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Triple Connected Domination Number of a Graph

G.Mahadevan*, Selvam Avadayappan†, J.Paulraj Joseph‡ and T.Subramanian*

*Department of Mathematics Anna University: Tirunelveli Region, India

†Department of Mathematics, VHNSN College, Virudhunagar, India

‡Department of Mathematics Manonmaniam Sundaranar University, Tirunelveli, India

E-mail: selvam_avadayappan@yahoo.co.in,jpaulraj_2003@yahoo.co.in

Abstract: The concept of triple connected graphs with real life application was introduced in [7] by considering the existence of a path containing any three vertices of a graph G. In this paper, we introduce a new domination parameter, called Smarandachely triple connected domination number of a graph. A subset S of V of a nontrivial graph G is said to be Smarandachely triple connected dominating set, if S is a dominating set and the induced sub graph $\langle S\rangle$ is triple connected. The minimum cardinality taken over all Smarandachely triple connected dominating sets is called the Smarandachely triple connected domination number and is denoted by γ_{tc}. We determine this number for some standard graphs and obtain bounds for general graphs. Its relationship with other graph theoretical parameters are also investigated.

Key Words: Domination number, triple connected graph, Smarandachely triple connected domination number.

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§1. Introduction

By a graph we mean a finite, simple, connected and undirected graph $G(V, E)$, where V denotes its vertex set and E its edge set. Unless otherwise stated, the graph G has p vertices and q edges. Degree of a vertex v is denoted by $d(v)$, the maximum degree of a graph G is denoted by $\Delta(G)$. We denote a cycle on p vertices by C_p, a path on p vertices by P_p, and a complete graph on p vertices by K_p. A graph G is connected if any two vertices of G are connected by a path. A maximal connected subgraph of a graph G is called a component of G. The number of components of G is denoted by $\omega(G)$. The complement $\overline{G}$ of G is the graph with vertex set V in which two vertices are adjacent if and only if they are not adjacent in G. A tree is a connected acyclic graph. A bipartite graph (or bigraph) is a graph whose vertex set can be divided into two disjoint sets V_1 and V_2 such that every edge has one end in V_1 and another end in V_2. A complete bipartite graph is a bipartite graph where every vertex of V_1 is adjacent to every

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vertex in V_2. The complete bipartite graph with partitions of order $|V_1| = m$ and $|V_2| = n$, is denoted by $K_{m,n}$. A star, denoted by $K_{1,p-1}$ is a tree with one root vertex and $p-1$ pendant vertices. A bistar, denoted by $B(m,n)$ is the graph obtained by joining the root vertices of the stars $K_{1,m}$ and $K_{1,n}$. A wheel graph, denoted by W_p is a graph with p vertices, formed by joining a single vertex to all vertices of C_{p-1}. A helm graph, denoted by H_n is a graph obtained from the wheel W_n by attaching a pendant vertex to each vertex in the outer cycle of W_n. Corona of two graphs G_1 and G_2, denoted by $G_1 \circ G_2$ is the graph obtained by taking one copy of G_1 and $|V(G_1)|$ copies of G_2 in which i^{th} vertex of G_1 is joined to every vertex in the i^{th} copy of G_2. If S is a subset of V, then $\langle S \rangle$ denotes the vertex induced subgraph of G induced by S. The open neighbourhood of a set S of vertices of a graph G, denoted by $N(S)$ is the set of all vertices adjacent to some vertex in S and $N(S) \cup S$ is called the closed neighbourhood of S, denoted by $N[S]$. The diameter of a connected graph is the maximum distance between two vertices in G and is denoted by $diam(G)$. A cut-vertex (cut edge) of a graph G is a vertex (edge) whose removal increases the number of components. A vertex cut, or separating set of a connected graph G is a set of vertices whose removal results in a disconnected graph. The connectivity or vertex connectivity of a graph G, denoted by $\kappa(G)$ (where G is not complete) is the size of a smallest vertex cut. A connected subgraph H of a connected graph G is called a H-cut if $\omega(G-H) \geq 2$. The chromatic number of a graph G, denoted by $\chi(G)$ is the smallest number of colors needed to colour all the vertices of a graph G in which adjacent vertices receive different colours. For any real number x, $\lfloor x \rfloor$ denotes the largest integer less than or equal to x. A Nordhaus-Gaddum-type result is a (tight) lower or upper bound on the sum or product of a parameter of a graph and its complement. Terms not defined here are used in the sense of [2].

A subset S of V is called a dominating set of G if every vertex in $V-S$ is adjacent to at least one vertex in S. The domination number $\gamma(G)$ of G is the minimum cardinality taken over all dominating sets in G. A dominating set S of a connected graph G is said to be a connected dominating set of G if the induced sub graph $\langle S \rangle$ is connected. The minimum cardinality taken over all connected dominating sets is the connected domination number and is denoted by γ_c.

Many authors have introduced different types of domination parameters by imposing conditions on the dominating set [11-12]. Recently, the concept of triple connected graphs has been introduced by Paulraj Joseph et. al. [7] by considering the existence of a path containing any three vertices of G. They have studied the properties of triple connected graphs and established many results on them. A graph G is said to be triple connected if any three vertices lie on a path in G. All paths, cycles, complete graphs and wheels are some standard examples of triple connected graphs. In this paper, we use this idea to develop the concept of Smarandachely triple connected dominating set and Smarandachely triple connected domination number of a graph.

