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Total Dominator Colorings in Paths

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Abstract: Let G be a graph without isolated vertices. A total dominator coloring of a graph G is a proper coloring of the graph G with the extra property that every vertex in the graph G properly dominates a color class. The smallest number of colors for which there exists a total dominator coloring of G is called the total dominator chromatic number of G and is denoted by $\chi_{td}(G)$. In this paper we determine the total dominator chromatic number in paths. Unless otherwise specified, n denotes an integer greater than or equal to 2.

Key Words: Total domination number, chromatic number and total dominator chromatic number, Smarandachely k-domination coloring, Smarandachely k-dominator chromatic number.

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§1. Introduction

All graphs considered in this paper are finite, undirected graphs and we follow standard definitions of graph theory as found in [2].

Let $G = (V, E)$ be a graph of order n with minimum degree at least one. The open neighborhood $N(v)$ of a vertex $v \in V(G)$ consists of the set of all vertices adjacent to v. The closed neighborhood of v is $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighborhood $N(S)$ is defined to be $\cup_{v \in S} N(v)$, and the closed neighborhood of S is $N[S] = N(S) \cup S$. A subset S of V is called a dominating (total dominating) set if every vertex in $V - S$ (V) is adjacent to some vertex in S. A dominating (total dominating) set is minimal dominating (total dominating) set if no proper subset of S is a dominating (total dominating) set of G. The domination number γ (total domination number γ_t) is the minimum cardinality taken over all minimal dominating (total dominating) sets of G. A γ-set (γ_t-set) is any minimal dominating (total dominating) set with cardinality γ (γ_t).

A proper coloring of G is an assignment of colors to the vertices of G, such that adjacent vertices have different colors. The smallest number of colors for which there exists a proper coloring of G is called chromatic number of G and is denoted by $\chi(G)$. Let $V = \{u_1, u_2, u_3, \cdots, u_p\}$ and $\mathcal{C} = \{C_1, C_2, C_3, \cdots, C_n\}$ be a collection of subsets $C_i \subset V$. A color represented in a vertex u is called a non-repeated color if there exists one color class $C_i \in \mathcal{C}$ such that $C_i = \{u\}$.

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a proper coloring of the graph G with the extra property that every vertex in the graph G properly dominates a color class. The smallest number of colors for which there exists a total dominator coloring of G is called the *total dominator chromatic number* of G and is denoted by $\chi_{td}(G)$. Generally, for an integer $k \geq 1$, a *Smarandachely k-dominator coloring* of G is a proper coloring on G such that every vertex in the graph G properly dominates a k color classes and the smallest number of colors for which there exists a Smarandachely k-dominator coloring of G is called the *Smarandachely k-dominator chromatic number* of G, denoted by $\chi_{td}^S(G)$. Clearly, if $k = 1$, such a Smarandachely 1-dominator coloring and Smarandachely 1-dominator chromatic number are nothing but the total dominator coloring and total dominator chromatic number of G.

In this paper we determine total dominator chromatic number in paths.

Throughout this paper, we use the following notations.

Notation 1.1 Usually, the vertices of P_n are denoted by $u_1, u_2, \ldots, u_n$ in order. We also denote a vertex $u_i \in V(P_n)$ with $i > \lceil \frac{n}{2} \rceil$ by $u_{i-(n+1)}$. For example, u_{n-1} by u_{-2}. This helps us to visualize the position of the vertex more clearly.

Notation 1.2 For $i < j$, we use the notation $\langle [i, j] \rangle$ for the subpath induced by $\langle u_i, u_{i+1}, \ldots, u_j \rangle$. For a given coloring C of P_n, $C|\langle [i, j] \rangle$ refers to the coloring C restricted to $\langle [i, j] \rangle$.

We have the following theorem from [1].

Theorem 1.3 *For any graph G with $\delta(G) \geqslant 1$,* $\max\{\chi(G), \gamma_t(G)\} \leqslant \chi_{td}(G) \leqslant \chi(G) + \gamma_t(G)$.

