International J.Math. Combin. Vol.3(2012), 58-63

Switching Equivalence in Symmetric n-Sigraphs-V

P.Siva Kota Reddy, M.C.Geetha and K.R.Rajanna

(Department of Mathematics, Acharya Institute of Technology, Soldevanahalli, Bangalore-560 090, India)

E-mail: pskreddy@acharya.ac.in

Abstract: An *n-tuple* $(a_1, a_2, ..., a_n)$ is *symmetric*, if $a_k = a_{n-k+1}, 1 \leq k \leq n$. Let $H_n = \{(a_1, a_2, \cdots, a_n) : a_k \in \{+, -\}, a_k = a_{n-k+1}, 1 \leq k \leq n\}$ be the set of all symmetric n-tuples. A *symmetric n-sigraph (symmetric n-marked graph)* is an ordered pair $S_n = (G, \sigma)$ ($S_n = (G, \mu)$), where $G = (V, E)$ is a graph called the *underlying graph* of S_n and $\sigma : E \to H_n$ ($\mu : V \to H_n$) is a function. In this paper, we introduced a new notion $\mathcal{S}$-antipodal symmetric n-sigraph of a symmetric n-sigraph and its properties are obtained. Also we give the relation between antipodal symmetric n-sigraphs and $\mathcal{S}$-antipodal symmetric n-sigraphs. Further, we discuss structural characterization of $\mathcal{S}$-antipodal symmetric n-sigraphs.

Key Words: Symmetric n-sigraphs, Smarandachely symmetric n-marked graph, symmetric n-marked graphs, balance, switching, antipodal symmetric n-sigraphs, $\mathcal{S}$-antipodal symmetric n-sigraphs, complementation.

AMS(2010): 05C22

§1. Introduction

Unless mentioned or defined otherwise, for all terminology and notion in graph theory the reader is refer to [1]. We consider only finite, simple graphs free from self-loops.

Let $n \geq 1$ be an integer. An *n-tuple* $(a_1, a_2, \cdots, a_n)$ is *symmetric*, if $a_k = a_{n-k+1}, 1 \leq k \leq n$. Let $H_n = \{(a_1, a_2, \cdots, a_n) : a_k \in \{+, -\}, a_k = a_{n-k+1}, 1 \leq k \leq n\}$ be the set of all symmetric n-tuples. Note that H_n is a group under coordinate wise multiplication, and the order of H_n is 2^m, where $m = \lceil \frac{n}{2} \rceil$.

A *Smarandachely k-marked graph* (*Smarandachely k-signed graph*) is an ordered pair $S = (G, \mu)$ ($S = (G, \sigma)$) where $G = (V, E)$ is a graph called *underlying graph* of S and $\mu : V \to \{\overline{e}_1, \overline{e}_2, ..., \overline{e}_k\}$ ($\sigma : E \to \{\overline{e}_1, \overline{e}_2, ..., \overline{e}_k\}$) is a function, where $\overline{e}_i \in \{+, -\}$. An *n-tuple* $(a_1, a_2, ..., a_n)$ is *symmetric*, if $a_k = a_{n-k+1}, 1 \leq k \leq n$. Let $H_n = \{(a_1, a_2, ..., a_n) : a_k \in \{+, -\}, a_k = a_{n-k+1}, 1 \leq k \leq n\}$ be the set of all symmetric n-tuples. A *Smarandachely symmetric n-marked graph* (*Smarandachely symmetric n-signed graph*) is an ordered pair $S_n = (G, \mu)$ ($S_n = (G, \sigma)$) where $G = (V, E)$ is a graph called the *underlying graph* of S_n and $\mu : V \to H_n$ ($\sigma : E \to H_n$) is a function. Particularly, a Smarandachely 1-marked graph (Smarandachely 1-signed graph) is called a *marked graph* (*signed graph*).

[1]Received February 25, 2012. Accepted September 9, 2012.

In this paper by an *n-tuple/n-sigraph/n-marked graph* we always mean a symmetric n-tuple/symmetric n-sigraph/symmetric n-marked graph.

An n-tuple $(a_1, a_2, \cdots, a_n)$ is the *identity n-tuple*, if $a_k = +$, for $1 \leq k \leq n$, otherwise it is a *non-identity n-tuple*. In an n-sigraph $S_n = (G, \sigma)$ an edge labelled with the identity n-tuple is called an *identity edge*, otherwise it is a *non-identity edge*. Further, in an n-sigraph $S_n = (G, \sigma)$, for any $A \subseteq E(G)$ the *n-tuple* $\sigma(A)$ is the product of the n-tuples on the edges of A.

