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A proof of Smarandache-Patrascu's theorem using barycentric coordinates

Claudiu Coanda

The National College "Carol I" Craiova, Romania

Abstract In this article we prove the Smarandache-Patrascu's theorem in relation to the inscribed orthohomological triangles using the barycentric coordinates.

Keywords Smarandache-Patrascu's theorem, barycentric coordinates.

Definition. Two triangles and ABC and $A_1B_1C_1$, where $A_1 \in BC$, $B_1 \in AC$, $C_1 \in AB$, are called inscribed ortho homological triangles if the perpendiculars in A_1, B_1, C_1 on BC, AC, AB respectively are concurrent.

Observation. The concurrency point of the perpendiculars on the triangle ABC's sides from above definition is the orthological center of triangles ABC and $A_1B_1C_1$.

Smarandache-Patrascu Theorem. If the triangles ABC and $A_1B_1C_1$ are orthohomological, then the pedal triangle $A_1^{'}B_1^{'}C_1^{'}$ of the second center of orthology of triangles ABC and $A_1B_1C_1$, and the triangle ABC are orthohomological triangles.

Proof. Let $P(\alpha, \beta, \gamma)$, $\alpha + \beta + \gamma = 1$, be the first orthologic center of triangles ABC and $A_1B_1C_1$ (See Figure 1).

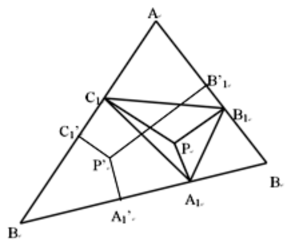

Fig. 1

The perpendicular vectors on the sides are:

$$U_{BC}^{\perp} = \left(2a^2, -a^2 - b^2 + c^2, -a^2 + b^2 - c^2\right),$$

$$U_{CA}^{\perp} = \left(-a^2 - b^2 + c^2, 2b^2, a^2 - b^2 - c^2\right),$$

$$U_{AB}^{\perp} = \left(-a^2 + b^2 - c^2, a^2 - b^2 - c^2, 2c^2\right).$$

We know that:

$$\frac{\overrightarrow{A_1B}}{\overrightarrow{A_1C}} \cdot \frac{\overrightarrow{B_1C}}{\overrightarrow{B_1A}} \cdot \frac{\overrightarrow{C_1A}}{\overrightarrow{C_1B}} = -1, \tag{1}$$

and we want to prove that:

$$\frac{\overrightarrow{A_1'B}}{\overrightarrow{A_1'C}} \cdot \frac{\overrightarrow{B_1'C}}{\overrightarrow{B_1'A}} \cdot \frac{\overrightarrow{C_1'A}}{\overrightarrow{C_1'B}} = -1. \tag{2}$$

We will show that:

$$\frac{\overrightarrow{A_1B}}{\overrightarrow{A_1C}} \cdot \frac{\overrightarrow{B_1C}}{\overrightarrow{B_1A}} \cdot \frac{\overrightarrow{C_1A}}{\overrightarrow{C_1B}} \cdot \frac{\overrightarrow{A_1'B}}{\overrightarrow{A_1'C}} \cdot \frac{\overrightarrow{B_1'C}}{\overrightarrow{B_1'A}} \cdot \frac{\overrightarrow{C_1'A}}{\overrightarrow{C_1'B}} = 1.$$

implies the relation (2)

The equation of the line BC is x=0, and the equation of the line PA_1 is

$$\begin{vmatrix} 0 & y & z \\ \alpha & \beta & \gamma \\ 2a^2 & -a^2-b^2+c^2 & -a^2+b^2-c^2 \end{vmatrix} = 0.$$

It results that:

$$y \cdot \begin{vmatrix} \alpha & \gamma \\ 2a^2 & -a^2+b^2+c^2 \end{vmatrix} = z \cdot \begin{vmatrix} \alpha & \beta \\ 2a^2 & -a^2-b^2+c^2 \end{vmatrix} = 0.$$

Because y+z=1, we find:

$$A_1\left(0, \frac{\alpha}{2a^2}(a^2+b^2-c^2)+\beta, \frac{\alpha}{2a^2}(a^2-b^2-c^2)+\gamma\right).$$