Definition 1.4 *We know from Theorem* 1.3 *that* $\chi_{td}(P_n) \in \{\gamma_t(P_n), \gamma_t(P_n) + 1, \gamma_t(P_n) + 2\}$. *We call the integer n, good (respectively bad, very bad) if $\chi_{td}(P_n) = \gamma_t(P_n) + 2$ (if respectively $\chi_{td}(P_n) = \gamma_t(P_n) + 1, \chi_{td}(P_n) = \gamma_t(P_n)$).*

§2. Determination of $\chi_{td}(P_n)$

First, we note the values of $\chi_{td}(P_n)$ for small n. Some of these values are computed in Theorems 2.7, 2.8 and the remaining can be computed similarly.

n	$\gamma_t(P_n)$	$\chi_{td}(P_n)$
2	2	2
3	2	2
4	2	3
5	3	4
6	4	4

n	$\gamma_t(P_n)$	$\chi_{td}(P_n)$
7	4	5
8	4	6
9	5	6
10	6	7

Thus $n = 2, 3, 6$ are very bad integers and we shall show that these are the only bad integers. First, we prove a result which shows that for large values of n, the behavior of $\chi_{td}(P_n)$ depends only on the residue class of $n \bmod 4$ [More precisely, if n is good, $m > n$ and $m \equiv n (\bmod\ 4)$ then m is also good]. We then show that $n = 8, 13, 15, 22$ are the least good integers in their respective residue classes. This therefore classifies the good integers.

Fact 2.1 Let $1 < i < n$ and let C be a td-coloring of P_n. Then, if either u_i has a repeated color or u_{i+2} has a non-repeated color, $C|\langle[i + 1, n]\rangle$ is also a td-coloring. This fact is used extensively in this paper.

Lemma 2.2 $\chi_{td}(P_{n+4}) \geqslant \chi_{td}(P_n) + 2$.

Proof For $2 \leqslant n \leqslant 5$, this is directly verified from the table. We may assume $n \geqslant 6$. Let $u_1, u_2, u_3, \ldots, u_{n+4}$ be the vertices of P_{n+4} in order. Let C be a minimal td-coloring of P_{n+4}. Clearly, u_2 and u_{-2} are non-repeated colors. First suppose u_4 is a repeated color. Then $C|\langle[5, n + 4]\rangle$ is a td-coloring of P_n. Further, $C|\langle[1, 4]\rangle$ contains at least two color classes of C. Thus $\chi_{td}(P_n + 4) \geqslant \chi_{td}(P_n) + 2$. Similarly the result follows if u_{-4} is a repeated color Thus we may assume u_4 and u_{-4} are non-repeated colors. But the $C|\langle[3, n + 2]\rangle$ is a td-coloring and since u_2 and u_{-2} are non-repeated colors, we have in this case also $\chi_{td}(P_{n+4}) \geqslant \chi_{td}(P_n) + 2$.□

Corollary 2.3 *If for any n, $\chi_{td}(P_n) = \gamma_t(P_n) + 2, \chi_{td}(P_m) = \gamma_t(P_m) + 2$, for all $m > n$ with $m \equiv n (\bmod\ 4)$.*

Proof By Lemma 2.2, $\chi_{td}(P_{n+4}) \geqslant \chi_{td}(P_n) + 2 = \gamma_t(P_n) + 2 + 2 = \gamma_t(P_{n+4}) + 2$. □

Corollary 2.4 *No integer $n \geqslant 7$ is a very bad integer.*

Proof For $n = 7, 8, 9, 10$, this is verified from the table. The result then follows from the Lemma 2.2. □

Corollary 2.5 *The integers $2, 3, 6$ are the only very bad integers.*

Next, we show that $n = 8, 13, 15, 22$ are good integers. In fact, we determine $\chi_{td}(P_n)$ for small integers and also all possible minimum td-colorings for such paths. These ideas are used more strongly in determination of $\chi_{td}(P_n)$ for $n = 8, 13, 15, 22$.

Definition 2.6 *Two td-colorings C_1 and C_2 of a given graph G are said to be equivalent if there exists an automorphism $f : G \to G$ such that $C_2(v) = C_1(f(v))$ for all vertices v of G. This is clearly an equivalence relation on the set of td-colorings of G.*

Theorem 2.7 *Let $V(P_n) = \{u_1, u_2, \ldots, u_n\}$ as usual. Then*

(1) $\chi_{td}(P_2) = 2$. The only minimum td-coloring is (given by the color classes) $\{\{u_1\}, \{u_2\}\}$

(2) $\chi_{td}(P_3) = 2$. The only minimum td-coloring is $\{\{u_1, u_3\}, \{u_2\}\}$.

(3) $\chi_{td}(P_4) = 3$ with unique minimum coloring $\{\{u_1, u_4\}, \{u_2\}, \{u_3\}\}$.