In [7], the authors defined two notions of balance in n-sigraph $S_n = (G, \sigma)$ as follows (See also R. Rangarajan and P.S.K.Reddy [4]):

Definition 1.1 *Let $S_n = (G, \sigma)$ be an n-sigraph. Then,*

(*i*) S_n *is* identity balanced *(or* i-balanced*), if product of n-tuples on each cycle of S_n is the identity n-tuple, and*

(*ii*) S_n *is* balanced*, if every cycle in S_n contains an even number of non-identity edges.*

Note 1.1 An i-balanced n-sigraph need not be balanced and conversely.

The following characterization of i-balanced n-sigraphs is obtained in [7].

Proposition 1.1 (E. Sampathkumar et al. [7]) *An n-sigraph $S_n = (G, \sigma)$ is i-balanced if, and only if, it is possible to assign n-tuples to its vertices such that the n-tuple of each edge uv is equal to the product of the n-tuples of u and v.*

Let $S_n = (G, \sigma)$ be an n-sigraph. Consider the n-marking μ on vertices of S_n defined as follows: each vertex $v \in V$, $\mu(v)$ is the n-tuple which is the product of the n-tuples on the edges incident with v. *Complement* of S_n is an n-sigraph $\overline{S_n} = (\overline{G}, \sigma^c)$, where for any edge $e = uv \in \overline{G}$, $\sigma^c(uv) = \mu(u)\mu(v)$. Clearly, $\overline{S_n}$ as defined here is an i-balanced n-sigraph due to Proposition 1.1 ([10]).

In [7], the authors also have defined switching and cycle isomorphism of an n-sigraph $S_n = (G, \sigma)$ as follows (See also [2,5,6,10]):

Let $S_n = (G, \sigma)$ and $S'_n = (G', \sigma')$, be two n-sigraphs. Then S_n and S'_n are said to be *isomorphic*, if there exists an isomorphism $\phi : G \to G'$ such that if uv is an edge in S_n with label $(a_1, a_2, \cdots, a_n)$ then $\phi(u)\phi(v)$ is an edge in S'_n with label $(a_1, a_2, \cdots, a_n)$.

Given an n-marking μ of an n-sigraph $S_n = (G, \sigma)$, *switching* S_n with respect to μ is the operation of changing the n-tuple of every edge uv of S_n by $\mu(u)\sigma(uv)\mu(v)$. The n-sigraph obtained in this way is denoted by $\mathcal{S}_\mu(S_n)$ and is called the *μ-switched n-sigraph* or just *switched n-sigraph*. Further, an n-sigraph S_n *switches* to n-sigraph S'_n (or that they are *switching equivalent* to each other), written as $S_n \sim S'_n$, whenever there exists an n-marking of S_n such that $\mathcal{S}_\mu(S_n) \cong S'_n$.

Two n-sigraphs $S_n = (G, \sigma)$ and $S'_n = (G', \sigma')$ are said to be *cycle isomorphic*, if there exists an isomorphism $\phi : G \to G'$ such that the n-tuple $\sigma(C)$ of every cycle C in S_n equals to the n-tuple $\sigma(\phi(C))$ in S'_n. We make use of the following known result (see [7]).

Proposition 1.2 (E. Sampathkumar et al. [7]) *Given a graph G, any two n-sigraphs with G* *as underlying graph are switching equivalent if, and only if, they are cycle isomorphic.*

Let $S_n = (G, \sigma)$ be an n-sigraph. Consider the n-marking μ on vertices of S defined as follows: each vertex $v \in V$, $\mu(v)$ is the product of the n-tuples on the edges incident at v. *Complement* of S is an n-sigraph $\overline{S_n} = (\overline{G}, \sigma')$, where for any edge $e = uv \in \overline{G}$, $\sigma'(uv) = \mu(u)\mu(v)$. Clearly, $\overline{S_n}$ as defined here is an i-balanced n-sigraph due to Proposition 1.1.

