where F_q is the q^{th} Fibonacci number, is said to be a super Fibonacci graceful graphs if the induced edge labeling $f^*(uv) = |f(u) - f(v)|$ is a bijection onto the set $\{F_1, F_2, \ldots, F_q\}$.*

Definition 2.2 *The graph $G = F_n \oplus P_m$ consists of a fan F_n and a Path P_m which is attached with the maximum degree of the vertex of F_n.*

The following theorem shows that the graph $F_n \oplus P_m$ is a super Fibonacci graceful graph.

Theorem 2.3 *The graph $G = F_n \oplus P_m$ is a super Fibonacci graceful graph.*

Proof Let $\{u_0 = v, u_1, u_2, \ldots, u_n\}$ be the vertex set of F_n and $v_1, v_2, \ldots, v_m$ be the vertices of P_m joined with the maximum degree of the vertex u_0 of F_n. Also, $|V(G)| = m + n + 1$ and $|E(G)| = 2n + m - 1$. Define $f : V(G) \to \{F_0, F_1, \ldots, F_q\}$ by $f(u_0) = F_0$, $f(u_i) = F_{2n+m-1-2(i-1)}$, $1 \leq i \leq n$, $f(v_i) = F_{m-2(i-1)}$, $1 \leq i \leq 2$,

$$f(v_m) = \begin{cases} F_2 & \text{if} m \equiv 0 (\text{mod} 3) \\ F_1 & \text{if} m \equiv 1, 2 (\text{mod} 3) \end{cases} \qquad f(v_{m-1}) = \begin{cases} F_3 & \text{if} m \equiv 1 (\text{mod} 3) \\ F_2 & \text{if} m \equiv 2 (\text{mod} 3) \end{cases}$$

and $f(v_{m-2}) = F_4$ if $m \equiv 2(mod 3)$.

For $l = 1, 2, \ldots, \dfrac{m-3}{3}$, or $\dfrac{m-4}{3}$, or $\dfrac{m-5}{3}$ according to $m \equiv 0(mod 3)$ or $m \equiv 1(mod 3)$

or $m \equiv 2(mod3)$, define

$$f(v_{i+2}) = F_{m-1-2(i-1)+3(l-1)}, \quad 3l-2 \leq i \leq 3l.$$

We claim that the edge labels are distinct. Let $E_1 = \{f^*(u_iu_{i+1}) : i = 1, 2, \ldots, n-1\}$. Then

$$\begin{aligned} E_1 &= \{|f(u_i) - f(u_{i+1})| : i = 1, 2, \ldots, n-1\} \\ &= \{|f(u_1) - f(u_2)|, |f(u_2) - f(u_3)|, \ldots, |f(u_{n-1}) - f(u_n)|\} \\ &= \{|F_{2n+m-1} - F_{2n+m-3}|, |F_{2n+m-3} - F_{2n+m-5}|, \ldots, |F_{m+3} - F_{m+1}|\} \\ &= \{F_{2n+m-2}, F_{2n+m-4}, \ldots, F_{m+4}, F_{m+2}\}, \end{aligned}$$

$$\begin{aligned} E_2 &= \{f^*(u_0u_i) : i = 1, 2, \ldots, n\} = \{|f(u_0) - f(u_i)| : i = 1, 2, \ldots, n\} \\ &= \{|f(u_0) - f(u_1)|, |f(u_0) - f(u_2)|, \ldots, |f(u_0) - f(u_n)|\} \\ &= \{|F_0 - F_{2n+m-1}|, |F_0 - F_{2n+m-3}|, \ldots, |F_0 - F_{m+1}|\} \\ &= \{F_{2n+m-1}, F_{2n+m-3}, \ldots, F_{m+3}, F_{m+1}\}, \end{aligned}$$

$$\begin{aligned} E_3 &= \{f^*(u_0v_1), f^*(v_1v_2)\} = \{|f(u_0) - f(v_1)|, |f(v_1) - f(v_2)|\} \\ &= \{|F_0 - F_m|, |F_m - F_{m-2}|\} = \{F_m, F_{m-1}\}. \end{aligned}$$

Let $E_4 = \{f^*(v_2v_3)\}$. The edge labeling between the vertex v_2 and starting vertex v_3 of the first loop is

$$E_4 = \{|f(v_2) - f(v_3)|\} = \{|F_{m-2} - F_{m-1}|\} = \{F_{m-3}\}.$$

For $l = 1$, let $E_5 = \{f^*(v_{i+2}v_{i+3}) : 1 \leq i \leq 2\}$. Then

$$\begin{aligned} E_5 &= \{|f(v_{i+2}) - f(v_{i+3})| : 1 \leq i \leq 2\} \\ &= \{|f(v_3) - f(v_4)|, |f(v_4) - f(v_5)|\} \\ &= \{|F_{m-1} - F_{m-3}|, |F_{m-3} - F_{m-5}|\} = \{F_{m-2}, F_{m-4}\}. \end{aligned}$$

Let $E_5^1 = \{f^*(v_5v_6)\}$. We find the edge labeling between the end vertex v_5 of the first loop and starting vertex v_6 of the second loop following.

$$E_5^1 = \{|f(v_5) - f(v_6)|\} = \{|F_{m-5} - F_{m-4}|\} = \{F_{m-6}\}.$$

For $l = 2$, let $E_6 = \{f^*(v_{i+2}v_{i+3}) : 4 \leq i \leq 5\}$. Then

$$\begin{aligned} E_6 &= \{|f(v_{i+2}) - f(v_{i+3})| : 4 \leq i \leq 5\} = \{|f(v_6) - f(v_7)|, |f(v_7) - f(v_8)|\} \\ &= \{|F_{m-4} - F_{m-6}|, |F_{m-6} - F_{m-8}|\} = \{F_{m-5}, F_{m-7}\}. \end{aligned}$$

For labeling between the end vertex v_8 of the second loop and starting vertex v_9 of the third loop, let $E_6^1 = \{f^*(v_8v_9)\}$. Then

$$E_6^1 = \{|f(v_8) - f(v_9)|\} = \{|F_{m-8} - F_{m-7}|\} = \{F_{m-9}\},$$

etc.. For $l = \dfrac{m-5}{3} - 1$, let $E_{\frac{m-5}{3}-1} = \{f^*(v_{i+2}v_{i+3}) : m-10 \le i \le m-9\}$. Then

$$\begin{aligned} E_{\frac{m-5}{3}-1} &= \{|f(v_{i+2}) - f(v_{i+3})| : m-10 \le i \le m-9\} \\ &= \{|f(v_{m-8}) - f(v_{m-7})|, |f(v_{m-7}) - f(v_{m-6})|\} \\ &= \{|F_{10} - F_8|, |F_8 - F_6|\} = \{F_9, F_7\}. \end{aligned}$$

For the edge labeling between the end vertex v_{m-6} of the $(\dfrac{m-5}{3} - 1)^{th}$ loop and starting vertex v_{m-5} of the $(\dfrac{m-5}{3})^{rd}$ loop, let $E^1_{\frac{m-5}{3}-1} = \{f^*(v_{m-6}v_{m-5})\}$. Then

$$E^1_{\frac{m-5}{3}-1} = \{|f(v_{m-6}) - f(v_{m-5})|\} = \{|F_6 - F_7|\} = \{F_5\},$$

$$\begin{aligned} E_{\frac{m-5}{3}} &= \{f^*(v_{i+2}v_{i+3}) : m-7 \le i \le m-6\} \\ &= \{|f(v_{i+2}) - f(v_{i+3})| : m-7 \le i \le m-6\} \\ &= \{|f(v_{m-5}) - f(v_{m-4})|, |f(v_{m-4}) - f(v_{m-3})|\} \\ &= \{|F_7 - F_5|, |F_5 - F_3|\} = \{F_6, F_4\}. \end{aligned}$$

For $l = \dfrac{m-4}{3} - 1$, let $E_{\frac{m-4}{3}-1} = \{f^*(v_{i+2}v_{i+3}) : m-9 \le i \le m-8\}$. Then

$$\begin{aligned} E_{\frac{m-4}{3}-1} &= \{|f(v_{i+2}) - f(v_{i+3})| : m-9 \le i \le m-8\} \\ &= \{|f(v_{m-7}) - f(v_{m-6})|, |f(v_{m-6}) - f(v_{m-5})|\} \\ &= \{|F_9 - F_7|, |F_7 - F_5|\} = \{F_8, F_6\}. \end{aligned}$$