(4) $\chi_{td}(P_5) = 4$. Any minimum coloring is equivalent to one of $\{\{u_1, u_3\}, \{u_2\}, \{u_4\}, \{u_5\}\}$ or $\{\{u_1, u_5\}, \{u_2\}, \{u_3\}, \{u_4\}\}$ or $\{\{u_1\}, \{u_2\}, \{u_4\}, \{u_3, u_5\}\}$.

(5) $\chi_{td}(P_6) = 4$ with unique minimum coloring $\{\{u_1, u_3\}, \{u_4, u_6\}, \{u_2\}, \{u_5\}\}$.

(6) $\chi_{td}(P_7) = 5$. Any minimum coloring is equivalent to one of $\{\{u_1, u_3\}, \{u_2\}, \{u_4, u_7\}, \{u_5\}, \{u_6\}\}$ or $\{\{u_1, u_4\}, \{u_2\}, \{u_3\}, \{u_5, u_7\}, \{u_6\}\}$ $\{\{u_1, u_4, u_7\}, \{u_2\}, \{u_3\}, \{u_5\}, \{u_6\}\}$.

Proof We prove only (vi). The rest are easy to prove. Now, $\gamma_t(P_7) = \lceil \frac{7}{2} \rceil = 4$. Clearly $\chi_{td}(P_7) \geqslant 4$. We first show that $\chi_{td}(P_7) \neq 4$ Let C be a td-coloring of P_7 with 4 colors. The vertices u_2 and $u_{-2} = u_6$ must have non-repeated colors. Suppose now that u_3 has a repeated color. Then $\{u_1, u_2, u_3\}$ must contain two color classes and $C| \langle [4, 7] \rangle$ must be a td-coloring which will require at least 3 new colors (by (3)). Hence u_3 and similarly u_{-3} must be non-repeated colors. But, then we require more than 4 colors. Thus $\chi_{td}(P_7) = 5$. Let C be a minimal td-coloring of P_7. Let u_2 and u_{-2} have colors 1 and 2 respectively. Suppose that both u_3 and u_{-3} are non-repeated colors. Then, we have the coloring $\{\{u_1, u_4, u_7\}, \{u_2\}, \{u_3\}, \{u_5\}, \{u_6\}\}$. If either u_3 or u_{-3} is a repeated color, then the coloring C can be verified to be equivalent to the coloring given by $\{\{u_1, u_3\}, \{u_2\}, \{u_4, u_7\}, \{u_5\}, \{u_6\}\}$, or by $\{\{u_1, u_4\}, \{u_2\}, \{u_3\}, \{u_5, u_7\}, \{u_6\}\}$. □

We next show that $n = 8, 13, 15, 22$ are good integers.

Theorem 2.8 $\chi_{td}(P_n) = \gamma_t(P_n) + 2$ *if* $n = 8, 13, 15, 22$.

Proof As usual, we always adopt the convention $V(P_n) = \{u_1, u_2, \ldots, u_n\}; u_{-i} = u_{n+1-i}$ for $i \geqslant \lceil \frac{n}{2} \rceil$; C denotes a minimum td-coloring of P_n.

We have only to prove $|C| > \gamma_t(P_n) + 1$. We consider the following four cases.

Case 1 $n = 8$

Let $|C| = 5$. Then, as before u_2, being the only vertex dominated by u_1 has a non-repeated color. The same argument is true for u_{-2} also. If now u_3 has a repeated color, $\{u_1, u_2, u_3\}$ contains 2-color classes. As $C|\langle [4, 8] \rangle$ is a td-coloring, we require at least 4 more colors. Hence, u_3 and similarly u_{-3} must have non-repeated colors. Thus, there are 4 singleton color classes and $\{u_2\}, \{u_3\}, \{u_{-2}\}$ and $\{u_{-3}\}$. The two adjacent vertices u_4 and u_{-4} contribute two more colors. Thus $|C|$ has to be 6.