§2. $\mathcal{S}$-Antipodal n-Sigraphs

Radhakrishnan Nair and Vijayakumar [3] has introduced the concept of $\mathcal{S}$-antipodal graph of a graph G as the graph $A^*(G)$ has the vertices in G with maximum eccentricity and two vertices of $A^*(G)$ are adjacent if they are at a distance of $diam(G)$ in G.

Motivated by the existing definition of complement of an n-sigraph, we extend the notion of $\mathcal{S}$-antipodal graphs to n-sigraphs as follows:

The *$\mathcal{S}$-antipodal n-sigraph* $A^*(S_n)$ of an n-sigraph $S_n = (G, \sigma)$ is an n-sigraph whose underlying graph is $A^*(G)$ and the n-tuple of any edge uv is $A^*(S_n)$ is $\mu(u)\mu(v)$, where μ is the canonical n-marking of S_n. Further, an n-sigraph $S_n = (G, \sigma)$ is called $\mathcal{S}$-antipodal n-sigraph, if $S_n \cong A^*(S'_n)$ for some n-sigraph S'_n. The following result indicates the limitations of the notion $A^*(S_n)$ as introduced above, since the entire class of i-unbalanced n-sigraphs is forbidden to be $\mathcal{S}$-antipodal n-sigraphs.

Proposition 2.1 *For any n-sigraph $S_n = (G, \sigma)$, its $\mathcal{S}$-antipodal n-sigraph $A^*(S_n)$ is i-balanced.*

Proof Since the n-tuple of any edge uv in $A^*(S_n)$ is $\mu(u)\mu(v)$, where μ is the canonical n-marking of S_n, by Proposition 1.1, $A^*(S_n)$ is i-balanced. □

For any positive integer k, the k^{th} iterated $\mathcal{S}$-antipodal n-sigraph $A^*(S_n)$ of S_n is defined as follows:

$$(A^*)^0(S_n) = S_n,\ (A^*)^k(S_n) = A^*((A^*)^{k-1}(S_n))$$

Corollary 2.2 *For any n-sigraph $S_n = (G, \sigma)$ and any positive integer k, $(A^*)^k(S_n)$ is i-balanced.*

In [3], the authors characterized those graphs that are isomorphic to their $\mathcal{S}$-antipodal graphs.

Proposition 2.3(Radhakrishnan Nair and Vijayakumar [3]) *For a graph $G = (V, E)$, $G \cong A^*(G)$ if, and only if, G is a regular self-complementary graph.*

We now characterize the n-sigraphs that are switching equivalent to their $\mathcal{S}$-antipodal n-sigraphs.

Proposition 2.4 *For any n-sigraph $S_n = (G, \sigma)$, $S_n \sim A^*(S_n)$ if, and only if, G is regular* *self-complementary graph and S_n is i-balanced n-sigraph.*

Proof Suppose $S_n \sim A^*(S_n)$. This implies, $G \cong A^*(G)$ and hence G is is a regular self-complementary graph. Now, if S_n is any n-sigraph with underlying graph as regular self-complementary graph, Proposition 2.1 implies that $A^*(S_n)$ is i-balanced and hence if S is i-unbalanced and its $A^*(S_n)$ being i-balanced can not be switching equivalent to S_n in accordance with Proposition 1.2. Therefore, S_n must be i-balanced.

Conversely, suppose that S_n is an i-balanced n-sigraph and G is regular self-complementary. Then, since $A^*(S_n)$ is i-balanced as per Proposition 2.1 and since $G \cong A^*(G)$, the result follows from Proposition 1.2 again. □

Proposition 2.5 *For any two vs S_n and S'_n with the same underlying graph, their $\mathcal{S}$-antipodal n-sigraphs are switching equivalent.*

Remark 2.6 If G is regular self-complementary graph, then $G \cong \overline{G}$. The above result is holds good for $\overline{S_n} \sim A^*(S_n)$.

In [16], P.S.K.Reddy et al. introduced antipodal n-sigraph of an n-sigraph as follows:

The *antipodal n-sigraph* $A(S_n)$ of an n-sigraph $S_n = (G, \sigma)$ is an n-sigraph whose underlying graph is $A(G)$ and the n-tuple of any edge uv in $A(S_n)$ is $\mu(u)\mu(v)$, where μ is the canonical n-marking of S_n. Further, an n-sigraph $S_n = (G, \sigma)$ is called antipodal n-sigraph, if $S_n \cong A(S'_n)$ for some n-sigraph S'_n.