For the edge labeling between the end vertex v_{m-5} of the $(\dfrac{m-4}{3} - 1)^{th}$ loop and starting vertex v_{m-4} of the $(\dfrac{m-4}{3})^{rd}$ loop, let $E^1_{\frac{m-4}{3}-1} = \{f^*(v_{m-5}v_{m-4})\}$. Then

$$E^1_{\frac{m-4}{3}-1} = \{|f(v_{m-5}) - f(v_{m-4})|\} = \{|F_5 - F_6|\} = \{F_4\}.$$

For $l = \dfrac{m-4}{3}$, let $E_{\frac{m-4}{3}} = \{f^*(v_{i+2}v_{i+3}) : m-6 \le i \le m-5\}$. Calculation shows that

$$\begin{aligned} E_{\frac{m-4}{3}} &= \{|f(v_{i+2}) - f(v_{i+3})| : m-6 \le i \le m-5\} \\ &= \{|f(v_{m-4}) - f(v_{m-3})|, |f(v_{m-3}) - f(v_{m-2})|\} \\ &= \{|F_6 - F_4|, |F_4 - F_2|\} = \{F_5, F_3\}. \end{aligned}$$

Now for $l = \dfrac{m-3}{3} - 1$, let $E_{\frac{m-3}{3}-1} = \{f^*(v_{i+2}v_{i+3}) : m-8 \le i \le m-7\}$. Then

$$\begin{aligned} E_{\frac{m-3}{3}-1} &= \{|f(v_{i+2}) - f(v_{i+3})| : m-8 \le i \le m-7\} \\ &= \{|f(v_{m-6}) - f(v_{m-5})|, |f(v_{m-5}) - f(v_{m-4})|\} \\ &= \{|F_8 - F_6|, |F_6 - F_4|\} = \{F_7, F_5\}. \end{aligned}$$

Similarly, for finding the edge labeling between the end vertex v_{m-4} of the $(\dfrac{m-3}{3} - 1)^{th}$ loop and starting vertex v_{m-3} of the $(\dfrac{m-3}{3})^{rd}$ loop, let $E^1_{\frac{m-3}{3}-1} = \{f^*(v_{m-4}v_{m-3})\}$. Then

$$E^1_{\frac{m-3}{3}-1} = \{|f(v_{m-4}) - f(v_{m-3})|\} = \{|F_4 - F_2|\} = \{F_3\}.$$

For $l = \dfrac{m-3}{3}$, let $E_{\frac{m-3}{3}} = \{f^*(v_{i+2}v_{i+3}) : m-5 \leq i \leq m-4\}$. Then

$$\begin{aligned} E_{\frac{m-3}{3}} &= \{|f(v_{i+2}) - f(v_{i+3})| : m-5 \leq i \leq m-4\} \\ &= \{|f(v_{m-3}) - f(v_{m-2})|, |f(v_{m-2}) - f(v_{m-1})|\} \\ &= \{|F_5 - F_3|, |F_3 - F_1|\} = \{F_4, F_2\}. \end{aligned}$$

Now let

$$E^{(1)} = \left(E_1 \bigcup E_2 \bigcup, \ldots, \bigcup E_{\frac{m-3}{3}}\right) \bigcup \left(E_5^1 \bigcup E_6^1 \bigcup, \ldots, \bigcup E^1_{\frac{m-3}{3}-1}\right),$$

$$E^{(2)} = \left(E_1 \bigcup E_2 \bigcup, \ldots, \bigcup E_{\frac{m-4}{3}}\right) \bigcup \left(E_5^1 \bigcup E_6^1 \bigcup, \ldots, \bigcup E^1_{\frac{m-4}{3}-1}\right),$$

$$E^{(3)} = \left(E_1 \bigcup E_2 \bigcup, \ldots, \bigcup E_{\frac{m-5}{3}}\right) \bigcup \left(E_5^1 \bigcup E_6^1 \bigcup, \ldots, \bigcup E^1_{\frac{m-5}{3}-1}\right).$$

If $m \equiv 0(mod3)$, let $E_1^* = \{f^*(v_{m-1}v_m)\}$, then $E_1^* = \{|f(v_{m-1} - f(v_m)|\} = \{|F_1 - F_2|\} = \{F_1\}$. Thus,

$$E = E_1^* \bigcup E^{(1)} = \{F_1, F_2, \ldots, F_{2n+m-1}\}.$$

For example the super Fibonacci graceful labeling of $F_4 \oplus P_6$ is shown in Fig.1.

$F_4 \oplus P_6$:

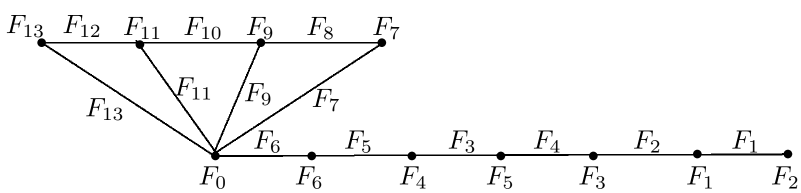

Fig.1

If $m \equiv 1(mod3)$, let $E_2^* = \{f^*(v_{m-2}v_{m-1}), f^*(v_{m-1}v_m)\}$, then

$$\begin{aligned} E_2^* &= \{|f(v_{m-2} - f(v_{m-1}|, |f(v_{m-1} - f(v_m)|\} \\ &= \{|F_2 - F_3|\,|F_3 - F_1|\} = \{F_1, F_2\}. \end{aligned}$$

Thus,

$$E = E_2^* \cup E^{(2)} = \{F_1, F_2, \ldots, F_{2n+m-1}\}.$$

For example the super Fibonacci graceful labeling of $F_4 \oplus P_7$ is shown in Fig.2.

$F_4 \oplus P_7$:

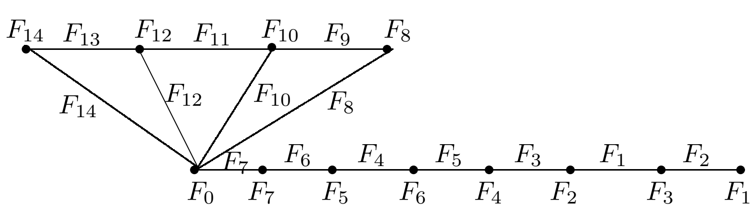

Fig.2

If $m \equiv 2(mod3)$, let $E_3^* = \{f^*(v_{m-3}v_{m-2}), f^*(v_{m-2}v_{m-1}), f^*(v_{m-1}v_m)\}$, then

$$\begin{aligned} E_3^* &= \{|f(v_{m-3}) - f(v_{m-2})|, |f(v_{m-2}) - f(v_{m-1})|, |f(v_{m-1} - f(v_m)|\} \\ &= \{|F_3 - F_4|, |F_4 - F_2|, |F_2 - F_1|\} = \{F_2, F_3, F_1\}. \end{aligned}$$

Thus,

$$E = E_3^* \cup E^{(3)} = \{F_1, F_2, \ldots, F_{2n+m-1}\}.$$

For example the super Fibonacci graceful labeling of $F_5 \oplus P_5$ is shown in Fig.3.