Case 2 $n = 13$

Let $|C| = 8 = \gamma_t(P_{13}) + 1$. As before u_2 and u_{-2} are non-repeated colors. Since $\chi_{td}(P_{10}) = 7 + 2 = 9$, u_3 can not be a repeated color, arguing as in case (i). Thus, u_3 and u_{-3} are also non-repeated colors. Now, if u_1 and u_{-1} have different colors, a diagonal of the color classes chosen as $\{u_1, u_{-1}, u_2, u_{-2}, u_3, u_{-3}, \ldots\}$ form a totally dominating set of cardinality $8 = \gamma_t(P_{13}) + 1$. However, clearly u_1 and u_{-1} can be omitted from this set without affecting total dominating set giving $\gamma_t(P_{13}) \leq 6$, a contradiction. Thus, u_1 and $u_{-1} = u_{13}$ have the same color say 1. Thus, $\langle[4, -4]\rangle = \langle[4, 10]\rangle$ is colored with 4 colors including the repeated color 1. Now, each of the pair of vertices $\{u_4, u_6\}, \{u_5, u_7\}, \{u_8, u_{10}\}$ contains a color classes. Thus $u_9 = u_{-5}$ must be colored with 1. Similarly, u_5. Now, if $\{u_4, u_6\}$ is not a color class, the vertex with repeated color must be colored with 1 which is not possible, since an adjacent vertex u_5 which also has color 1. Therefore $\{u_4, u_6\}$ is a color class. Similarly $\{u_8, u_{10}\}$ is also a color class. But then, u_7 will not dominate any color class. Thus $|C| = 9$.

Case 3 $n = 15$

Let $|C| = 9$. Arguing as before, u_2, u_{-2}, u_3 and u_{-3} have non-repeated colors $[\chi_{td}(P_{12}) = 8]$; u_1 and u_{-1} have the same color, say 1. The section $\langle[4, -4]\rangle = \langle[4, 12]\rangle$ consisting of 9 vertices is colored by 5 colors including the color 1. An argument similar to the one used in Case (2), gives u_4 (and u_{-4}) must have color 1. Thus, $C|\langle[5, -5]\rangle$ is a td-coloring with 4 colors including 1. Now, the possible minimum td-coloring of P_7 are given by Theorem 2.7. We can check that 1 can not occur in any color class in any of the minimum colorings given. e.g. take the coloring given by $\{u_5, u_8\}, \{u_6\}, \{u_7\}, \{u_9, u_{11}\}, \{u_{10}\}$. If u_6 has color 1, u_5 can not dominate a color class. Since u_4 has color 1, $\{u_5, u_8\}$ can not be color class 1 and so on. Thus $\chi_{td}(P_{15}) = 10$.

Case 4 $n = 22$

Let $|C| = \gamma_t(P_{22}) + 1 = 13$. We note that $\chi_{td}(P_{19}) = \gamma_t(P_{19}) + 2 = 12$. Then, arguing as in previous cases, we get the following facts.

Fact 1 u_2, u_{-2}, u_3, u_{-3} have non-repeated colors.

Fact 2 u_1 and u_{-1} have the same color, say 1.

Fact 3 u_7 is a non-repeated color.

This follows from the facts, otherwise $C|\langle[8, 22]\rangle$ will be a td-coloring; The section $\langle[1, 7]\rangle$ contain 4 color classes which together imply $\chi_{td}(P_{22}) \geqslant 4 + \chi_{td}(P_{15}) = 4 + 10 = 14$. In particular $\{u_5, u_7\}$ is not a color class.

Fact 4 The Facts 1 and 2, it follows that $C|\langle[4, -4]\rangle = C|\langle[4, 19]\rangle$ is colored with 9 colors including 1. Since each of the pair $\{\{u_4, u_6\}, \{u_5, u_7\}, \{u_8, u_{10}\}, \{u_9, u_{11}\}, \{u_{12}, u_{14}\}, \{u_{13}, u_{15}\}, \{u_{16}, u_{18}\}, \{u_{17}, u_{19}\}\}$ contain a color class, if any of these pairs is not a color class, one of the vertices must have a non-repeated color and the other colored with 1. From Fact 3, it then follows that the vertex u_5 must be colored with 1. It follows that $\{u_4, u_6\}$ must be a color class, since otherwise either u_4 or u_6 must be colored with 1.

Since $\{u_4, u_6\}$ is a color class, u_7 must dominate the color class $\{u_8\}$.

We summarize:

- u_2, u_3, u_7, u_8 have non-repeated colors.
- $\{u_4, u_6\}$ is a color class

- u_1 and u_5 are colored with color 1.

Similarly,

- $u_{-2}, u_{-3}, u_{-7}, u_{-8}$ have non-repeated colors.
- $\{u_{-4}, u_{-6}\}$ is a color class.
- u_{-1} and u_{-5} are colored with color 1.