Proposition 2.7(P.S.K.Reddy et al. [16]) *For any n-sigraph $S_n = (G, \sigma)$, its antipodal n-sigraph $A(S_n)$ is i-balanced.*

We now characterize n-sigraphs whose $\mathcal{S}$-antipodal n-sigraphs and antipodal n-sigraphs are switching equivalent. In case of graphs the following result is due to Radhakrishnan Nair and Vijayakumar [3].

Proposition 2.8 *For a graph $G = (V, E)$, $A^*(G) \cong A(G)$ if, and only if, G is self-centred.*

Proposition 2.9 *For any n-sigraph $S_n = (G, \sigma)$, $A^*(S_n) \sim A(S_n)$ if, and only if, G is self-centred.*

Proof Suppose $A^*(S_n) \sim A(S_n)$. This implies, $A^*(G) \cong A(G)$ and hence by Proposition 2.8, we see that the graph G must be self-centred.

Conversely, suppose that G is self centred. Then $A^*(G) \cong A(G)$ by Proposition 2.8. Now, if S_n is an n-sigraph with underlying graph as self centred, by Propositions 2.1 and 2.7, $A^*(S_n)$ and $A(S_n)$ are i-balanced and hence, the result follows from Proposition 1.2.

In [3], the authors shown that $A^*(G) \cong A^*(\overline{G})$ if G is either complete or totally disconnected. We now characterize n-sigraphs whose $A^*(S_n)$ and $A^*(\overline{S_n})$ are switching equivalent.

Proposition 2.10 *For any signed graph $S = (G, \sigma)$, $A^*(S_n) \sim A^*(\overline{S_n})$ if, and only if, G is either complete or totally disconnected.*

The following result characterize n-sigraphs which are $\mathcal{S}$-antipodal n-sigraphs.

Proposition 2.11 *An n-sigraph $S_n = (G, \sigma)$ is a $\mathcal{S}$-antipodal n-sigraph if, and only if, S_n is i-balanced n-sigraph and its underlying graph G is a $\mathcal{S}$-antipodal graph.*

Proof Suppose that S_n is i-balanced and G is a $A(G)$. Then there exists a graph H such that $A^*(H) \cong G$. Since S_n is i-balanced, by Proposition 1.1, there exists an n-marking μ of G such that each edge uv in S_n satisfies $\sigma(uv) = \mu(u)\mu(v)$. Now consider the n-sigraph $S'_n = (H, \sigma')$, where for any edge e in H, $\sigma'(e)$ is the n-marking of the corresponding vertex in G. Then clearly, $A^*(S'_n) \cong S_n$. Hence S_n is a $\mathcal{S}$-antipodal n-sigraph.

Conversely, suppose that $S_n = (G, \sigma)$ is a $\mathcal{S}$-antipodal n-sigraph. Then there exists an n-sigraph $S'_n = (H, \sigma')$ such that $A^*(S'_n) \cong S_n$. Hence G is the $A^*(G)$ of H and by Proposition 2.1, S_n is i-balanced. □

§3. Complementation

In this section, we investigate the notion of complementation of a graph whose edges have signs (a *sigraph*) in the more general context of graphs with multiple signs on their edges. We look at two kinds of complementation: complementing some or all of the signs, and reversing the order of the signs on each edge.

For any $m \in H_n$, the *m-complement* of $a = (a_1, a_2, \cdots., a_n)$ is: $a^m = am$. For any $M \subseteq H_n$, and $m \in H_n$, the *m-complement* of M is $M^m = \{a^m : a \in M\}$. For any $m \in H_n$, the *m-complement* of an n-sigraph $S_n = (G, \sigma)$, written (S_n^m), is the same graph but with each edge label $a = (a_1, a_2, ..., a_n)$ replaced by a^m. For an n-sigraph $S_n = (G, \sigma)$, the $A^*(S_n)$ is i-balanced (Proposition 2.1). We now examine, the condition under which m-complement of $A(S_n)$ is i-balanced, where for any $m \in H_n$.