Theorem 1.1([7]) *A tree T is triple connected if and only if $T \cong P_p; p \geq 3$.*

Theorem 1.2([7]) *A connected graph G is not triple connected if and only if there exists a H-cut with $\omega(G-H) \geq 3$ such that $|V(H) \cap N(C_i)| = 1$ for at least three components C_1, C_2 and C_3 of $G-H$.*

Notation 1.3 Let G be a connected graph with m vertices $v_1, v_2, \ldots, v_m$. The graph obtained from G by attaching n_1 times a pendant vertex of P_{l_1} on the vertex v_1, n_2 times a pendant vertex of P_{l_2} on the vertex v_2 and so on, is denoted by $G(n_1P_{l_1}, n_2P_{l_2}, n_3P_{l_3}, \ldots, n_mP_{l_m})$ where $n_i, l_i \geq 0$ and $1 \leq i \leq m$.

Example 1.4 Let v_1, v_2, v_3, v_4, be the vertices of K_4. The graph $K_4(2P_2, P_3, P_4, P_3)$ is obtained from K_4 by attaching 2 times a pendant vertex of P_2 on v_1, 1 time a pendant vertex of P_3 on v_2, 1 time a pendant vertex of P_4 on v_3 and 1 time a pendant vertex of P_3 on v_4 and is shown in Figure 1.1.

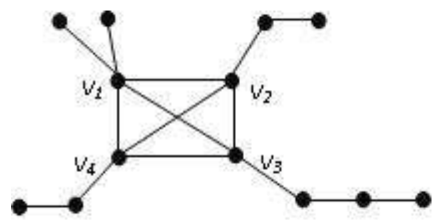

Figure 1.1 $K_4(2P_2, P_3, P_4, P_3)$

§2. Triple Connected Domination Number

Definition 2.1 *A subset S of V of a nontrivial connected graph G is said to be a Smarandachely triple connected dominating set, if S is a dominating set and the induced subgraph $\langle S \rangle$ is triple connected. The minimum cardinality taken over all Smarandachely triple connected dominating sets is called the Smarandachely triple connected domination number of G and is denoted by $\gamma_{tc}(G)$. Any Smarandachely triple connected dominating set with γ_{tc} vertices is called a γ_{tc}-set of G.*

Example 2.2 For the graph G_1 in Figure 2.1, $S = \{v_1, v_2, v_5\}$ forms a γ_{tc}-set of G_1. Hence $\gamma_{tc}(G_1) = 3$.

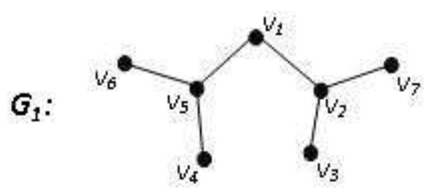

Figure 2.1 Graph with $\gamma_{tc} = 3$

Observation 2.3 Triple connected dominating set (tcd-set) does not exist for all graphs and if exists, then $\gamma_{tc}(G) \geq 3$.

Example 2.4 For the graph G_2 in Figure 2.2, any minimum dominating set must contain all the supports and any connected subgraph containing these supports is not triple connected and hence γ_{tc} does not exist.

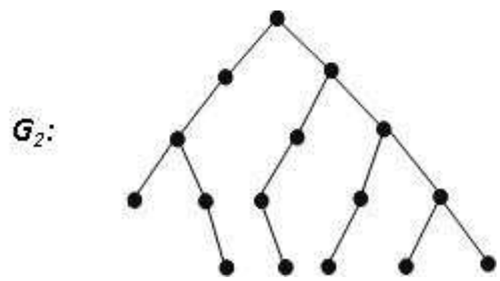

Figure 2.2 Graph with no tcd-set

Throughout this paper we consider only connected graphs for which triple connected dominating set exists.

Observation 2.5 The complement of the triple connected dominating set need not be a triple connected dominating set.

Example 2.6 For the graph G_3 in Figure 2.3, $S = \{v_1, v_2, v_3\}$ forms a triple connected dominating set of G_3. But the complement $V - S = \{v_4, v_5, v_6, v_7, v_8, v_9\}$ is not a triple connected dominating set.

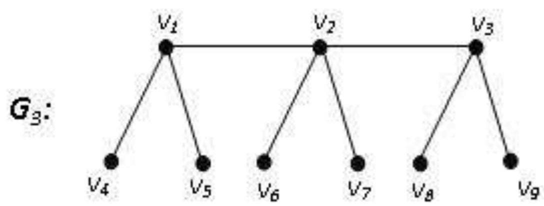

Figure 2.3 Graph in which $V - S$ is not a tcd-set

Observation 2.7 Every triple connected dominating set is a dominating set but not conversely.

Observation 2.8 For any connected graph $G, \gamma(G) \leq \gamma_c(G) \leq \gamma_{tc}(G)$ and the bounds are sharp.