$F_5 \oplus P_5$:

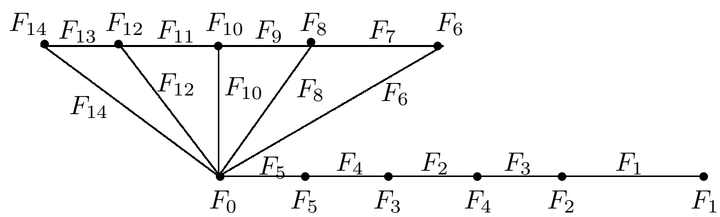

Fig.3

Therefore, $F_n \oplus P_m$ admits a super Fibonacci graceful labeling. Hence, $F_n \oplus P_m$ is a super Fibonacci graceful graph. □

Definition 2.4 *An (n, m)-kite consists of a cycle of length n with m-edge path attached to one vertex and it is denoted by $C_n \oplus P_m$.*

Theorem 2.5 *The graph $G = C_n \oplus P_m$ is a super Fibonacci graceful graph when $n \equiv 0(mod3)$.*

Proof Let $\{u_1, u_2, \ldots, u_n = v\}$ be the vertex set of C_n and $\{v = u_n, v_1, v_2, \ldots, v_m\}$ be the vertex set of P_m joined with the vertex u_n of C_n. Also, $|V(G)| = |E(G)| = m + n$. Define $f : V(G) \to \{F_0, F_1, \ldots, F_q\}$ by $f(u_n) = F_0$, $f(u_1) = F_{m+n}$, $f(u_2) = F_{m+n-2}$ and for $l = 1, 2, \ldots, \dfrac{n-3}{3}$, $f(u_{i+2}) = F_{m+n-1-2(i-1)+3(l-1)}$, and for $3l-2 \leq i \leq 3l$, $f(v_i) = F_{m-2(i-1)}$, and for $1 \leq i \leq 2$,

$$f(v_m) = \begin{cases} F_2 & \text{if m} \equiv 0(\text{mod}3) \\ F_1 & \text{if m} \equiv 1, 2(\text{mod}3), \end{cases} \qquad f(v_{m-1}) = \begin{cases} F_3 & \text{if m} \equiv 1(\text{mod}3) \\ F_2 & \text{if m} \equiv 2(\text{mod}3) \end{cases}$$

and $f(v_{m-2}) = F_4$ when $m \equiv 2(mod3)$. For $l = 1, 2, \ldots, \dfrac{m-3}{3}$ or $\dfrac{m-4}{3}$ or $\dfrac{m-5}{3}$ according to $m \equiv 0(mod3)$ or $m \equiv 1(mod3)$ or $m \equiv 2(mod3)$, let $f(v_{i+2}) = F_{m-1-2(i-1)+3(l-1)}$ for $3l - 2 \leq i \leq 3l$.

We claim that all these edge labels are distinct. Let $E_1 = \{f^*u_nu_1), f^*(u_1u_2)\}$. Then

$$\begin{aligned} E_1 &= \{|f(u_n) - f(u_1)|, |f(u_1) - f(u_2)|\} \\ &= \{|F_0 - F_{m+n}|, |F_{m+n} - F_{m+n-2}|\} = \{F_{m+n}, F_{m+n-1}\}. \end{aligned}$$

For the edge labeling between the vertex u_2 and starting vertex u_3 of the first loop, let $E_2 = \{f^*(u_2u_3)\}$. Then

$$E_2 = \{|f(u_2) - f(u_3)|\} = \{|F_{m+n-2} - F_{m+n\_1}|\} = \{F_{m+n-3}\}.$$

For $l = 1$, let $E_3 = \{f^*(u_{i+2}u_{i+3}) : 1 \leq i \leq 2\}$. Then

$$\begin{aligned} E_3 &= \{|f(u_{i+2}) - f(u_{i+3})| : 1 \leq i \leq 2\} \\ &= \{|f(u_3) - f(u_4)|, |f(u_4) - f(u_5)|\} \\ &= \{|F_{m+n-1} - F_{m+n-3}|, |F_{m+n-3} - F_{m+n-5})|\} = \{F_{m+n-2}, F_{m+n-4}\}. \end{aligned}$$

For the edge labeling between the end vertex u_5 of the first loop and starting vertex u_6 of the second loop, let $E_3^{(1)} = \{f^*(u_5u_6)\}$. Then

$$E_3^{(1)} = \{|f(u_5) - f(u_6)|\} = \{|F_{m+n-5} - F_{m+n-4}|\} = \{F_{m+n-6}\}.$$

For $l = 2$, let $E_4 = \{f^*(u_{i+2}u_{i+3}) : 4 \leq i \leq 5\}$. Then

$$\begin{aligned} E_4 &= \{|f(u_{i+2}) - f(u_{i+3})| : 4 \leq i \leq 5\} = \{|f(u_6) - f(u_7)|, |f(u_7) - f(u_8)|\} \\ &= \{|F_{m+n-4} - F_{m+n-6}|, |F_{m+n-6} - F_{m+n-8}|\} = \{F_{m+n-5}, F_{m+n-7}\}. \end{aligned}$$

For the edge labeling between the end vertex u_8 of the second loop and starting vertex u_9 of the third loop, let $E_4^{(1)} = \{f^*(u_8u_9)\}$. Then

$$E_4^{(1)} = \{|f(u_8) - f(u_9)|\} = \{|F_{m+n-8} - F_{m+n-7}|\} = \quad \{F_{m+n-9}\},$$

etc.. For $l = \dfrac{n-3}{3} - 1$, let $E_{\frac{n-3}{3}-1} = \{f^*(u_{i+2}u_{i+3}) : n-8 \leq i \leq n-7\}$. Then

$$\begin{aligned} E_{\frac{n-3}{3}-1} &= \{|f(u_{i+2}) - f(u_{i+3})| : n-8 \leq i \leq n-7\} \\ &= \{|f(u_{n-6}) - f(u_{n-5})|, |f(u_{n-5}) - f(u_{n-4})|\} \\ &= \{|F_{m+8} - F_{m+6}|, |F_{m+6} - F_{m+4})|\} = \{F_{m+7}, F_{m+5}\}. \end{aligned}$$

For finding the edge labeling between the end vertex u_{n-4} of the $(\dfrac{n-3}{3} - 1)^{th}$ loop and starting vertex u_{n-3} of the $(\dfrac{n-3}{3})^{rd}$ loop, let $E^{(1)}_{\frac{n-3}{3}-1} = \{f^*(u_{n-4}u_{n-3})\}$. Then

$$E^{(1)}_{\frac{n-3}{3}-1} = \{|f(u_{n-4}) - f(u_{n-3})|\} = \{|F_{m+4} - F_{m+5}|\} = \{F_{m+3}\}.$$

For $l = \dfrac{n-3}{3}$, let $E_{\frac{n-3}{3}} = \{f^*(u_{i+2}u_{i+3}) : n-5 \leq i \leq n-4\}$. Then

$$\begin{aligned} E_{\frac{n-3}{3}} &= \{|f(u_{i+2}) - f(u_{i+3})| : n-5 \leq i \leq n-4\} \\ &= \{|f(u_{n-3}) - f(u_{n-2})|, |f(u_{n-2}) - f(u_{n-1})|\} \\ &= \{|F_{m+5} - F_{m+3}|, |F_{m+3} - F_{m+1}|\} = \{F_{m+4}, F_{m+2}\}. \end{aligned}$$

Let $E_1^* = \{f^*(u_{n-1}u_n)\}$ and $E_2^* = \{f^*(u_nv_1), f^*(v_1v_2)\}$. Then

$$E_1^* = \{|f(u_{n-1}) - f(u_n)|\} = \{|F_{m+1} - F_0|\} = \{F_{m+1}\},$$

$$\begin{aligned} E_2^* &= \{|f(u_n) - f(v_1)|, |f(v_1) - f(v_2)|\} \\ &= \{|F_0 - F_m|, |F_m - F_{m-2}|\} = \{F_m, F_{m-1}\}. \end{aligned}$$

For finding the edge labeling between the vertex v_2 and starting vertex v_3 of the first loop, let $E_3^* = \{f^*(v_2v_3)\}$. Then

$$E_3^* = \{|f(v_2) - f(v_3)|\} = \{|F_{m-2} - F_{m-1}|\} = \{F_{m-3}\}.$$

For $l = 1$, let $E_4^* = \{f^*(v_{i+2}v_{i+3}) : 1 \leq i \leq 2\}$. Then

$$\begin{aligned} E_4^* &= \{|f(v_{i+2}) - f(v_{i+3})| : 1 \leq i \leq 2\} \\ &= \{|f(v_3) - f(v_4)|, |f(v_4) - f(v_5)|\} \\ &= \{|F_{m-1} - F_{m-3}|, |F_{m-3} - F_{m-5}|\} = \{F_{m-2}, F_{m-4}\}. \end{aligned}$$

Now let $E_4^{(*1)} = \{f^*(v_5v_6)\}$. Then

$$E_4^{(*1)} = \{|f(v_5) - f(v_6)|\} = \{|F_{m-5} - F_{m-4}|\} = \{F_{m-6}\}.$$

For $l = 2$, let $E_5^* = \{f^*(v_{i+2}v_{i+3}) : 4 \leq i \leq 5\}$. Calculation shows that

$$\begin{aligned} E_5^* &= \{|f(v_{i+2}) - f(v_{i+3})| : 4 \leq i \leq 5\} \\ &= \{|f(v_6) - f(v_7)|, |f(v_7) - f(v_8)|\} \\ &= \{|F_{m-4} - F_{m-6}|, |F_{m-6} - F_{m-8}|\} = \{F_{m-5}, F_{m-7}\}. \end{aligned}$$