Proposition 3.1 *Let $S_n = (G, \sigma)$ be an n-sigraph. Then, for any $m \in H_n$, if $A^*(G)$ is bipartite then $(A^*(S_n))^m$ is i-balanced.*

Proof Since, by Proposition 2.1, $A^*(S_n)$ is i-balanced, for each k, $1 \leq k \leq n$, the number of n-tuples on any cycle C in $A^*(S_n)$ whose k^{th} co-ordinate are $-$ is even. Also, since $A^*(G)$ is bipartite, all cycles have even length; thus, for each k, $1 \leq k \leq n$, the number of n-tuples on any cycle C in $A^*(S_n)$ whose k^{th} co-ordinate are $+$ is also even. This implies that the same thing is true in any m-complement, where for any $m, \in H_n$. Hence $(A^*(S_n))^t$ is i-balanced. □

Problem 3.2 *Characterize these n-sigraphs for which*

(1) $(S_n)^m \sim A^*(S_n)$;
(2) $(\overline{S_n})^m \sim A(S_n)$;
(3) $(A^*(S_n))^m \sim A(S_n)$;
(4) $A^*(S_n) \sim (A(S_n))^m$;
(5) $(A^*(S))^m \sim A^*(\overline{S_n})$;
(6) $A^*(S_n) \sim (A^*(\overline{S_n}))^m$.

References

[1] F. Harary, *Graph Theory*, Addison-Wesley Publishing Co., 1969.

[2] V.Lokesha, P.S.K.Reddy and S. Vijay, The triangular line n-sigraph of a symmetric n-sigraph, *Advn. Stud. Contemp. Math.*, 19(1) (2009), 123-129.

[3] R.Radhakrishnan Nair and A.Vijaykumar, S-Antipodal graphs, *Indian J. Pure Appl. Math.*, 28(5) (1997), 641-645.

[4] R.Rangarajan and P.S.K.Reddy, Notions of balance in symmetric n-sigraphs, *Proceedings of the Jangjeon Math. Soc.*, 11(2) (2008), 145-151.

[5] R.Rangarajan, P.S.K.Reddy and M.S.Subramanya, Switching Equivalence in Symmetric n-Sigraphs, *Adv. Stud. Comtemp. Math.*, 18(1) (2009), 79-85. R.

[6] R.Rangarajan, P.S.K.Reddy and N.D.Soner, Switching equivalence in symmetric n-sigraphs-II, *J. Orissa Math. Sco.*, 28 (1 & 2) (2009), 1-12.

[7] E.Sampathkumar, P.S.K.Reddy, and M.S.Subramanya, Jump symmetric n-sigraph, *Proceedings of the Jangjeon Math. Soc.*, 11(1) (2008), 89-95.

[8] E.Sampathkumar, P.S.K.Reddy, and M.S.Subramanya, The Line n-sigraph of a symmetric n-sigraph, *Southeast Asian Bull. Math.*, 34(5) (2010), 953-958.

[9] R. Singleton, There is no irregular moore graph, *Amer. Math. Monthly*, 75(1968), 42-43.

[10] P.S.K.Reddy and B.Prashanth, Switching equivalence in symmetric n-sigraphs-I, *Advances and Applications in Discrete Mathematics*, 4(1) (2009), 25-32.

[11] P.S.K.Reddy, S.Vijay and B.Prashanth, The edge C_4 n-sigraph of a symmetric n-sigraph, *Int. Journal of Math. Sci. & Engg. Appls.*, 3(2) (2009), 21-27.

[12] P.S.K.Reddy, V.Lokesha and Gurunath Rao Vaidya, The Line n-sigraph of a symmetric n-sigraph-II, *Proceedings of the Jangjeon Math. Soc.*, 13(3) (2010), 305-312.

[13] P.S.K.Reddy, V.Lokesha and Gurunath Rao Vaidya, The Line n-sigraph of a symmetric n-sigraph-III, *Int. J. Open Problems in Computer Science and Mathematics*, 3(5) (2010), 172-178.

[14] P.S.K.Reddy, V.Lokesha and Gurunath Rao Vaidya, Switching equivalence in symmetric n-sigraphs-III, *Int. Journal of Math. Sci. & Engg. Appls.*, 5(1) (2011), 95-101.

[15] P.S.K.Reddy, B. Prashanth and Kavita. S. Permi, A Note on Switching in Symmetric n-Sigraphs, *Notes on Number Theory and Discrete Mathematics*, 17(3) (2011), 22-25.

[16] P.S.K.Reddy, M. C. Geetha and K. R. Rajanna, Switching Equivalence in Symmetric n-Sigraphs-IV, *Scientia Magna*, 7(3) (2011), 34-38.