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The Smarandache Perfect Numbers

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Abstract In this paper we prove that 12 is the only Smarandache perfect number.

Keywords Smarandache function, Smarandache perfect number, divisor function.

§1. Introduction and result

Let N be the set of all positive integer. For any positive integer a, let $S(a)$ denote the Smarandache function of a. Let n be a postivie integer. If n satisfy

$$\sum_{d|n} S(d) = n + 1 + S(n), \tag{1}$$

then n is called a Smarandache perfect number. Recently, Ashbacher [1] showed that if $n \leq 10^6$, then 12 is the only Smarandache perfect number. In this paper we completely determine all Smarandache perfect number as follows:

Theorem. 12 is the only Smarandache perfect number.

§2. Proof of the theorem

The proof of our theorem depends on the following lemmas.

Lemma 1 ([2]). For any positive integer n with $n > 1$, if

$$n = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k} \tag{2}$$

is the factorization of n, then we have

$$S(n) = \max\left(S(p_1^{r_1}), S(p_2^{r_2}), \cdots, S(p_k^{r_k})\right).$$

Lemma 2 ([2]). For any prime p and any positive integer r, we have $S(p^r) \leq pr$.

Lemma 3 ([3], Theorem 274). Let $d(n)$ denote the divisor function of n. Then $d(n)$ is a multiplicative function. Namely, if (2) is the factorization of n, then

$$d(n) = (r_1 + 1)(r_2 + 1) \cdots (r_k + 1).$$

Lemma 4. The inequality

$$\frac{n}{d(n)} < 2, n \in N. \tag{3}$$

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has only the solutions $n = 1, 2, 3, 4$ and 6.

Proof. For any positive integer n, let

$$f(n) = \frac{n}{d(n)}.$$

Since $f(1) = 1$, $f(2) = 1$, $f(3) = 3/2$, $f(4) = 4/3$, and $f(6) = 3/2$, (3) has solutions $n = 1, 2, 3, 4$ and 6.

Let n be a solution of (3) with $n \neq 1, 2, 3, 4$ or 6. Since $f(5) = \frac{5}{2} > 2$, we have $n > 6$. Let (2) be the factorization of n. If $k = 1$ and $r_1 = 1$, then $n = p_1 \geq 7$ and $2 > f(n) = \frac{p_1}{2} \geq \frac{7}{2}$, a contradiction. If $k = 1$ and $r_1 = 2$, then $n = p_1^2$, where $p_1 \geq 3$. So we have $2 > f(n) = \frac{p_1^{r_1}}{(r_1+1)} \geq \frac{2^3}{4} \geq 2$, a contradiction. If $k = 2$, since $n > 6$, then we get

$$2 > f(n) = \frac{p_1^{r_1}}{r_1 + 1} \cdot \frac{p_2^{r_2}}{r_2 + 1} \geq \begin{cases} \frac{5}{2} & \text{if } p_1 = 2 \text{ and } r_1 = 1, \\ 2 & \text{if } p_1 = 2 \text{ and } r_1 > 1, \\ \frac{15}{4} & \text{if } p_1 > 2, \end{cases}$$

a contradiction. If $k \geq 3$, then

$$2 > f(n) = \frac{p_1^{r_1}}{(r_1 + 1)} \frac{p_2^{r_2}}{(r_2 + 1)} \frac{p_3^{r_3}}{(r_3 + 1)} \geq \frac{15}{4},$$

a contradiction. To sum up, (3) has no solution n with $n \neq 1, 2, 3, 4$ or 6. The Lemma is proved.

Proof of Theorem. Let n be a Smarandache perfect number with $n \neq 12$. By [1] we have $n > 10^6$. By Lemma 1, if (2) is the factorization of n, Then

$$S(n) = S(p^r), \tag{4}$$

where

$$p = p_j, \quad r = r_j, \quad 1 \leq j \leq k. \tag{5}$$

From (2) and (5), we get

$$n = p^r m, \quad m \in N, \quad \texttt{gcd}(p^r, m) = 1. \tag{6}$$

For any positive integer n, let

$$g(n) = \sum_{d|n} S(d). \tag{7}$$

Then, by (1), the Smarandache perfect number n satisfies

$$g(n) = n + 1 + S(n). \tag{8}$$

We see from (4) that $n|S(p^r)!$. Therefore, for any divisor d of n, we have

$$S(d) \leq S(p^r). \tag{9}$$

Thus, if (8) holds, then from (7) and (9) we obtain

$$d(n)S(p^r) > n. \tag{10}$$

where $d(n)$ is the divisor function of n. Further, by Lemma 3, we get from (4), (6) and (10) that

$$\frac{(r+1)S(p^r)}{p^r} > f(m). \tag{11}$$

If $r = 1$, since $S(p) = p$, then from (11) we get $2 > f(m)$. Hence, by Lemma 4, we obtain $m = 1, 2, 3, 4$ or 6. When $m = 1$, we get from (8) that

$$g(n) = g(p) = S(1) + S(p) = 1 + p = p + 1 + S(p) = 1 + 2p,$$

a contradiction. When $m = 2$, we have $p > 2$ and

$$g(n) = g(p) = S(1) + S(2) + S(p) + S(2p) = 3 + 2p = 3p + 1, \tag{12}$$

whence we get $p = 2$, a contradiction. By the same method, we can prove that if $r = 1$ and $m = 3, 4$ or 6, then (8) is false.