Example 2.9 For the graph G_4 in Figure 2.4, $\gamma(G_4) = 4, \gamma_c(G_4) = 6$ and $\gamma_{tc}(G_4) = 8$. For the graph G_5 in Figure 2.4, $\gamma(G_5) = \gamma_c(G_5) = \gamma_{tc}(G_5) = 3$.

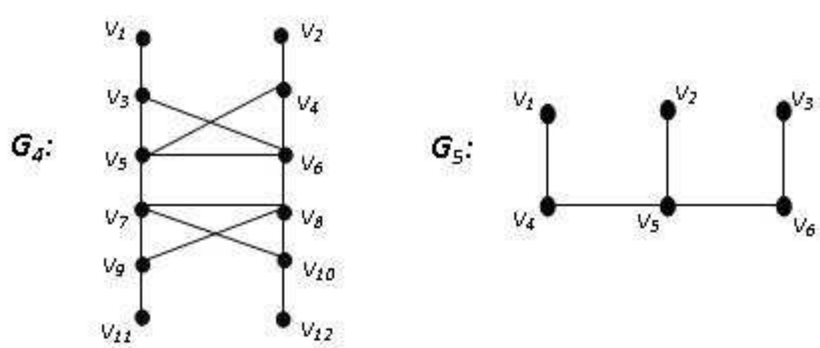

Figure 2.4

Theorem 2.10 *If the induced subgraph of each connected dominating set of G has more than two pendant vertices, then G does not contain a triple connected dominating set.*

Proof The proof follows from Theorem 1.2. □

Some exact value for some standard graphs are listed in the following:

1. Let P be the petersen graph. Then $\gamma_{tc}(P) = 5$.
2. For any triple connected graph G with p vertices, $\gamma_{tc}(G \circ K_1) = p$.
3. For any path of order $p \geq 3, \gamma_{tc}(P_p) = \begin{cases} 3 & \text{if } p < 5 \\ p-2 & \text{if } p \geq 5. \end{cases}$
4. For any cycle of order $p \geq 3, \gamma_{tc}(C_p) = \begin{cases} 3 & \text{if } p < 5 \\ p-2 & \text{if } p \geq 5. \end{cases}$
5. For any complete bipartite graph of order $p \geq 4, \gamma_{tc}(K_{m,n}) = 3$. (where $m, n \geq 2$ and $m + n = p$).
6. For any star of order $p \geq 3, \gamma_{tc}(K_{1,p-1}) = 3$.
7. For any complete graph of order $p \geq 3, \gamma_{tc}(K_p) = 3$.
8. For any wheel of order $p \geq 4, \gamma_{tc}(W_p) = 3$.
9. For any helm graph of order $p \geq 7, \gamma_{tc}(H_n) = \frac{p-1}{2}$ (where $2n - 1 = p$).
10. For any bistar of order $p \geq 4, \gamma_{tc}(B(m, n)) = 3$ (where $m, n \geq 1$ and $m + n + 2 = p$).

Example 2.11 For the graph G_6 in Figure 2.5, $S = \{v_6, v_2, v_3, v_4\}$ is a unique minimum connected dominating set so that $\gamma_c(G_6) = 4$. Here we notice that the induced subgraph of S has three pendant vertices and hence G does not contain a triple connected dominating set.

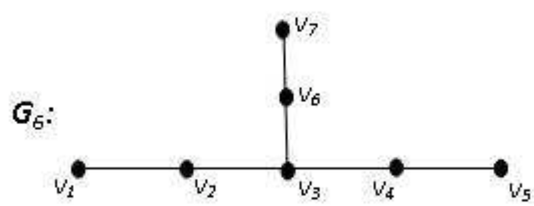

Figure 2.5 Graph having cd set and not having tcd-set

Observation 2.12 If a spanning sub graph H of a graph G has a triple connected dominating set, then G also has a triple connected dominating set.

Observation 2.13 Let G be a connected graph and H be a spanning sub graph of G. If H has a triple connected dominating set, then $\gamma_{tc}(G) \leq \gamma_{tc}(H)$ and the bound is sharp.

Example 2.14 Consider the graph G_7 and its spanning subgraphs G_8 and G_9 shown in Figure 2.6.

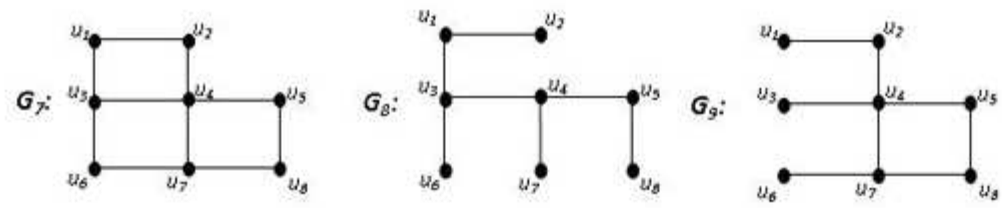

Figure 2.6

For the graph $G_7, S = \{u_2, u_4, u_7\}$ is a minimum triple connected dominating set and so $\gamma_{tc}(G_7) = 3$. For the spanning subgraph G_8 of $G_7, S = \{u_1, u_3, u_4, u_5\}$ is a minimum triple connected dominating set so that $\gamma_{tc}(G_8) = 4$. Hence $\gamma_{tc}(G_7) < \gamma_{tc}(G_8)$. For the spanning subgraph G_9 of $G_7, S = \{u_2, u_4, u_7\}$ is a minimum triple connected dominating set so that $\gamma_{tc}(G_9) = 3$. Hence $\gamma_{tc}(G_7) = \gamma_{tc}(G_9)$.