Let $E_5^{(*1)} = \{f^*(v_8v_9)\}$. We find the edge labeling between the end vertex v_8 of the second loop and starting vertex v_9 of the third loop. In fact,

$$E_5^{(*1)} = \{|f(v_8) - f(v_9)|\} = \{|F_{m-8} - F_{m-7}|\} = \{F_{m-9}\}$$

etc.. For $l = \dfrac{m-5}{3} - 1$, let $E^*_{\frac{m-5}{3}-1} = \{f^*(v_{i+2}v_{i+3}) : m - 10 \leq i \leq m - 9\}$. Then

$$\begin{aligned} E^*_{\frac{m-5}{3}-1} &= \{|f(v_{i+2}) - f(v_{i+3})| : m - 10 \leq i \leq m - 9\} \\ &= \{|f(v_{m-8}) - f(v_{m-7})|, |f(v_{m-7}) - f(v_{m-6})|\} \\ &= \{|F_{10} - F_8|, |F_8 - F_6|\} = \{F_9, F_7\}. \end{aligned}$$

Similarly, for finding the edge labeling between the end vertex v_{m-6} of the $(\dfrac{m-5}{3} - 1)^{th}$ loop and starting vertex v_{m-5} of the $(\dfrac{m-5}{3})^{rd}$ loop, let $E^{(*1)}_{\frac{m-5}{3}-1} = \{f^*(v_{m-6}v_{m-5})\}$. Then

$$E^{(*1)}_{\frac{m-5}{3}-1} = \{|f(v_{m-6}) - f(v_{m-5})|\} = \{|F_6 - F_7|\} = \{F_5\}.$$

For $l = \dfrac{m-5}{3}$, let $E^*_{\frac{m-5}{3}} = \{f^*(v_{i+2}v_{i+3}) : m-7 \le i \le m-6\}$. Then

$$\begin{aligned} E^*_{\frac{m-5}{3}} &= \{|f(v_{i+2}) - f(v_{i+3})| : m-7 \le i \le m-6\} \\ &= \{|f(v_{m-5}) - f(v_{m-4})|, |f(v_{m-4}) - f(v_{m-3})|\} \\ &= \{|F_7 - F_5|, |F_5 - F_3|\} = \{F_6, F_4\}. \end{aligned}$$

For $l = \dfrac{m-4}{3} - 1$, let $E^*_{\frac{m-4}{3}-1} = \{f^*(v_{i+2}v_{i+3}) : m-9 \le i \le m-8\}$. We find that

$$\begin{aligned} E^*_{\frac{m-4}{3}-1} &= \{|f(v_{i+2}) - f(v_{i+3})| : m-9 \le i \le m-8\} \\ &= \{|f(v_{m-7}) - f(v_{m-6})|, |f(v_{m-6}) - f(v_{m-5})|\} \\ &= \{|F_9 - F_7|, |F_7 - F_5|\} = \{F_8, F_6\}. \end{aligned}$$

For getting the edge labeling between the end vertex v_{m-5} of the $(\dfrac{m-4}{3} - 1)^{th}$ loop and starting vertex v_{m-4} of the $(\dfrac{m-4}{3})^{rd}$ loop, let $E^{(*1)}_{\frac{m-4}{3}-1} = \{f^*(v_{m-5}v_{m-4})\}$. Then

$$E^{(*1)}_{\frac{m-4}{3}-1} = \{|f(v_{m-5}) - f(v_{m-4})|\} = \{|F_5 - F_6|\} = \{F_4\}.$$

For $l = \dfrac{m-4}{3}$, let $E^*_{\frac{m-4}{3}} = \{f^*(v_{i+2}v_{i+3}) : m-6 \le i \le m-5\}$. Then

$$\begin{aligned} E^*_{\frac{m-4}{3}} &= \{|f(v_{i+2}) - f(v_{i+3})| : m-6 \le i \le m-5\} \\ &= \{|f(v_{m-4}) - f(v_{m-3})|, |f(v_{m-3}) - f(v_{m-2})|\} \\ &= \{|F_6 - F_4|, |F_4 - F_2|\} = \{F_5, F_3\}. \end{aligned}$$

For $l = \dfrac{m-3}{3} - 1$, let $E^*_{\frac{m-3}{3}-1} = \{f^*(v_{i+2}v_{i+3}) : m-8 \le i \le m-7\}$. Then

$$\begin{aligned} E^*_{\frac{m-3}{3}-1} &= \{|f(v_{i+2}) - f(v_{i+3})| : m-8 \le i \le m-7\} \\ &= \{|f(v_{m-5}) - f(v_{m-4})|, |f(v_{m-4}) - f(v_{m-3})|\} \\ &= \{|F_8 - F_6|, |F_6 - F_4|\} = \{F_7, F_5\}. \end{aligned}$$

For the edge labeling between the end vertex v_{m-3} of the $(\dfrac{m-3}{3} - 1)^{th}$ loop and starting vertex v_{m-2} of the $(\dfrac{m-3}{3})^{rd}$ loop, let $E^{(*1)}_{\frac{m-3}{3}-1} = \{f^*(v_{m-3}v_{m-2})\}$. Then

$$E^{(*1)}_{\frac{m-3}{3}-1} = \{|f(v_{m-3}) - f(v_{m-2})|\} = \{|F_4 - F_5|\} = \{F_3\}.$$

Similarly, for $l = \dfrac{m-3}{3}$, let $E^*_{\frac{m-3}{3}} = \{f^*(v_{i+2}v_{i+3}) : m-5 \le i \le m-4\}$. Then

$$\begin{aligned} E^*_{\frac{m-3}{3}} &= \{|f(v_{i+2}) - f(v_{i+3})| : m-5 \le i \le m-4\} \\ &= \{|f(v_{m-3}) - f(v_{m-2})|, |f(v_{m-2}) - f(v_{m-1})|\} \\ &= \{|F_5 - F_3|, |F_3 - F_1|\} = \{F_4, F_2\}. \end{aligned}$$

Now let

$$\begin{aligned} E^{(1)} &= \left(E_1 \bigcup E_2 \bigcup \cdots \bigcup E_{\frac{n-3}{3}}\right) \bigcup \left(E^1_3 \bigcup E^1_4 \bigcup \cdots \bigcup E^1_{\frac{n-3}{3}}\right) \\ & \bigcup \left(E^*_1 \bigcup E^*_2 \bigcup \cdots \bigcup E^*_{\frac{m-3}{3}}\right) \bigcup \left(E^{(*1)}_4 \bigcup E^{(*1)}_5 \bigcup \cdots \bigcup E^{(*1)}_{\frac{m-3}{3}-1}\right), \end{aligned}$$

$$\begin{aligned} E^{(2)} &= \Big(E_1 \bigcup E_2 \bigcup \cdots \bigcup E_{\frac{n-3}{3}}\Big) \bigcup \Big(E_3^1 \bigcup E_4^1 \bigcup \cdots \bigcup E^1_{\frac{n-3}{3}}\Big) \\ &\quad \bigcup \Big(E_1^* \bigcup E_2^* \bigcup \cdots \bigcup E^*_{\frac{m-4}{3}}\Big) \bigcup \Big(E_4^{(*1)} \bigcup E_5^{(*1)} \bigcup \cdots \bigcup E^{(*1)}_{\frac{m-4}{3}-1}\Big) \end{aligned}$$

and

$$\begin{aligned} E^{(3)} &= \Big(E_1 \bigcup E_2 \bigcup \cdots \bigcup E_{\frac{n-3}{3}}\Big) \bigcup \Big(E_3^1 \bigcup E_4^1 \bigcup \cdots \bigcup E^1_{\frac{n-3}{3}}\Big) \\ &\quad \bigcup \Big(E_1^* \bigcup E_2^* \bigcup \cdots \bigcup E^*_{\frac{m-5}{3}}\Big) \bigcup \Big(E_4^{(*1)} \bigcup E_5^{(*1)} \bigcup \cdots \bigcup E^{(*1)}_{\frac{m-5}{3}-1}\Big). \end{aligned}$$

If $m \equiv 0(mod3)$, let $E_1^{**} = \{f^*(v_{m-1}v_m)\}$, then

$$E_1^{**} = \{|f(v_{m-1} - f(v_m)|\} = \{|F_1 - F_2|\} = \{F_1\}.$$

Thus,

$$E = E_1^{**} \cup E^{(1)} = \{F_1, F_2, \ldots, F_{m+n}\}.$$

For example the super Fibonacci graceful labeling of $C_6 \oplus P_6$ is shown in Fig.4.

