ON THE PRIMITIVE NUMBERS OF POWER P AND ITS ASYMPTOTIC PROPERTY *

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Abstract Let p be a prime, n be any positive integer, $S_p(n)$ denotes the smallest integer $m \in N^+$, where $p^n|m!$. In this paper, we study the mean value properties of $S_p(n)$, and give an interesting asymptotic formula for it.

Keywords: Smarandache function; Primitive numbers; Asymptotic formula

§1. Introduction and results

Let p be a prime, n be any positive integer, $S_p(n)$ denotes the smallest integer such that $S_p(n)!$ is divisible by p^n. For example, $S_3(1) = 3$, $S_3(2) = 6$, $S_3(3) = 9$, $S_3(4) = 9$, $\cdots\cdots$. In problem 49 of book [1], Professor F. Smarandache ask us to study the properties of the sequence $\{S_p(n)\}$. About this problem, Professor Zhang and Liu in [2] have studied it and obtained an interesting asymptotic formula. That is, for any fixed prime p and any positive integer n,

$$S_p(n) = (p-1)n + O\left(\frac{p}{\ln p} \cdot \ln n\right).$$

In this paper, we will use the elementary method to study the asymptotic properties of $S_p(n)$ in the following form:

$$\frac{1}{p}\sum_{n\leq x}\left|S_p(n+1) - S_p(n)\right|,$$

where x be a positive real number, and give an interesting asymptotic formula for it. In fact, we shall prove the following result:

Theorem. *For any real number $x \geq 2$, let p be a prime and n be any positive integer. Then we have the asymptotic formula*

$$\frac{1}{p}\sum_{n\leq x}\left|S_p(n+1) - S_p(n)\right| = x\left(1 - \frac{1}{p}\right) + O\left(\frac{\ln x}{\ln p}\right).$$

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§2. Proof of the Theorem

In this section, we shall complete the proof of the theorem. First we need following one simple Lemma. That is,

Lemma. *Let p be a prime and n be any positive integer, then we have*

$$|S_p(n+1) - S_p(n)| = \begin{cases} p, & \textit{if } p^n \parallel m!; \\ 0, & \textit{otherwise}, \end{cases}$$

where $S_p(n) = m$, $p^n \parallel m!$ denotes that $p^n | m!$ and $p^{n+1} \dagger m!$.

Proof. Now we will discuss it in two cases.

(i) Let $S_p(n) = m$, if $p^n \parallel m!$, then we have $p^n | m!$ and $p^{n+1} \dagger m!$. From the definition of $S_p(n)$ we have $p^{n+1} \dagger (m+1)!$, $p^{n+1} \dagger (m+2)!$, $\cdots$, $p^{n+1} \dagger (m+p-1)!$ and $p^{n+1} | (m+p)!$, so $S_p(n+1) = m+p$, then we get

$$|S_p(n+1) - S_p(n)| = p. \tag{1}$$

(ii) Let $S_p(n) = m$, if $p^n | m!$ and $p^{n+1} | m!$, then we have $S_p(n+1) = m$, so

$$|S_p(n+1) - S_p(n)| = 0. \tag{2}$$

Combining (1) and (2), we can easily get

$$|S_p(n+1) - S_p(n)| = \begin{cases} p, & \text{if } p^n \parallel m!; \\ 0, & \text{otherwise.} \end{cases}$$

This completes the proof of Lemma.

Now we use above Lemma to complete the proof of Theorem. For any real number $x \geq 2$, by the definition of $S_p(n)$ and Lemma we have

$$\frac{1}{p} \sum_{n \leq x} |S_p(n+1) - S_p(n)| = \frac{1}{p} \sum_{\substack{n \leq x \\ p^n \parallel m!}} p = \sum_{\substack{n \leq x \\ p^n \parallel m!}} 1, \tag{3}$$

where $S_p(n) = m$. Note that if $p^n \parallel m!$, then we have (see reference [3], Theorem 1.7.2)

$$\begin{aligned} n &= \sum_{i=1}^{\infty} \left[\frac{m}{p^i}\right] = \sum_{i \leq \log_p m} \left[\frac{m}{p^i}\right] \\ &= m \cdot \sum_{i \leq \log_p m} \frac{1}{p^i} + O\left(\log_p m\right) \\ &= \frac{m}{p-1} + O\left(\frac{\ln m}{\ln p}\right). \end{aligned} \tag{4}$$

From (4), we can deduce that

$$m = (p-1)n + O\left(\frac{p \ln n}{\ln p}\right). \tag{5}$$

So that

$$1 \leq m \leq (p-1)\cdot x + O\left(\frac{p \ln x}{\ln p}\right), \quad \text{if} \quad 1 \leq n \leq x.$$

Note that for any fixed positive integer n, if there has one m such that $p^n \parallel m!$, then $p^n \parallel (m+1)!$, $p^n \parallel (m+2)!$, $\cdots$, $p^n \parallel (m+p-1)!$. Hence there have p times of m such that $n = \sum_{i=1}^{\infty} \left[\frac{m}{p^i}\right]$ in the interval $1 \leq m \leq (p-1)\cdot x + O\left(\frac{p \ln x}{\ln p}\right)$. Then from this and (3), we have

$$\begin{aligned}
\frac{1}{p}\sum_{n\leq x} |S_p(n+1) - S_p(n)| &= \frac{1}{p}\sum_{\substack{n\leq x \\ p^n\|m!}} p = \sum_{\substack{n\leq x \\ p^n\|m!}} 1 \\
&= \frac{1}{p}\left((p-1)\cdot x + O\left(\frac{p\ln x}{\ln p}\right)\right) \\
&= x\cdot\left(1-\frac{1}{p}\right) + O\left(\frac{\ln x}{\ln p}\right).
\end{aligned}$$

This completes the proof of Theorem.

References

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