Observation 2.15 For any connected graph G with p vertices, $\gamma_{tc}(G) = p$ if and only if $G \cong P_3$ or C_3.

Theorem 2.16 *For any connected graph G with p vertices, $\gamma_{tc}(G) = p - 1$ if and only if $G \cong P_4, C_4, K_4, K_{1,3}, K_4 - \{e\}, C_3(P_2)$.*

Proof Suppose $G \cong P_4, C_4, K_4 - \{e\}, K_4, K_{1,3}, C_3(P_2)$, then $\gamma_{tc}(G) = 3 = p - 1$. Conversely, let G be a connected graph with p vertices such that $\gamma_{tc}(G) = p - 1$. Let $S = \{u_1, u_2, \ldots, u_{p-1}\}$ be a γ_{tc}-set of G. Let x be in $V - S$. Since S is a dominating set, there exists a vertex v_i in S such that v_i is adjacent to x. If $p \geq 5$, by taking the vertex v_i, we can construct a triple connected dominating set S with fewer elements than $p - 1$, which is a contradiction. Hence $p \leq 4$. Since $\gamma_{tc}(G) = p - 1$, by Observation 2.5, we have $p = 4$. Let $S = \{v_1, v_2, v_3\}$ and $V - S = \{v_4\}$. Since S is a γ_{tc}-set of $G, \langle S \rangle = P_3$ or C_3.

Case *i* $\langle S \rangle = P_3 = v_1v_2v_3$

Since G is connected, v_4 is adjacent to v_1 (or v_3) or v_4 is adjacent to v_2. Hence $G \cong P_4$ or $K_{1,3}$.

Case *ii* $\langle S \rangle = C_3 = v_1v_2v_3v_1$

Since G is connected, v_4 is adjacent to v_1 (or v_2 or v_3). Hence $G \cong C_3(P_2)$. Now by adding edges to $P_4, K_{1,3}$ or $C_3(P_2)$ without affecting γ_{tc}, we have $G \cong C_4, K_4 - \{e\}, K_4$. □

Theorem 2.17 *For any connected graph G with $p \geq 5$, we have $3 \leq \gamma_{tc}(G) \leq p - 2$ and the bounds are sharp.*

Proof The lower bound follows from Definition 2.1 and the upper bound follows from Observation 2.15 and Theorem 2.16. Consider the dodecahedron graph G_{10} in Figure 2.7, the path P_5 and the cycle C_9.

G_{10}:

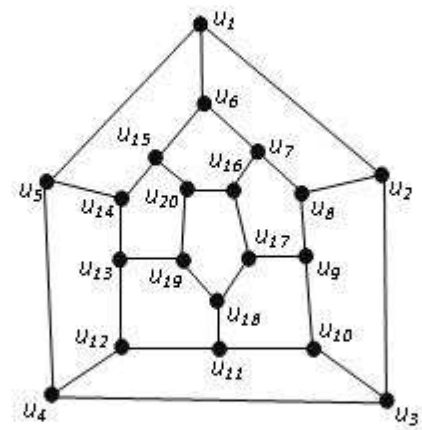

Figure 2.7

One can easily check that $S = \{u_6, u_7, u_8, u_9, u_{10}, u_{11}, u_{12}, u_{13}, u_{14}, u_{15}\}$ is a minimum triple connected dominating set of G_{10} and $\gamma_{tc}(G_{10}) = 10 > 3$. In addition, $\gamma_{tc}(G_{10}) = 10 < p - 2$. For P_5, the lower bound is attained and for C_9 the upper bound is attained. □

Theorem 2.18 *For a connected graph G with 5 vertices, $\gamma_{tc}(G) = p - 2$ if and only if G is isomorphic to $P_5, C_5, W_5, K_5, K_{1,4}, K_{2,3}, K_1 \circ 2K_2, K_5 - \{e\}, K_4(P_2), C_4(P_2), C_3(P_3), C_3(2P_2), C_3(P_2, P_2, 0)$, $P_4(0, P_2, 0, 0)$ or any one of the graphs shown in Figure* 2.8.