$C_6 \oplus P_6$:

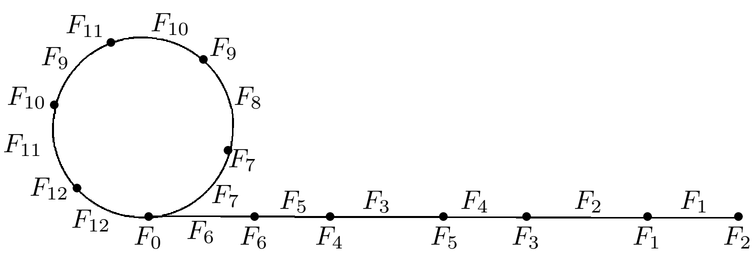

Fig.4

If $m \equiv 1(mod3)$, let $E_2^{**} = \{f^*(v_{m-2}v_{m-1}), f^*(v_{m-1}v_m)\}$, then

$$\begin{aligned} E_2^{**} &= \{|f(v_{m-2} - f(v_{m-1}|, |f(v_{m-1} - f(v_m)|\} \\ &= \{|F_2 - F_3|\, |F_3 - F_1|\} = \{F_1, F_2\}. \end{aligned}$$

Thus,

$$E = E_2^{**} \cup E^{(2)} = \{F_1, F_2, \ldots, F_{m+n}\}.$$

For example the super Fibonacci graceful labeling of $C_6 \oplus P_7$ is shown in Fig.5.

$C_6 \oplus P_7$:

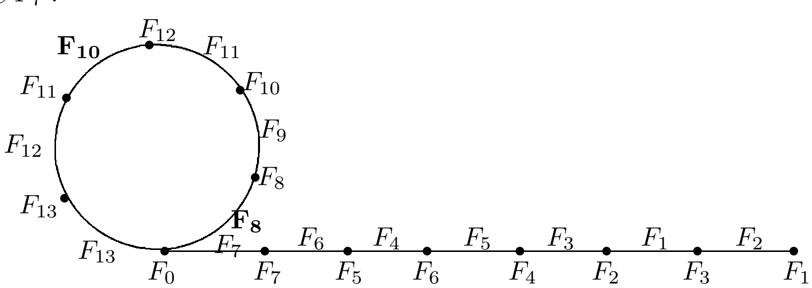

Fig.5

If $m \equiv 2(mod3)$, let $E_3^{**} = \{f^*(v_{m-3}v_{m-2}), f^*(v_{m-2}v_{m-1}), f^*(v_{m-1}v_m)\}$, then

$$\begin{aligned} E_3^{**} &= \{|f(v_{m-3}) - f(v_{m-2})|, |f(v_{m-2}) - f(v_{m-1})|, |f(v_{m-1} - f(v_m)|\} \\ &= \{|F_3 - F_4|, |F_4 - F_2|, |F_2 - F_1|\} = \{F_2, F_3, F_1\}. \end{aligned}$$

Thus,

$$E = E_3^{**} \cup E^{(3)} = \{F_1, F_2, \ldots, F_{m+n}\}.$$

For example the super Fibonacci graceful labeling of $C_6 \oplus P_5$ is shown in Fig.6.

$C_6 \oplus P_5$:

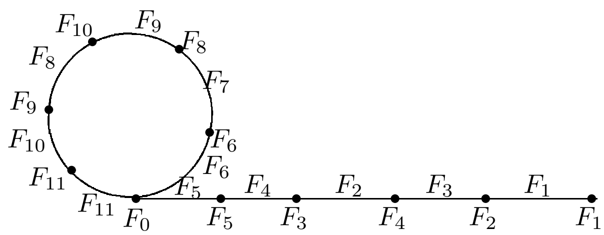

Fig.6

Therefore, $C_n \oplus P_m$ admits a super Fibonacci graceful labeling. Hence, $C_n \oplus P_m$ is a super Fibonacci graceful graph. □

Definition 2.6 *The graph $G = F_n \oplus K_{1,m}^+$ consists of a fan F_n and the extension graph of $K_{1,m}^+$ which is attached with the maximum degree of the vertex of F_n.*

Theorem 2.7 *The graph $G = F_n \oplus K_{1,m}^+$ is a super Fibonacci graceful graph.*

Proof Let $V(G) = U \cup V$, where $U = \{u_0, u_1, \ldots, u_n\}$ be the vertex set of F_n and $V = (V_1, V_2)$ be the bipartion of $K_{1,m}$, where $V_1 = \{v = u_0\}$ and $V_2 = \{v_1, v_2, \ldots, v_m\}$ and $w_1, w_2, \ldots, w_m$ be the pendant vertices joined with $v_1, v_2, \ldots, v_m$ respectively. Also, $|V(G)| = 2m + n + 1$ and $|E(G)| = 2m + 2n - 1$.

Case 1 m, n is even.

Define $f : V(G) \to \{F_0, F_1, \ldots, F_q\}$ by $f(u_0) = F_0$, $f(u_i) = F_{2m+2n-1-2(i-1)}$ if $1 \le i \le n$; $f(v_{2i-1}) = F_{2m-4(i-1)}$ if $1 \le i \le \dfrac{m}{2}$; $f(v_{2i}) = F_{2m-3-4(i-1)}$ if $1 \le i \le \dfrac{m}{2}$; $f(w_{2i-1}) = F_{2m-2-4(i-1)}$ if $1 \le i \le \dfrac{m}{2}$ and $f(w_{2i}) = F_{2m-1-4(i-1)}$ if $1 \le i \le \dfrac{m}{2}$.

We claim that all these edge labels are distinct. Calculation shows that

$$\begin{aligned} E_1 &= \{f^*(u_iu_{i+1}) : i = 1, 2, \ldots, n-1\} \\ &= \{|f(u_i) - f(u_{i+1})| : i = 1, 2, \ldots, n-1\} \\ &= \{|f(u_1) - f(u_2)|, |f(u_2) - f(u_3)|, \ldots, |f(u_{n-1}) - f(u_n)|\} \\ &= \{|F_{2n+2m-1} - F_{2n+2m-3}|, |F_{2n+2m-3} - F_{2n+2m-5}|, \ldots, |F_{2m+3} - F_{2m+1}|\} \\ &= \{F_{2n+2m-2}, F_{2n+2m-4}, \ldots, F_{2m+4}, F_{2m+2}\}, \end{aligned}$$

$$\begin{aligned} E_2 &= \{f^*(u_0u_i) : i = 1, 2, \ldots, n\} \\ &= \{|f(u_0) - f(u_i)| : i = 1, 2, \ldots, n\} \\ &= \{|f(u_0) - f(u_1)|, |f(u_0) - f(u_2)|, \ldots, |f(u_0) - f(u_{n-1})|, |f(u_0) - f(u_n)|\} \\ &= \{|F_0 - F_{2n+2m-1}|, |F_0 - F_{2n+2m-3}|, \ldots, |F_0 - F_{2m+3}|, |F_0 - F_{2m+1}|\} \\ &= \{F_{2n+2m-1}, F_{2n+2m-3}, \ldots, F_{2m+3}, F_{2m+1}\}, \end{aligned}$$

$$\begin{aligned} E_3 &= \{f^*(u_0v_{2i-1}) : 1 \le i \le \frac{m}{2}\} \\ &= \{|f(u_0) - f(v_{2i-1})| : 1 \le i \le \frac{m}{2}\} \\ &= \{|f(u_0) - f(v_1)|, |f(u_0) - f(v_3)|, \ldots, |f(u_o) - f(v_{m-3})|, |f(u_0) - f(v_{m-1})|\} \\ &= \{|F_0 - F_{2m}|, |F_0 - F_{2m-4}|, \ldots, |F_0 - F_8|, |F_0 - F_4|\} \\ &= \{F_{2m}, F_{2m-4}, \ldots, F_8, F_4\}, \end{aligned}$$