Similarly:

$$B_1\left(\frac{-\beta}{2b^2}(-a^2-b^2+c^2)+\alpha, 0, \frac{-\beta}{2b^2}(a^2-b^2-c^2)+\gamma\right),$$

$$C_1\left(\frac{-\gamma}{2c^2}(-a^2+b^2-c^2)+\alpha, \frac{\gamma}{2c^2}(a^2-b^2-c^2)+\beta, 0\right).$$

We will make the following notations:

$$-a^2+b^2-c^2=i, -a^2-b^2+c^2=j, a^2-b^2-c^2=k,$$

And we have:

$$A_1\left(0, \frac{-\alpha}{2a^2}j+\beta, \frac{-\alpha}{2a^2}i+\gamma\right),$$

$$B_1\left(\frac{-\beta}{2b^2}j+\alpha, 0, \frac{-\beta}{2b^2}k+\gamma\right),$$

$$C_1\left(\frac{-\gamma}{2c^2}i+\alpha, \frac{-\gamma}{2c^2}k+\beta, 0\right),$$

$$\frac{\overrightarrow{A_1B}}{\overrightarrow{A_1C}} = -\frac{\frac{-\alpha}{2a^2}i+\gamma}{\frac{-\alpha}{2a^2}j+\beta};$$

$$\frac{\overrightarrow{B_1C}}{\overrightarrow{B_1A}} = -\frac{\frac{-\beta}{2b^2}j+\alpha}{\frac{-\beta}{2b^2}k+\gamma};$$

$$\frac{\overrightarrow{C_1A}}{\overrightarrow{C_1B}} = -\frac{\frac{-\gamma}{2c^2}k + \beta}{\frac{-\gamma}{2c^2}i + \alpha}.$$

If $P^{'}\left(\alpha^{'}, \beta^{'}, \gamma^{'}\right)$ is the second center of orthology of the triangles ABC and $A_1B_1C_1$, and $A_1^{'}, B_1^{'}, C_1^{'}$ are the projections of $P^{'}$ on BC, AC, AB respectively, similarly, we will find:

$$\frac{\overrightarrow{A_1^{'}B}}{\overrightarrow{A_1^{'}C}} = -\frac{\frac{-\alpha^{'}}{2a^2}i + \gamma^{'}}{\frac{-\alpha^{'}}{2a^2}j + \beta^{'}};$$

$$\frac{\overrightarrow{B_1^{'}C}}{\overrightarrow{B_1^{'}A}} = -\frac{\frac{-\beta^{'}}{2b^2}j + \alpha^{'}}{\frac{-\beta^{'}}{2b^2}k + \gamma^{'}};$$

$$\frac{\overrightarrow{C_1^{'}A}}{\overrightarrow{C_1^{'}B}} = -\frac{\frac{-\gamma^{'}}{2c^2}k + \beta^{'}}{\frac{-\gamma^{'}}{2c^2}i + \alpha^{'}}.$$

It is known the theorem [2].

Theorem. Given two isogonal conjugated points $P(\alpha, \beta, \gamma)$ and $P^{'}(\alpha^{'}, \beta^{'}, \gamma^{'})$ with respect to the triangle ABC (BC=a, CA=b, AB=c), then:

$$\frac{\alpha\alpha^{'}}{a^2} = \frac{\beta\beta^{'}}{b^2} = \frac{\gamma\gamma^{'}}{c^2}.$$

On the other side:

$$\frac{\overrightarrow{A_1B}}{\overrightarrow{A_1C}} \cdot \frac{\overrightarrow{A_1^{'}B}}{\overrightarrow{A_1^{'}C}} = \frac{\left(\frac{-\alpha}{2a^2}i + \gamma\right)\left(\frac{-\alpha^{'}}{2a^2}i + \gamma^{'}\right)}{\left(\frac{-\alpha}{2a^2}j + \beta\right)\left(\frac{-\alpha^{'}}{2a^2}j + \beta^{'}\right)} = \frac{\frac{\alpha\alpha^{'}}{4a^4}i^2 - \frac{\alpha\gamma^{'}}{2a^2}i - \frac{\alpha^{'}\gamma}{2a^2}i + \gamma\gamma^{'}}{\frac{\alpha\alpha^{'}}{4a^4}j^2 - \frac{\alpha\beta^{'}}{2a^2}j - \frac{\alpha^{'}\beta}{2a^2}j + \beta\beta^{'}} = \frac{U_1}{V_1};$$