Thus the section $\langle[9,-9]\rangle = \langle[9,14]\rangle$ must be colored with 3 colors including 1. This is easily seen to be not possible, since for instance this will imply both u_{13} and u_{14} must be colored with color1. Thus, we arrive at a contradiction. Thus $\chi_{td}(P_{22}) = 14$. □

Theorem 2.9 *Let n be an integer. Then,*

(1) *any integer of the form $4k$, $k \geqslant 2$ is good;*
(2) *any integer of the form $4k+1$, $k \geqslant 3$ is good;*
(3) *any integer of the form $4k+2$, $k \geqslant 5$ is good;*
(4) *any integer of the form $4k+3$, $k \geqslant 3$ is good.*

Proof The integers $n = 2, 3, 6$ are very bad and $n = 4, 5, 7, 9, 10, 11, 14, 18$ are bad. □

Remark 2.10 Let C be a minimal td-coloring of G. We call a color class in C, a non-dominated color class (n-d color class) if it is not dominated by any vertex of G. These color classes are useful because we can add vertices to these color classes without affecting td-coloring.

Lemma 2.11 *Suppose n is a good number and P_n has a minimal td-coloring in which there are two non-dominated color class. Then the same is true for $n+4$ also.*

Proof Let $C_1, C_2, \ldots, C_r$ be the color classes for P_n where C_1 and C_2 are non-dominated color classes. Suppose u_n does not have color C_1. Then $C_1 \cup \{u_{n+1}\}, C_2 \cup \{u_{n+4}\}, \{u_{n+2}\}, \{u_{n+3}\}, C_3, C_4, \cdots, C_r$ are required color classes for P_{n+4}. i.e. we add a section of 4 vertices with middle vertices having non-repeated colors and end vertices having C_1 and C_2 with the coloring being proper. Further, suppose the minimum coloring for P_n, the end vertices have different colors. Then the same is true for the coloring of P_{n+4} also. If the vertex u_1 of P_n does not have the color C_2, the new coloring for P_{n+4} has this property. If u_1 has color C_2, then u_n does not have the color C_2. Therefore, we can take the first two color classes of P_{n+4} as $C_1 \cup \{u_{n+4}\}$ and $C_2 \cup \{u_{n+1}\}$. □

Corollary 2.12 *Let n be a good number. Then P_n has a minimal td-coloring in which the end vertices have different colors. [It can be verified that the conclusion of the corollary is true for all $n \neq 3, 4, 11$ and 18].*

Proof We claim that P_n has a minimum td-coloring in which: (1)there are two non-dominated color classes; (2)the end vertices have different colors.

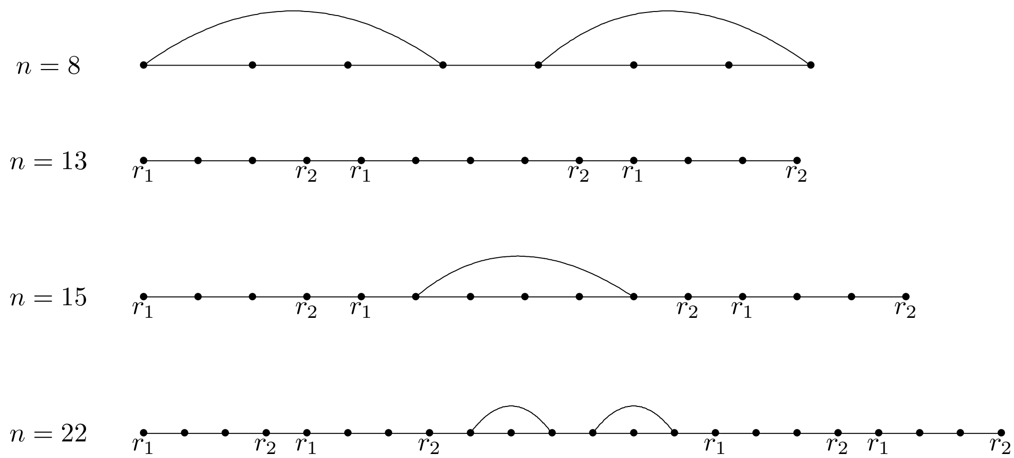

Fig.1

Now, it follows from the Lemma 2.11 that (1) and (2) are true for every good integer. □

Corollary 2.13 *Let n be a good integer. Then, there exists a minimum td-coloring for P_n with two n-d color classes.*

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