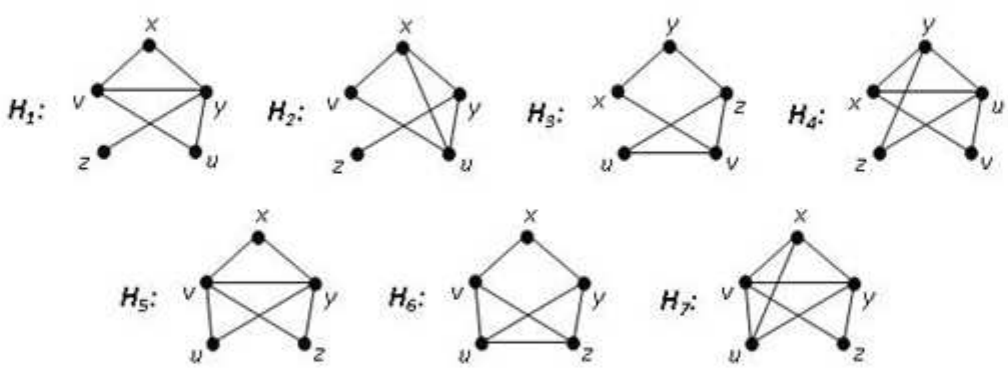

Figure 2.8 Graphs with $\gamma_{tc} = p - 2$

Proof Suppose G is isomorphic to $P_5, C_5, W_5, K_5, K_{1,4}, K_{2,3}, K_1 \circ 2K_2, K_5 - \{e\}$, $K_4(P_2), C_4(P_2), C_3(P_3), C_3(2P_2), C_3(P_2, P_2, 0), P_4(0, P_2, 0, 0)$ or any one of the graphs H_1 to H_7 given in Figure 2.8., then clearly $\gamma_{tc}(G) = p - 2$. Conversely, let G be a connected graph with 5 vertices and $\gamma_{tc}(G) = 3$. Let $S = \{x, y, z\}$ be a γ_{tc}-set. Then clearly $\langle S \rangle = P_3$ or C_3. Let $V - S = V(G) - V(S) = \{u, v\}$. Then $\langle V - S \rangle = K_2$ or $\overline{K}_2$.

Case *i* $\langle S \rangle = P_3 = xyz$

Subcase *i* $\langle V-S\rangle = K_2 = uv$

Since G is connected, there exists a vertex say x (or z) in P_3 which is adjacent to u (or v) in K_2. Then $S=\{x,y,u\}$ is a minimum triple connected dominating set of G so that $\gamma_{tc}(G)=p-2$. If $d(x)=d(y)=2, d(z)=1$, then $G\simeq P_5$. Since G is connected, there exists a vertex say y in P_3 is adjacent to u (or v) in K_2. Then $S=\{y,u,v\}$ is a minimum triple connected dominating set of G so that $\gamma_{tc}(G)=p-2$. If $d(x)=d(z)=1, d(y)=3$, then $G\cong P_4(0,P_2,0,0)$. Now by increasing the degrees of the vertices, by the above arguments, we have $G\cong C_5, W_5, K_5, K_{2,3}, K_5-\{e\}, K_4(P_2), C_4(P_2), C_3(P_3), C_3(2P_2), C_3(P_2,P_2,0)$ and H_1 to H_7 in Figure 2.8. In all the other cases, no new graph exists.

Subcase *ii* $\langle V-S\rangle = 2$

Since G is connected, there exists a vertex say x (or z) in P_3 is adjacent to u and v in $\overline{K}_2$. Then $S=\{x,y,z\}$ is a minimum triple connected dominating set of G so that $\gamma_{tc}(G)=p-2$. If $d(x)=3, d(y)=2, d(z)=1$, then $G\cong P_4(0,P_2,0,0)$. In all the other cases, no new graph exists. Since G is connected, there exists a vertex say y in P_3 which is adjacent to u and v in $\overline{K}_2$. Then $S=\{x,y,z\}$ is a minimum triple connected dominating set of G so that $\gamma_{tc}(G)=p-2$. If $d(x)=d(z)=1, d(y)=4$, then $G\cong K_{1,4}$. In all the other cases, no new graph exists. Since G is connected, there exists a vertex say x in P_3 which is adjacent to u in $\overline{K}_2$ and y in P_3 is adjacent to v in $\overline{K}_2$. Then $S=\{x,y,z\}$ is a minimum triple connected dominating set of G so that $\gamma_{tc}(G)=p-2$. If $d(x)=2, d(y)=3, d(z)=1$, then $G\cong P_4(0,P_2,0,0)$. In all the other cases, no new graph exists. Since G is connected, there exists a vertex say x in P_3 which is adjacent to u in $\overline{K}_2$ and z in P_3 which is adjacent to v in $\overline{K}_2$. Then $S=\{x,y,z\}$ is a minimum triple connected dominating set of G so that $\gamma_{tc}(G)=p-2$. If $d(x)=d(y)=d(z)=2$, then $G\cong P_5$. In all the other cases, no new graph exists.

Case *ii* $\langle S\rangle = C_3 = xyzx$

Subcase *i* $\langle V-S\rangle = K_2 = uv$

Since G is connected, there exists a vertex say x (or y,z) in C_3 is adjacent to u (or v) in K_2. Then $S=\{x,y,u\}$ is a minimum triple connected dominating set of G so that $\gamma_{tc}(G)=p-2$. If $d(x)=3, d(y)=d(z)=2$, then $G\cong C_3(P_3)$. If $d(x)=4, d(y)=d(z)=2$, then $G\cong K_1\circ 2K_2$. In all the other cases, no new graph exists.