If $r = 2$, since $S(p^2) = 2p$, then from (11) we get

$$\frac{6}{p} > f(m). \tag{13}$$

Since $n > 10^6$, by (4) we have $S(p^2) = S(n) \geq 10$ it implies that $p \geq 5$. Hence, by (13) we get $f(m) < \frac{6}{5}$. Further, by Lemma 4 we get $m \leq 6$. Since $n = p^2 m \leq 6p^2$, we obtain $p \geq 7$. Therefore, by (13) it is impossible. By the same method, we can prove that if $r = 3, 4, 5$ or 6, then (11) is false.

If $r \geq 7$, then we have $S(p^r) \leq pr$ and

$$\frac{(r+1)r}{p^{r-1}} > \frac{(r+1)S(p^r)}{p^r} > f(m) \geq 1, \tag{14}$$

by (11). From (14), we get

$$(r+1)r > p^{r-1} \geq 2^{r-1} \geq 2\left(\binom{r-1}{0} + \binom{r-1}{1} + \binom{r-1}{2} + \binom{r-1}{3} \right), \tag{15}$$

whence we obtain

$$0 > r^2 - 6r + 5 = (r-1)(r-5) > 0, \tag{16}$$

a contradiction. To sum up, there has no Smarandache perfect number n with $n > 10^6$. Thus, the theorem is proved.

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On the solutions of an equation involving the Smarandache dual function

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Abstract In this paper, we use the elementary method to study the solutions of an equation involving the Smarandache dual function $\bar{s}_k(n)$, and give its all solutions.

Keywords Smarandache dual function, the positive integer solutions.

§1. Introduction

For any positive integer n, the famous Smarandache function $S(n)$ is defined by

$$S(n) = \max\{m : n \mid m!\}.$$

For example, $S(1) = 1$, $S(2) = 2$, $S(3) = 3$, $S(4) = 4$, $S(5) = 5$, $S(6) = 3$, $S(7) = 7$, $S(8) = 4$, $\cdots$. About the arithmetical properties of $S(n)$, many scholars have show their interest on it, see [1], [2] and [3]. For example, Farris Mark and Mitchell Patrick [2] studied the bounding of Smarandache function, and they gave an upper and lower bound for $S(p^\alpha)$, i.e.

$$(p-1)\alpha + 1 \le S(p^\alpha) \le (p-1)[\alpha + 1 + \log_p \alpha] + 1.$$

Wang Yongxing [3] studied the mean value of $\sum\limits_{n\le x} S(n)$ and obtained an asymptotic formula by using the elementary methods. He proved that

$$\sum_{n\le x} S(n) = \frac{\pi^2}{12}\frac{x^2}{\ln x} + O\left(\frac{x^2}{\ln^2 x}\right).$$

Similarly, many scholars studied another function which have close relations with the Smarandache function. It is called the Smarandache dual function $S^*(n)$ which defined by

$$S^*(n) = \max\{m : m! \mid n\}.$$

About this function, J. Sandor in [4] conjectured that

$$S^*((2k-1)!(2k+1)!) = q - 1,$$

where k is a positive integer, q is the first prime following $2k+1$. This conjecture was proved by Le Maohua [5].

Li Jie [6] studied the mean value property of $\sum_{n\leq x} S^*(n)$ by using the elementary methods, and obtained an interesting asymptotic formula:

$$\sum_{n\leq x} S^*(n) = ex + O\left(\ln^2 x(\ln\ln x)^2\right).$$

In this paper, we introduce another Smarandache dual function $\bar{s}_k(n)$ which denotes the greatest positive integer m such that $m^k|n$, where n denotes any positive integer. That is,

$$\bar{s}_k(n) = \max\{m : m^k|n\}.$$

On the other hand, we let $\Omega(n)$ denotes the number of the prime divisors of n, including multiple numbers. If $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}$ denotes the factorization of n into prime powers, then

$$\Omega(n) = \alpha_1 + \alpha_2 \cdots + \alpha_r.$$

In this paper, we shall study the positive integer solutions of the equation

$$\bar{s}_3(1) + \bar{s}_3(2) + \cdots + \bar{s}_3(n) = 3\Omega(n),$$

and give its all solutions. That is, we shall prove the following conclusions:

Theorem. For all positive integer n, the equation

$$\bar{s}_3(1) + \bar{s}_3(2) + \cdots + \bar{s}_3(n) = 3\Omega(n)$$

has only three solutions. They are $n = 3,\ 6,\ 8$.