$$\begin{aligned} E_4 &= \{f^*(u_0v_{2i}) : 1 \le i \le \frac{m}{2}\} \\ &= \{|f(u_0) - f(v_{2i})| : 1 \le i \le \frac{m}{2}\} \\ &= \{|f(u_0) - f(v_2)|, |f(u_0) - f(v_4)|, \ldots, |f(u_0) - f(v_{m-2})|, |f(u_0) - f(v_m)|\} \\ &= \{|F_0 - F_{2m-3}|, |F_0 - F_{2m-7}|, \ldots, |F_0 - F_5|, |F_0 - F_1|\} \\ &= \{F_{2m-3}, F_{2m-7}, \ldots, F_5, F_1\}, \end{aligned}$$

$$\begin{aligned} E_5 &= \{f^*(v_{2i-1}w_{2i-1}) : 1 \le i \le \frac{m}{2}\} \\ &= \{|f(v_{2i-1}) - f(w_{2i-1})| : 1 \le i \le \frac{m}{2}\} \\ &= \{|f(v_1) - f(w_1)|, |f(v_3) - f(w_3)|, \ldots, |f(v_{m-3}) - f(w_{m-3})|, |f(v_{m-1}) - f(w_{m-1})|\} \\ &= \{|F_{2m} - F_{2m-2}|, |F_{2m-4} - F_{2m-6}|, \ldots, |F_8 - F_6|, |F_4 - F_2|\} \\ &= \{F_{2m-1}, F_{2m-5}. \ldots, F_7, F_3\}, \end{aligned}$$

$$\begin{aligned} E_6 &= \{f^*(v_{2i}w_{2i}) : 1 \le i \le \frac{m}{2}\} \\ &= \{|f(v_{2i}) - f(w_{2i})| : 1 \le i \le \frac{m}{2}\} \\ &= \{|f(v_2) - f(w_2)|, |f(v_4) - f(w_4)|, \ldots, |f(v_{m-2}) - f(w_{m-2})|, |f(v_m) - f(w_m)| \\ &= \{|F_{2m-3} - F_{2m-1}|, |F_{2m-7} - F_{2m-5}|, \ldots, |F_5 - F_7|, |F_1 - F_3|\} \\ &= \{F_{2m-2}, F_{2m-6}. \ldots, F_6, F_2\}. \end{aligned}$$

Therefore,

$$E = E_1 \bigcup E_2 \bigcup \cdots \bigcup E_6 = \{F_1, F_2, \ldots, F_{2m+2n-1}\}.$$

Thus, the edge labels are distinct. Therefore, $F_n \oplus K_{1,m}^+$ admits super Fibonacci graceful labeling. Hence, $F_n \oplus K_{1,m}^+$ is a super Fibonacci graceful graph.

For example the super Fibonacci graceful labeling of $F_4 \oplus K_{1,4}^+$ is shown in Fig.7.

$F_4 \oplus K_{1,4}^{+}$:

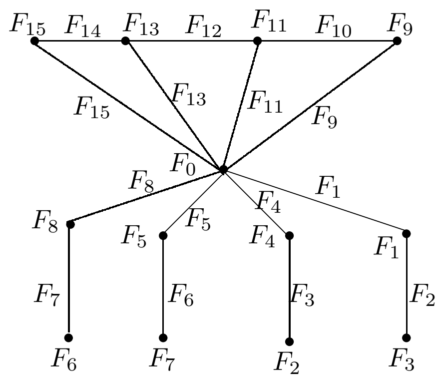

Fig.7

Case 2 m even, n odd.

Proof of this case is analogous to case(i).

For example the super Fibonacci graceful labeling of $F_5 \oplus K_{1,4}^{+}$ is shown in Fig.8.

$F_5 \oplus K_{1,4}^{+}$:

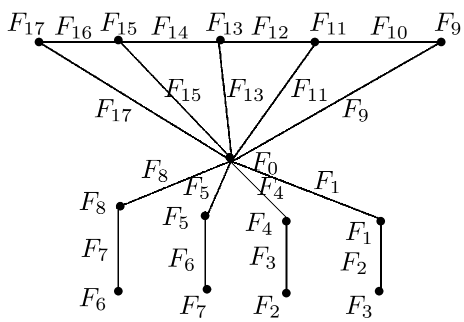

Fig.8

Case 3 m, n is odd.

Define $f : V(G) \to \{F_0, F_1, \ldots, F_q\}$ by $f(u_0) = F_0$; $f(u_i) = F_{2m+2n-1-2(i-1)}$ if $1 \leq i \leq n$; $f(w_m) = F_1$; $f(v_{2i-1}) = F_{2m-4(i-1)}$ if $1 \leq i \leq \dfrac{m+1}{2}$; $f(v_{2i}) = F_{2m-3-4(i-1)}$ if $1 \leq i \leq \dfrac{m-1}{2}$; $f(w_{2i-1}) = F_{2m-2-4(i-1)}$ if $1 \leq i \leq \dfrac{m-1}{2}$ and $f(w_{2i}) = F_{2m-1-4(i-1)}$ if $1 \leq i \leq \dfrac{m-1}{2}$.

We claim that the edge labels are distinct. Calculation shows that

$$\begin{aligned} E_1 &= \{f^*(u_i u_{i+1}) : i = 1, 2, \ldots, n-1\} \\ &= \{|f(u_i) - f(u_{i+1})| : i = 1, 2, \ldots, n-1\} \\ &= \{|f(u_1) - f(u_2)|, |f(u_2) - f(u_3)|, \ldots, |f(u_{n-2}) - f(u_{n-1})|, |f(u_{n-1}) - f(u_n)|\} \\ &= \{|F_{2n+2m-1} - F_{2n+2m-3}|, |F_{2n+2m-3} - F_{2n+2m-5}|, \ldots, |F_{2m+5} - F_{2m+3}|, \\ &\quad |F_{2m+3} - F_{2m+1}|\} = \{F_{2n+2m-2}, F_{2n+2m-4}, \ldots, F_{2m+4}, F_{2m+2}\}, \end{aligned}$$

$$\begin{aligned}
E_2 &= \{f^*(u_0u_i) : i = 1, 2, \ldots, n\}\\
&= \{|f(u_0) - f(u_i)| : i = 1, 2, \ldots, n\}\\
&= \{|f(u_0) - f(u_1)|, |f(u_0) - f(u_2)|, \ldots, |f(u_0) - f(u_{n-1})|, |f(u_0) - f(u_n)|\}\\
&= \{|F_0 - F_{2n+2m-1}|, |F_0 - F_{2n+2m-3}|, \ldots, |F_0 - F_{2m+3}|, |F_0 - F_{2m+1}|\}\\
&= \{F_{2n+2m-1}, F_{2n+2m-3}, \ldots, F_{2m+3}, F_{2m+1}\},
\end{aligned}$$

$$\begin{aligned}
E_3 &= \{f^*(u_0v_{2i-1}) : 1 \le i \le \frac{m+1}{2}\}\\
&= \{|f(u_0) - f(v_{2i-1})| : 1 \le i \le \frac{m+1}{2}\}\\
&= \{|f(u_0) - f(v_1)|, |f(u_0) - f(v_3)|, \ldots, |f(u_o) - f(v_{m-2})|, |f(u_0) - f(v_m)|\}\\
&= \{|F_0 - F_{2m}|, |F_0 - F_{2m-4}|, \ldots, |F_0 - F_6|, |F_0 - F_2|\}\\
&= \{F_{2m}, F_{2m-4}, \ldots, F_6, F_2\},
\end{aligned}$$

$$\begin{aligned}
E_4 &= \{f^*(u_0v_{2i}) : 1 \le i \le \frac{m-1}{2}\}\\
&= \{|f(u_0) - f(v_{2i})| : 1 \le i \le \frac{m-1}{2}\}\\
&= \{|f(u_0) - f(v_2)|, |f(u_0) - f(v_4)|, \ldots, |f(u_o) - f(v_{m-3})|, |f(u_0) - f(v_{m-1})|\}\\
&= \{|F_0 - F_{2m-3}|, |F_0 - F_{2m-7}|, \ldots, |F_0 - F_7|, |F_0 - F_3|\}\\
&= \{F_{2m-3}, F_{2m-7}. \ldots, F_7, F_3\},
\end{aligned}$$

