ON THE ODD SIEVE SEQUENCE

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Abstract The odd sieve sequence is the sequence, which is composed of all odd numbers that are not equal to the difference of two primes. In this paper, we use analytic method to study the mean value properties of this sequence, and give two interesting asymptotic formulae.

Keywords: The odd sieve sequence; Mean value; Asymptotic formula.

S1.Introduction

The odd sieve sequence is the sequence, which is composed of all odd numbers that are not equal to the difference of two primes. For example: $7, 11, 19, 23, 25, \cdots$. In problem 94 of [1], Professor F.Smarandache asked us to study this sequence. About this problem, it seems that none had studied it before. Let $\mathcal{A}$ denotes the set of the odd sieve numbers. In this paper, we use analytic method to study the mean value properties of this sequence, and give two interesting asymptotic formulae. That is, we shall prove the following:

Theorem 1. For any positive number $x > 1$, we have the asymptotic formula

$$\sum_{\substack{n\le x\\ n\in\mathcal{A}}} n = \frac{x^2}{4} - \frac{x^2}{2\ln x} + O\left(\frac{x^2}{\ln^2 x}\right).$$

Theorem 2. For any positive number $x > 1$, we have the asymptotic formula

$$\sum_{\substack{n\le x\\ n\in\mathcal{A}}} \frac{1}{n} = \frac{1}{2}\ln\frac{x}{2} - \ln\ln(x+2) + \frac{1}{2}\gamma - A + B + O\left(\frac{1}{\ln x}\right),$$

where A, B are computable constants, γ is the Euler's constant.

§2. Proof of Theorems

In this section, we shall complete the proof of Theorems. Firstly we prove Theorem 1, let

$$a(n) = \begin{cases} 1, & n \text{ is a prime}, \\ 0, & \text{otherwise}, \end{cases}$$

and note that

$$\pi(x) = \sum_{n\le x} a(n) = \frac{x}{\ln x} + O\left(\frac{x}{\ln^2 x}\right).$$

Therefore if we take $f(n) = n$ in Abel's identity, we can get the estimate

$$\begin{aligned}\sum_{p\le x+2} p &= (x+2)\pi(x+2) - 2\pi(2) - \int_2^{x+2} \pi(t)f'(t)dt \\ &= \frac{(x+2)^2}{\ln(x+2)} + O\left(\frac{(x+2)^2}{\ln^2(x+2)}\right) - \int_2^{x+2}\left(\frac{t}{\ln t} + O\left(\frac{t}{\ln^2 t}\right)\right)dt \\ &= \frac{(x+2)^2}{2\ln(x+2)} + O\left(\frac{(x+2)^2}{\ln^2(x+2)}\right).\end{aligned}$$

Then from the definition of the odd sieve sequence and the Euler's summation formula, we have

$$\begin{aligned}\sum_{\substack{n\le x\\ n\in\mathcal{A}}} n &= \sum_{2n-1\le x}(2n-1) - \sum_{p-2\le x}(p-2) \\ &= \frac{(x+1)(x+3)}{4} + O(x) - \sum_{p\le x+2} p + 2\sum_{p\le x+2} 1 \\ &= \frac{(x+1)(x+3)}{4} - \frac{(x+2)^2}{2\ln(x+2)} + O\left(\frac{(x+2)^2}{\ln^2(x+2)}\right) \\ &= \frac{x^2}{4} - \frac{x^2}{2\ln x} + O\left(\frac{x^2}{\ln^2 x}\right).\end{aligned}$$

This completes the proof of Theorem 1.

Now we prove Theorem 2. From the Euler's summation formula, we have

$$\sum_{n\le x}\frac{1}{n} = \ln x + \gamma + O\left(\frac{1}{x}\right), \tag{1}$$

where γ is the Euler's constant.

Since

$$\sum_{n\le x}\frac{1}{2n(2n-1)} \le \sum_{n=1}^{\infty}\frac{1}{(n-1)^2},$$

we have

$$\sum_{n\le x}\frac{1}{2n(2n-1)} = O(1). \tag{2}$$

Note

$$\sum_{n\le x}\frac{1}{p} = \ln\ln x + A + O\left(\frac{1}{\ln x}\right), \tag{3}$$

where A is a constant.

From the definition of odd sieve and formulae (1), (2) and (3), we can obtain

$$\begin{aligned}
&\sum_{\substack{n\leq x\\ n\in\mathcal{A}}}\frac{1}{n}=\sum_{2n-1\leq x}\frac{1}{2n-1}-\sum_{p-2\leq x}\frac{1}{p}\\
&=\sum_{n\leq\frac{x+1}{2}}\left(\frac{1}{2n}+\frac{1}{2n(2n-1)}\right)-\sum_{3\leq p\leq x+2}\frac{1}{p-2}\\
&=\frac{1}{2}\sum_{n\leq\frac{x+1}{2}}\frac{1}{n}-\sum_{p\leq x+2}\frac{1}{p}+\sum_{n\leq\frac{x+1}{2}}\frac{1}{2n(2n-1)}-\sum_{3\leq p\leq x+2}\frac{2}{p(p-2)}+\frac{1}{2}\\
&=\frac{1}{2}\ln\frac{x}{2}-\ln\ln(x+2)+\frac{1}{2}\gamma-A+B+O\left(\frac{1}{\ln x}\right),
\end{aligned}$$

where B is a computable constant.

This completes the proof of Theorem 2.

References

[1] F. Smarandache, Only Problems, Not Solutions, Xiquan Publishing House, Chicago, 1993.

[2] Tom M. Apostol, Introduction to Analytic Number Theory, New York, 1976.