Subcase *ii* $\langle V-S\rangle = \overline{K}_2$

Since G is connected, there exists a vertex say x (or y,z) in C_3 is adjacent to u and v in $\overline{K}_2$. Then $S=\{x,y,z\}$ is a minimum triple connected dominating set of G so that $\gamma_{tc}(G)=p-2$. If $d(x)=4, d(y)=d(z)=2$, then $G\cong C_3(2P_2)$. In all the other cases, no new graph exists. Since G is connected, there exists a vertex say x(or y,z) in C_3 is adjacent to u in $\overline{K}_2$ and y (or z) in C_3 is adjacent to v in $\overline{K}_2$. Then $S=\{x,y,z\}$ is a minimum triple connected dominating set of G so that $\gamma_{tc}(G)=p-2$. If $d(x)=d(y)=3, d(z)=2$, then $G\cong C_3(P_2,P_2,0)$. In all other cases, no new graph exists. □

Theorem 2.19 *For a connected graph G with $p>5$ vertices, $\gamma_{tc}(G)=p-2$ if and only if G*

is isomorphic to P_p or C_p.

Proof Suppose G is isomorphic to P_p or C_p, then clearly $\gamma_{tc}(G) = p-2$. Conversely, let G be a connected graph with $p > 5$ vertices and $\gamma_{tc}(G) = p-2$. Let $S = \{v_1, v_2, \ldots, v_{p-2}\}$ be a γ_{tc}-set and let $V - S = V(G) - V(S) = \{v_{p-1}, v_p\}$. Then $\langle V - S\rangle = K_2, \overline{K}_2$.

Claim. *$\langle S\rangle$ is a tree.*

Suppose $\langle S\rangle$ is not a tree. Then $\langle S\rangle$ contains a cycle. Without loss of generality, let $C = v_1v_2\cdots v_qv_1, q \leq p-2$ be a cycle of shortest length in $\langle S\rangle$. Now let $\langle V-S\rangle = K_2 = v_{p-1}v_p$. Since G is connected and S is a γ_{tc}-set of G, v_{p-1}(or v_p) is adjacent to a vertex v_k in $\langle S\rangle$. If v_k is in C, then $S = \{v_{p-1}, v_i, v_{i+1}, \ldots, v_{i-3}\} \cup \{x \in V(G) : x \notin C\}$ forms a γ_{tc}-set of G so that $\gamma_{tc}(G) < p-2$, which is a contradiction. Suppose v_{p-1} (or v_p) is adjacent to a vertex v_i in $\langle S\rangle - C$, then we can construct a γ_{tc}-set which contains v_{p-1}, v_i with fewer elements than $p-2$, which is a contradiction. Similarly if $\langle V - S\rangle = \overline{K}_2$, we can prove that no graph exists. Hence $\langle S\rangle$ is a tree. But S is a triple connected dominating set. Therefore by Theorem 1.1, we have $\langle S\rangle \cong P_{p-2}$.

Case *i* $\langle V - S\rangle = K_2 = v_{p-1}v_p$

Since G is connected and S is a γ_{tc}-set of G, there exists a vertex, say, v_i in P_{p-2} which is adjacent to a vertex, say, v_{p-1} in K_2. If $v_i = v_1$ (or) v_{p-2}, then $G \cong P_p$. If $v_i = v_1$ is adjacent to v_{p+1} and v_{p-2} is adjacent to v_p, then $G \cong C_p$. If $v_i = v_j$ for $j = 2, 3, \ldots, p-3$, then $S_1 = S - \{v_1, v_{p-2}\} \cup \{v_{p-1}\}$ is a triple connected dominating set of cardinality $p-3$ and hence $\gamma_{tc} \leq p-3$, which is a contradiction.

Case *ii* $\langle V - S\rangle = \overline{K}_2$

Since G is connected and S is a γ_{tc}-set of G, there exists a vertex say v_i in P_{p-2} which is adjacent to both the vertices v_{p-1} and v_p in $\overline{K}_2$. If $v_i = v_1$ (or v_{p-2}), then by taking the vertex v_1 (or v_{p-2}), we can construct a triple connected dominating set which contains fewer elements than $p-2$, which is a contradiction. Hence no graph exists. If $v_i = v_j$ for $j = 2, 3, \ldots, n-3$, then by taking the vertex v_j, we can construct a triple connected dominating set which contains fewer elements than $p-2$, which is a contradiction. Hence no graph exists. Suppose there exists a vertex say v_i in P_{p-2} which is adjacent to v_{p-1} in $\overline{K}_2$ and a vertex $v_j (i \neq j)$ in P_{p-2} which is adjacent to v_p in $\overline{K}_2$. If $v_i = v_1$ and $v_j = v_{p-2}$, then $S = \{v_1, v_2, \ldots, v_{p-2}\}$ is a γ_{tc}-set of G and hence $G \cong P_p$. If $v_i = v_1$ and $v_j = v_k$ for $k = 2, 3, \ldots, n-3$, then by taking the vertex v_1 and v_k, we can construct a triple connected dominating set which contains fewer elements than $p-2$, which is a contradiction. Hence no graph exists. If $v_i = v_k$ and $v_j = v_l$ for $k, l = 2, 3, \ldots, n-3$, then by taking the vertex v_k and v_l, we can construct a triple connected dominating set which contains fewer elements than $p-2$, which is a contradiction. □