$$E_5 = \{f^*(v_mw_m)\} = \{|f(v_m) - f(w_m)|\} = \{|F_2 - F_1|\} = \{F_1\},$$

$$\begin{aligned}
E_6 &= \{f^*(v_{2i-1}w_{2i-1}) : 1 \le i \le \frac{m-1}{2}\}\\
&= \{|f(v_{2i-1}) - f(w_{2i-1})| : 1 \le i \le \frac{m-1}{2}\}\\
&= \{|f(v_1) - f(w_1)|, |f(v_3) - f(w_3)|, \ldots, |f(v_{m-4}) - f(w_{m-4})|, |f(v_{m-2}) - f(w_{m-2})|\}\\
&= \{|F_{2m} - F_{2m-2}|, |F_{2m-4} - F_{2m-6}|, \ldots, |F_6 - F_8|, |F_6 - F_4|\}\\
&= \{F_{2m-1}, F_{2m-5}. \ldots, F_9, F_5\},
\end{aligned}$$

$$\begin{aligned}
E_7 &= \{f^*(v_{2i}w_{2i}) : 1 \le i \le \frac{m-1}{2}\}\\
&= \{|f(v_{2i}) - f(w_{2i})| : 1 \le i \le \frac{m-1}{2}\}\\
&= \{|f(v_2) - f(w_2)|, |f(v_4) - f(w_4)|, \ldots, |f(v_{m-3}) - f(w_{m-3})|, |f(v_{m-1}) - f(w_{m-1})|\}\\
&= \{|F_{2m-3} - F_{2m-1}|, |F_{2m-7} - F_{2m-5}|, \ldots, |F_7 - F_9|, |F_3 - F_5|\}\\
&= \{F_{2m-2}, F_{2m-6}. \ldots, F_8, F_4\}.
\end{aligned}$$

Therefore,

$$E = E_1 \bigcup E_2 \bigcup \cdots \bigcup E_7 = \{F_1, F_2, \ldots, F_{2m+2n-1}\}.$$

Thus, the edge labels are distinct.Therefore, $F_n \oplus K_{1,m}^+$ admits super Fibonacci graceful labeling. Whence, $F_n \oplus K_{1,m}^+$ is a super Fibonacci graceful graph.

For example the super Fibonacci graceful labeling of $F_5 \oplus K_{1,3}^+$ is shown in Fig.9.

$F_5 \oplus K_{1,3}^+$:

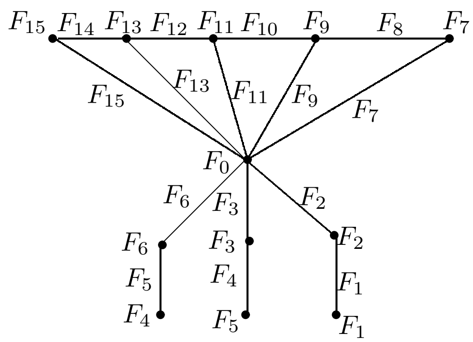

Fig.9

Case 4 m odd, n even.

Proof of this case is analogous to Case 4.

For example the super Fibonacci graceful labeling of $F_4 \oplus K_{1,3}^+$ is shown in Fig.10.

$F_4 \oplus K_{1,3}^+$:

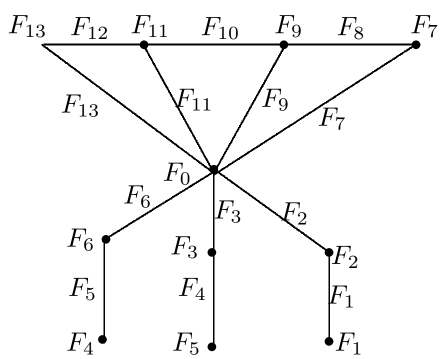

Fig.10

□

Definition 2.8 *The graph $G = C_n \oplus K_{1,m}$ consists of a cycle C_n of length n and a star $K_{1,m}$ is attached with the vertex u_n of C_n.*

Theorem 2.9 *The graph $G = C_n \oplus K_{1,m}$ is a super Fibonacci graceful graph when $n \equiv 0(mod 3)$.*

Proof Let $V(G) = V_1 \cup V_2$, where $V_1 = \{u_1, u_2, \ldots, u_n\}$ be the vertex set of C_n and $V_2 = \{v = u_n, v_1, v_2, \ldots, v_m\}$ be the vertex set of $K_{1,m}$. Also, $|V(G)| = |E(G)| = m + n$. Define $f : V(G) \to \{F_0, F_1, F_2, \ldots, F_q\}$ by $f(u_n) = F_0$; $f(u_i) = F_{m+n-2(i-1)}$ if $1 \le i \le 2$; $f(v_i) = F_i$ if $1 \le i \le m$ and for $l = 1, 2, \ldots, \dfrac{n-3}{3}$, $f(u_{i+2}) = F_{m+n-1-2(i-1)+3(l-1)}$ if $3l - 2 \le i \le 3l$.

We claim that the edge labels are distinct. Calculation shows that

$$\begin{aligned} E_1 &= \{f^*(u_n v_i) : 1 \le i \le m\} \\ &= \{|f(u_n) - f(v_i)| : 1 \le i \le m\} \\ &= \{|f(u_n) - f(v_1)|, |f(u_n) - f(v_2)|, \ldots, |f(u_n) - f(v_{m-1})|, |f(u_n) - f(v_m)|\} \\ &= \{|F_0 - F_1|, |F_0 - F_2|, \ldots, |F_0 - F_{m-1}|, |F_0 - F_m|\} \\ &= \{F_1, F_2, \ldots, F_{m-1}, F_m\}, \end{aligned}$$

$$\begin{aligned} E_2 &= \{f^*(u_n u_1), f^*(u_1 u_2)\} = \{|f(u_n) - f(u_1)|, |f(u_1) - f(u_2)|\} \\ &= \{|F_0 - F_{m+n}|, |F_{m+n} - F_{m+n-2}|\} = \{F_{m+n}, F_{m+n-1}\}. \end{aligned}$$

For the edge labeling between the vertex u_2 and starting vertex u_3 of the first loop, let $E_3 = \{f^*(u_2u_3)\}$. Then

$$E_3 = \{|f(u_2) - f(u_3)|\} = \{|F_{m+n-2} - F_{m+n-1}\} = \{F_{m+n-3}\}.$$

For $l = 1$, let $E_4 = \{f^*(u_{i+2}u_{i+3}) : 1 \le i \le 2\}$. Then

$$\begin{aligned} E_4 &= \{|f(u_{i+2}) - f(u_{i+3})| : 1 \le i \le 2\} \\ &= \{|f(u_3) - f(u_4)|, |f(u_4) - f(u_5)|\} \\ &= \{|F_{m+n-1} - F_{m+n-3}|, |F_{m+n-3} - F_{m+n-5})|\} \\ &= \{F_{m+n-2}, F_{m+n-4}\}. \end{aligned}$$

Let $E_4^{(1)} = \{f^*(u_5u_6)\}$. Then

$$E_4^{(1)} = \{|f(u_5) - f(u_6)|\} = \{|F_{m+n-5} - F_{m+n-4}|\} = \{F_{m+n-6}\}.$$

For $l = 2$, let $E_5 = \{f^*(u_{i+2}u_{i+3}) : 4 \le i \le 5\}$. Then

$$\begin{aligned} E_5 &= \{|f(u_{i+2}) - f(u_{i+3})| : 4 \le i \le 5\} \\ &= \{|f(u_6) - f(u_7)|, |f(u_7) - f(u_8)|\} \\ &= \{|F_{m+n-4} - F_{m+n-6}|, |F_{m+n-6} - F_{m+n-8})|\} \\ &= \{F_{m+n-5}, F_{m+n-7}\}. \end{aligned}$$

For finding the edge labeling between the end vertex u_8 of the second loop and starting vertex u_9 of the third loop, let $E_5^{(1)} = \{f^*(u_8u_9)\}$. Then

$$E_5^{(1)} = \{|f(u_8) - f(u_9)|\} = \{|F_{m+n-8} - F_{m+n-7}|\} = \{F_{m+n-9}\}$$

etc.. Similarly, for $l = \dfrac{n-3}{3} - 1$, let $E_{\frac{n-3}{3}-1} = \{f^*(u_{i+2}u_{i+3}) : n-8 \le i \le n-7\}$. Then

$$\begin{aligned} E_{\frac{n-3}{3}-1} &= \{|f(u_{i+2}) - f(u_{i+3})| : n-8 \le i \le n-7\} \\ &= \{|f(u_{n-6}) - f(u_{n-5})|, |f(u_{n-5}) - f(u_{n-4})|\} \\ &= \{|F_{m+8} - F_{m+6}|, |F_{m+6} - F_{m+4})|\} = \{F_{m+7}, F_{m+5}\}. \end{aligned}$$