For general positive integer $k > 3$, whether there exists finite solutions for the equation

$$\bar{s}_k(1) + \bar{s}_k(2) + \cdots + \bar{s}_k(n) = k\Omega(n).$$

It is an unsolved problem. We believe that it is true.

§2. Proof of the theorem

In this section, we will complete the proof of Theorem. First we will separate all positive integer into two cases.

1. If $n \leq 8$, then from the definition of $\bar{s}_k(n)$ and $\Omega(n)$, we have

$$\bar{s}_3(1) = 1, \quad \bar{s}_3(2) = 1, \quad \bar{s}_3(3) = 1, \quad \bar{s}_3(4) = 1,$$

$$\bar{s}_3(5) = 1, \quad \bar{s}_3(6) = 1, \quad \bar{s}_3(7) = 1, \quad \bar{s}_3(8) = 2.$$

$$\Omega(1) = 0, \quad \Omega(2) = 1, \quad \Omega(3) = 1, \quad \Omega(4) = 2,$$

$$\Omega(5) = 1, \quad \Omega(6) = 2, \quad \Omega(7) = 1, \quad \Omega(8) = 3.$$

So that we have

$$\bar{s}_3(1) + \bar{s}_3(2) + \bar{s}_3(3) = 3\Omega(3);$$

$$\bar{s}_3(1) + \bar{s}_3(2) + \cdots + \bar{s}_3(6) = 3\Omega(6);$$

$$\bar{s}_3(1) + \bar{s}_3(2) + \cdots + \bar{s}_3(8) = 3\Omega(8).$$

Hence $n = 3,\ 6,\ 8$ are the positive integer solutions of the equation.

2. If $n > 8$, then we have the following:

Lemma. For all positive integer $n > 8$, we have

$$\bar{s}_3(1) + \bar{s}_3(2) + \cdots + \bar{s}_3(n) > 3\Omega(n).$$

Proof. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ is the factorization of n into prime powers, then we have

$$\bar{s}_3(1) + \bar{s}_3(2) + \cdots + \bar{s}_3(n) > n \quad \text{if} \quad n > 8.$$

From the definition of $\Omega(n)$, we have

$$\Omega(n) = \alpha_1 + \alpha_2 \cdots + \alpha_r.$$

So to complete the proof of the lemma, we only prove the following inequality:

$$p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} > 3(\alpha_1 + \alpha_2 \cdots + \alpha_r). \tag{1}$$

Now we prove (1) by mathematical induction on r.

i) If $r = 1$, then $n = p_1^{\alpha_1}$.

a. If $p_1 = 2$, then we have $\alpha_1 \geq 4$, hence

$$2^4 > 3 \cdot 4, \quad 2^{\alpha_1} > 3\alpha_1.$$

b. If $p_1 = 3$, 5 and 7, then we have $\alpha_1 \geq 2$, hance

$$i^4 > 3 \cdot 2, \quad i^{\alpha_1} > 3\alpha_1, \quad i = 3,\ 5,\ 7.$$

c. If $p_1 \geq 11$, then we have $\alpha_1 \geq 1$, hence

$$p_1^{\alpha_1} > 3\alpha_1.$$

This proved that Lemma holds for $r = 1$.

ii) Now we assume (1) holds for r (≥ 2), and prove that it is also holds for $r + 1$.

From the inductive hypothesis, we have

$$p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} p_{r+1}^{\alpha_{r+1}} > 3(\alpha_1 + \alpha_2 \cdots + \alpha_r) \cdot p_{r+1}^{\alpha_{r+1}}.$$

Since p_{r+1} is a prime, then

$$p_{r+1}^{\alpha_{r+1}} > \alpha_{r+1} + 1.$$

From above we obtain

$$p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} p_{r+1}^{\alpha_{r+1}} > 3(\alpha_1 + \alpha_2 \cdots + \alpha_r) \cdot (\alpha_{r+1} + 1).$$

Note that if $a > 1$, $b > 1$, then $a \cdot b \geq a + b$, so we have

$$(\alpha_1 + \alpha_2 \cdots + \alpha_r) \cdot (\alpha_{r+1} + 1) \geq \alpha_1 + \alpha_2 \cdots + \alpha_r + \alpha_{r+1} + 1 > \alpha_1 + \alpha_2 \cdots + \alpha_r + \alpha_{r+1}.$$

So

$$p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} p_{r+1}^{\alpha_{r+1}} > 3(\alpha_1 + \alpha_2 \cdots + \alpha_r + \alpha_{r+1}).$$

This completes the proof of the lemma.

Now we complete the proof of Theorem. From the lemma we know that the equation has no positive solutions if $n > 8$. In other words, the equation

$$\bar{s}_3(1) + \bar{s}_3(2) + \cdots + \bar{s}_3(n) = 3\Omega(n)$$

has only three solutions. They are $n = 3, 6, 8$.

This completes the proof of Theorem.

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