Corollary 2.20 *Let G be a connected graph with $p > 5$ vertices. If $\gamma_{tc}(G) = p-2$, then $\kappa(G) = 1$ or 2, $\Delta(G) = 2, \chi(G) = 2$ or 3, and diam$(G) = p-1$ or $\lfloor \frac{p}{2} \rfloor$.*

Proof Let G be a connected graph with $p > 5$ vertices and $\gamma_{tc}(G) = p-2$. Since $\gamma_{tc}(G) = p-2$, by Theorem 2.19, G is isomorphic to P_p or C_p. We know that for $P_p, \kappa(G) = 1, \Delta(G) =$

$2, \chi(G) = 2$ and $diam(G) = p-1$. For $C_p, \kappa(G) = 2, \Delta(G) = 2$, $diam(G) = \lfloor \frac{p}{2} \rfloor$ and

$$\chi(G) = \begin{cases} 2 & \text{if p is even,} \\ 3 & \text{if p is odd.} \end{cases}$$ □

Observation 2.21 Let G be a connected graph with $p \geq 3$ vertices and $\Delta(G) = p-1$. Then $\gamma_{tc}(G) = 3$.

For, let v be a full vertex in G. Then $S = \{v, v_i, v_j\}$ is a minimum triple connected dominating set of G, where v_i and v_j are in $N(v)$. Hence $\gamma_{tc}(G) = 3$.

Theorem 2.22 *For any connected graph G with $p \geq 3$ vertices and $\Delta(G) = p-2, \gamma_{tc}(G) = 3$.*

Proof Let G be a connected graph with $p \geq 3$ vertices and $\Delta(G) = p-2$. Let v be a vertex of maximum degree $\Delta(G) = p-2$. Let $v_1, v_2, \ldots$ and v_{p-2} be the vertices which are adjacent to v, and let v_{p-1} be the vertex which is not adjacent to v. Since G is connected, v_{p-1} is adjacent to a vertex v_i for some i. Then $S = \{v, v_i, v_j | i \neq j\}$ is a minimum triple connected dominating set of G. Hence $\gamma_{tc}(G) = 3$. □

Theorem 2.23 *For any connected graph G with $p \geq 3$ vertices and $\Delta(G) = p-3, \gamma_{tc}(G) = 3$.*

Proof Let G be a connected graph with $p \geq 3$ vertices and $\Delta(G) = p-3$ and let v be the vertex of G with degree $p-3$. Suppose $N(v) = \{v_1, v_2, \ldots, v_{p-3}\}$ and $V - N(v) = \{v_{p-2}, v_{p-1}\}$. If v_{p-1} and v_{p-2} are not adjacent in G, then since G is connected, there are vertices v_i and v_j for some $i, j, 1 \leq i, j \leq p-3$, which are adjacent to v_{p-2} and v_{p-1} respectively. Here note that i can be equal to j. If $i = j$, then $\{v, v_i, v_{p-1}\}$ is a required triple connected dominating set of G. If $i \neq j$, then $\{v_i, v, v_j\}$ is a required triple connected dominating set of G. If v_{p-2} and v_{p-1} are adjacent in G, then there is a vertex v_j, for some $j, 1 \leq j \leq p-3$, which is adjacent to v_{p-1} or to v_{p-1} or to both. In this case, $\{v, v_i, v_{p-1}\}$ or $\{v, v_i, v_{p-2}\}$ is a triple connected dominating set of G. Hence in all the cases, $\gamma_{tc}(G) = 3$. □

The Nordhaus - Gaddum type result is given below:

Theorem 2.24 *Let G be a graph such that G and $\overline{G}$ are connected graphs of order $p \geq 5$. Then $\gamma_{tc}(G) + \gamma_{tc}(\overline{G}) \leq 2(p-2)$ and the bound is sharp.*

Proof The bound directly follows from Theorem 2.17. For the cycle $C_5, \gamma_{tc}(G) + \gamma_{tc}(\overline{G}) = 2(p-2)$. □

§3. Relation with Other Graph Parameters

Theorem 3.1 *For any connected graph G with $p \geq 5$ vertices, $\gamma_{tc}(G) + \kappa(G) \leq 2p-3$ and the bound is sharp if and only if $G \cong K_5$.*

Proof Let G be a connected graph with $p \geq 5$ vertices. We know that $\kappa(G) \leq p-1$ and by Theorem 2.17, $\gamma_{tc}(G) \leq p-2$. Hence $\gamma_{tc}(G) + \kappa(G) \leq 2p-3$. Suppose G is isomorphic to K_5. Then clearly $\gamma_{tc}(G) + \kappa(G) = 2p - 3$. Conversely, let $\gamma_{tc}(G) + \kappa(G) = 2p - 3$. This is possible only if $\gamma_{tc}(G) = p - 2$ and $\kappa(G) = p - 1$. But $\kappa(G) = p - 1$, and so $G \cong K_p$ for which $\gamma_{tc}(G) = 3 = p - 2$ so that $p = 5$. Hence $G \cong K_5$. □

Theorem 3.2 *For any connected graph G with $p \geq 5$ vertices, $\gamma_{tc}(G) + \chi(G) \leq 2p - 2$ and the bound is sharp if and only if $G \cong K_5$.*