For finding the edge labeling between the end vertex u_{n-4} of the $(\frac{n-3}{3}-1)^{th}$ loop and starting vertex u_{n-3} of the $(\frac{n-3}{3})^{rd}$ loop, let $E^{(1)}_{\frac{n-3}{3}-1} = \{f^*(u_{n-4}u_{n-3})\}$. Then

$$E^{(1)}_{\frac{n-3}{3}-1} = \{|f(u_{n-4}) - f(u_{n-3})|\} = \{|F_{m+4} - F_{m+5}|\} = \{F_{m+3}\}.$$

For $l = \frac{n-3}{3}$, let $E_{\frac{n-3}{3}} = \{f^*(u_{i+2}u_{i+3}) : n-5 \leq i \leq n-4\}$. Then

$$\begin{aligned} E_{\frac{n-3}{3}} &= \{|f(u_{i+2}) - f(u_{i+3})| : n-5 \leq i \leq n-4\} \\ &= \{|f(u_{n-3}) - f(u_{n-2})|, |f(u_{n-2}) - f(u_{n-1})|\} \\ &= \{|F_{m+5} - F_{m+3}|, |F_{m+3} - F_{m+1})|\} = \{F_{m+4}, F_{m+2}\}. \end{aligned}$$

Let $E^*_1 = \{f^*(u_{n-1}u_n)\}$. Then

$$E^*_1 = \{|f(u_{n-1}) - f(u_n)|\} = \{|F_{m+1} - F_0|\} = \{F_{m+1}\}.$$

Therefore,

$$\begin{aligned} E &= \left(E_1 \bigcup E_2 \bigcup \cdots \bigcup E_{\frac{n-3}{3}}\right) \bigcup \left(E^{(1)}_4 \bigcup E^{(1)}_5 \bigcup \cdots \bigcup E^{(1)}_{\frac{n-3}{3}-1}\right) \bigcup E^*_1 \\ &= \{F_1, F_2, \ldots, F_{m+n}\}. \end{aligned}$$

Thus, all edge labels are distinct. Therefore, the graph $G = C_n \oplus K_{1,m}$ admits super Fibonacci graceful labeling. Whence, it is a super Fibonacci graceful graph. □

Example 2.10 This example shows that the graph $C_6 \oplus K_{1,4}$ is a super Fibonacci graceful graph.

$C_6 \oplus K_{1,4}$:

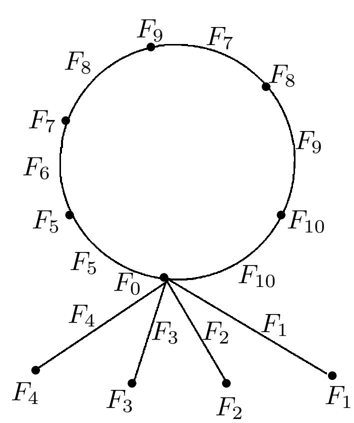

Fig.11

Definition 2.11 *$G = K_{1,n} \oslash K_{1,2}$ is a graph in which $K_{1,2}$ is joined with each pendant vertex of $K_{1,n}$.*

Theorem 2.12 *The graph $G = K_{1,n} \oslash K_{1,2}$ is a super Fibonacci graceful graph.*

Proof Let $\{u_0, u_1, u_2, \ldots, u_n\}$ be the vertex set of $K_{1,n}$ and $v_1, v_2, \ldots, v_n$ and $w_1, w_2, \ldots, w_n$ be the vertices joined with the pendant vertices $u_1, u_2, \ldots, u_n$ of $K_{1,n}$ respectively. Also, $|V(G)| = 3n + 1$ and $|E(G)| = 3n$. Define $f : V(G) \to \{F_0, F_1, F_2, \ldots, F_q\}$ by $f(u_0) = F_0$, $f(u_i) = F_{3n-3(i-1)}$, $1 \le i \le n$, $f(v_i) = F_{3n-1-3(i-1)}$, $1 \le i \le n$, $f(w_i) = F_{3n-2-3(i-1)}$, $1 \le i \le n$.

We claim that the edge labels are distinct. Calculation shows that

$$\begin{aligned} E_1 &= \{f^*(u_0u_i) : i = 1, 2, \ldots, n\} \\ &= \{|f(u_0) - f(u_i)| : i = 1, 2, \ldots n\} \\ &= \{|f(u_0) - f(u_1)|, |f(u_0) - f(u_2)|, \ldots, |f(u_0) - f(u_{n-1})|, |f(u_0) - f(u_n)|\} \\ &= \{|F_0 - F_{3n}|, |F_0 - F_{3n-3}|, \ldots, |F_0 - F_6|, |F_0 - F_3|\} \\ &= \{F_{3n}, F_{3n-3}, \ldots, F_6, F_3\}, \end{aligned}$$

$$\begin{aligned} E_2 &= \{f^*(u_iv_i) : i = 1, 2, \ldots, n\} \\ &= \{|f(u_i) - f(v_i)| : i = 1, 2, \ldots, n\} \\ &= \{|f(u_1) - f(v_1)|, |f(u_2) - f(v_2)|, \ldots, |f(u_{n-1}) - f(v_{n-1})|, = |f(u_n) - f(v_n)|\} \\ &= \{|F_{3n} - F_{3n-1}|, |F_{3n-3} - F_{3n-4}|, \ldots, |F_6 - F_5|, |F_3 - F_2|\} \\ &= \{F_{3n-2}, F_{3n-5}, \ldots, F_4, F_1\}, \end{aligned}$$

$$\begin{aligned} E_3 &= \{f^*(u_iw_i) : i = 1, 2, \ldots, n\} \\ &= \{|f(u_i) - f(w_i)| : i = 1, 2, \ldots, n\} \\ &= 1\{|f(u_1) - f(w_1)|, |f(u_2) - f(w_2)|, \ldots, |f(u_{n-1}) - f(w_{n-1})|, |f(u_n) - f(w_n)|\} \\ &= \{|F_{3n} - F_{3n-2}|, |F_{3n-3} - F_{3n-5}|, \ldots, |F_6 - F_4|, |F_3 - F_1|\} \\ &= \{F_{3n-1}, F_{3n-4}, \ldots, F_5, F_2\}. \end{aligned}$$

Therefore,

$$E = E_1 \bigcup E_2 \bigcup E_3 = \{F_1, F_2, \ldots, F_{3n}\}.$$

Thus, all edge labels are distinct. Therefore, $K_{1,n} \oslash K_{1,2}$ admits super Fibonacci graceful labeling. Whence, it is a super Fibonacci graceful graph. □

Example 2.13 This example shows that the graph $K_{1,3} \oslash K_{1,2}$ is a super Fibonacci graceful graph.

$K_{1,3} \oslash K_{1,2}$:

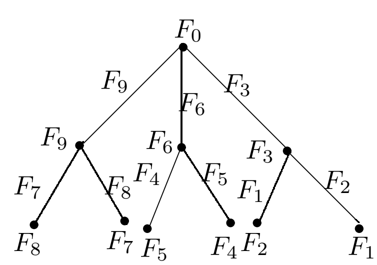

Fig.12

References

[1] G.J.Gallian, A Dynamic survey of graph labeling, *The electronic Journal of Combinotorics*, 16(2009), #DS6, PP 219.

[2] Henry Ibstedt, *Surfing on the Ocean of Numbers a Few Smarandache Notions and Similar Topics*, Ethus University Press, Vail 1997.

[3] Ian Anderson, *A First Course in Combinatorial Mathematics*, Claridon Press-Oxford, 28(1989) 3-13.

[4] K.M.Kathiresan and S.Amutha, *The Existence and Construction of Certain Types of Labelings for Graphs*, PhD. thesis, Madurai Kamaraj University, October 2006.

[5] A.Rosa, On certain valuations of the vertices of a graph, *Theory of Graphs* (International Symposium, Rome), July (1966).