$$\frac{\overrightarrow{B_1C}}{\overrightarrow{B_1A}} \cdot \frac{\overrightarrow{B_1^{'}C}}{\overrightarrow{B_1^{'}A}} = \frac{\frac{\beta\beta^{'}}{4b^4}j^2 - \frac{\beta\alpha^{'}}{2b^2}j - \frac{\alpha\beta^{'}}{2b^2}j + \alpha\alpha^{'}}{\frac{\beta\beta^{'}}{4b^4}k^2 - \frac{\beta\gamma^{'}}{2b^2}k - \frac{\beta^{'}\gamma}{2b^2}k + \gamma\gamma^{'}} = \frac{U_2}{V_2};$$

$$\frac{\overrightarrow{C_1A}}{\overrightarrow{C_1B}} \cdot \frac{\overrightarrow{C_1^{'}A}}{\overrightarrow{C_1^{'}B}} = \frac{\frac{\gamma\gamma^{'}}{4c^4}k^2 - \frac{\gamma\beta^{'}}{2c^2}k - \frac{\gamma^{'}\beta}{2c^2}k + \beta\beta^{'}}{\frac{\gamma\gamma^{'}}{4c^4}i^2 - \frac{\gamma\alpha^{'}}{2c^2}i - \frac{\gamma^{'}\alpha}{2c^2}i + \alpha\alpha^{'}} = \frac{U_3}{V_3}.$$

The only thing left to be proved is that:

$$\frac{U_1}{V_1} \cdot \frac{U_2}{V_2} \cdot \frac{U_3}{V_3} = 1$$

if and only if

$$\frac{\frac{a^2}{c^2}U_1}{V_1} \cdot \frac{\frac{b^2}{a^2}U_2}{V_2} \cdot \frac{\frac{c^2}{b^2}U_3}{V_3} = 1.$$

We show that

$$\frac{b^2}{a^2}U_2 = V_1, \frac{c^2}{b^2}U_3 = V_2, \frac{a^2}{c^2}U_1 = V_3;$$

$$\frac{b^2}{a^2}U_2 = \frac{\beta\beta^{'}}{4a^2b^2}j^2 - \frac{\beta\alpha^{'}}{2a^2}j - \frac{\alpha\beta^{'}}{2a^2}j + \frac{b^2}{a^2}\alpha\alpha^{'} = \frac{\alpha\alpha^{'}}{4a^4}j^2 - \frac{\beta\alpha^{'}}{2a^2}j - \frac{\beta^{'}\alpha}{2a^2}j + \beta\beta^{'} = V_1;$$

$$\frac{c^2}{b^2}U_3 = \frac{\gamma\gamma^{'}}{4c^2b^2}k^2 - \frac{\gamma\beta^{'}}{2b^2}k - \frac{\gamma^{'}\beta}{2b^2}k + \frac{c^2}{b^2}\beta\beta^{'} = \frac{\beta\beta^{'}}{4b^4}k^2 - \frac{\gamma\beta^{'}}{2b^2}k - -\frac{\gamma^{'}\beta}{2b^2}k + \gamma\gamma^{'} = V_2;$$

$$\frac{a^2}{c^2}U_1 = \frac{\alpha\alpha^{'}}{4a^2c^2}i^2 - \frac{\alpha\gamma^{'}}{2c^2}i - \frac{\alpha^{'}\gamma}{2c^2}i + \frac{a^2}{c^2}\gamma\gamma^{'} = \frac{\gamma\gamma^{'}}{4c^4}i^2 - \frac{\alpha\gamma^{'}}{2c^2}i - -\frac{\alpha^{'}\gamma}{2c^2}i + \alpha\alpha^{'} = V_3.$$

References

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