Proof Let G be a connected graph with $p \geq 5$ vertices. We know that $\chi(G) \leq p$ and by Theorem 2.17, $\gamma_{tc}(G) \leq p - 2$. Hence $\gamma_{tc}(G) + \chi(G) \leq 2p - 2$. Suppose G is isomorphic to K_5. Then clearly $\gamma_{tc}(G) + \chi(G) = 2p - 2$. Conversely, let $\gamma_{tc}(G) + \chi(G) = 2p - 2$. This is possible only if $\gamma_{tc}(G) = p - 2$ and $\chi(G) = p$. Since $\chi(G) = p, G$ is isomorphic to K_p for which $\gamma_{tc}(G) = 3 = p - 2$ so that $p = 5$. Hence $G \cong K_5$. □

Theorem 3.3 *For any connected graph G with $p \geq 5$ vertices, $\gamma_{tc}(G) + \Delta(G) \leq 2p - 3$ and the bound is sharp if and only if G is isomorphic to $W_5, K_5, K_{1,4}, K_1 \circ 2K_2, K_5 - \{e\}, K_4(P_2), C_3(2P_2)$ or any one of the graphs shown in Figure 3.1.*

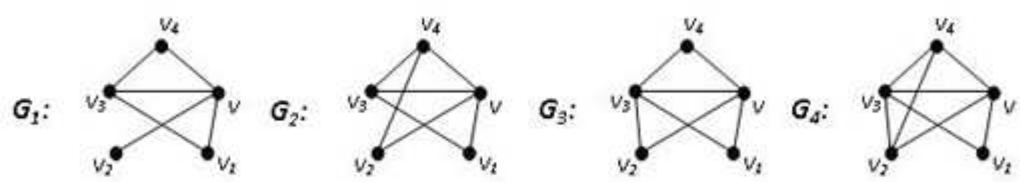

Figure 3.1 Graphs with $\gamma_{tc} + \Delta = 2p - 3$

Proof Let G be a connected graph with $p \geq 5$ vertices. We know that $\Delta(G) \leq p - 1$ and by Theorem 2.17, $\gamma_{tc}(G) \leq p - 2$. Hence $\gamma_{tc}(G) + \Delta(G) \leq 2p - 3$. Let G be isomorphic to $W_5, K_5, K_{1,4}, K_1 \circ 2K_2, K_5 - \{e\}, K_4(P_2), C_3(2P_2)$ or any one of the graphs G_1 to G_4 given in Figure 3.1. Then clearly $\gamma_{tc}(G) + \Delta(G) = 2p - 3$. Conversely, let $\gamma_{tc}(G) + \Delta(G) = 2p - 3$. This is possible only if $\gamma_{tc}(G) = p - 2$ and $\Delta(G) = p - 1$. Since $\Delta(G) = p - 1$, by Observation 2.21, we have $\gamma_{tc}(G) = 3$ so that $p = 5$. Let v be the vertex having a maximum degree and let v_1, v_2, v_3, v_4 be the vertices which are adjacent to the vertex v. If $d(v) = 4, d(v_1) = d(v_2) = d(v_3) = d(v_4) = 1$, then $G \cong K_{1,4}$. Now by adding edges to $K_{1,4}$ without affecting the value of γ_{tc}, we have $G \cong W_5, K_5, K_1 \circ 2K_2, K_5 - \{e\}, K_4(P_2), C_3(2P_2)$ and the graphs G_1 to G_4 given in Figure 3.1. □

References

[1] Cokayne E. J. and Hedetniemi S. T., Total domination in graphs, *Networks*, Vol.10 (1980), 211-219.

[2] John Adrian Bondy, Murty U.S.R., *Graph Theory*, Springer, 2008.

[3] Nordhaus E. A. and Gaddum J. W. (1956), On complementary graphs, *Amer. Math.Monthly*, 63: 175-177.

[4] Paulraj Joseph J. and Arumugam. S., Domination and connectivity in graphs, *International Journal of Management Systems*, 8 (3)(1992), 233-236.

[5] Paulraj Joseph J. and Arumugam.S., Domination in graphs, *International Journal of Management Systems*, 11 (1995): 177-182.

[6] Paulraj Joseph J. and Arumugam.S., Domination and coloring in graphs, *International Journal of Management Systems*, 8 (1)(1997), 37-44.

[7] Paulraj Joseph J., Angel Jebitha M.K., Chithra Devi P. and Sudhana G., Triple connected graphs, *Indian Journal of Mathematics and Mathematical Sciences*, Vol. 8, No. 1 (2012), 61-75.

[8] Paulraj Joseph J. and Mahadevan G., On complementary perfect domination number of a graph, *Acta Ciencia Indica*, Vol. XXXI M, No. 2(2006),847-853.

[9] Sampathkumar, E., Walikar, H.B, The connected domination number of a graph, *J. Math. Phys. Sci*, 13 (6)(1979), 607-613.

[10] Teresa W. Haynes, Stephen T. Hedetniemi and Peter J. Slater, *Domination in graphs—Advanced Topics*, Marcel Dekker, New York, 1998.

[11] Teresa W. Haynes, Stephen T. Hedetniemi and Peter J. Slater, *Fundamentals of domination in graphs*, Marcel Dekker, New York, 1998.