The p-Arm Theory

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Abstract
We introduce the p-Arm theory which give rise to a new mathematical object that we call the "p-exponential" which is invariant under p derivation. We calculate its derivate and we use this new function to solve differential equations. Next, we define its real and imaginary part which are the p-cosinus and the p-sinus respectively.

Introduction

The Arm theory [1] gives a developpment on any p-th power function basis in changing of variable in the Arm formula. But for functions in $\mathbb{C}[(u(z)-z_0)^p], p \in \mathbb{N}^*$ there is an other way (the p-Arm formula) to make this developpment : instead of changing the variable at the p-th powers, you can also derivate p times which will finally give the same result. This is the main idea behind the p-Arm theory.

The exponential function is the function which leaves invariant the operator in the Taylor formula i.e. :

$$\frac{\partial e^x}{\partial x} = e^x \tag{0.1}$$

So in constructing the p-Arm theory, we see that we need a "p-exponential" ${e_p}^x$ function which leaves the operator of the p-Arm formula invariant :

$$\frac{\partial^p {e_p}^x}{\partial x^p} = {e_p}^x \quad ; \quad \frac{\partial^k {e_p}^x}{\partial x^k} \neq {e_p}^x \tag{0.2}$$

for $1 \leq k < p$. The answer to the question (0.2) is the definition of the p-exponential as follow :

$$e_p^x = \sum_{k=0}^{\infty} \frac{x^{pk}}{(pk)!} \tag{0.3}$$

The p-Arm formula is not so much interesting itself because we already have the developpment by the Arm-theory, but this formula give rise to the p-exponential which is very interesting to study.

In studying the derivate of the p-exponential, we see that this operator acts like a shift operator on the p-exponential and we need a generalization of the "p-exponential" to also include its derivate. This generalized exponential function is :

$$e_{p,\mu}{}^x = \sum_{k=0}^{\infty} \frac{x^{pk+\mu}}{(pk+\mu)!} \tag{0.4}$$

for $p, \mu \in \mathbb{N}^*$. I know that there is already a generalized exponential function in the theory of the fractional calculus (see [2]) which is given by

$$E_\mu^y \equiv \sum_{k=0}^{\infty} \frac{t^{k-\mu}}{\Gamma(k+1-\mu)} \tag{0.5}$$

but which one I introduce here is more generalized because (0.4) has a multiplication and a shift whereas (0.5) has only a shift.

In the first section, we give the equivalent of the Arm formula for the p-Arm theory which we naturally call the p-Arm formula for function in $\mathbb{C}[(u(z)-z_0)^p]$.

In the second section, we give the equivalent shifted Arm formula for the p-Arm theory which we call the shifted p-Arm formula.

In the third section, we give the definition of the generalized exponential function. Next, we draw the six first real p-exponentials which is a beautiful graph. In effect, we explain why the p-th derivate of the p-exponential is itself. In this case, we calculate the derivate of the p-exponential. Thereby, we give the relation between the p-exponential and the traditional exponential. This is why we use this result to show that every function solving that its p-th derivate is itself can be expressed as a linear combination of p-exponential and we give the example of $p = 2$. Then defining the complex p-exponential, we give its real part called the p-cosinus and we draw the six first p-cosinus. Furthermore, we define the p-sinus which is the imaginary part of the complex p-exponential and we draw the six first of it. Finally, we define the p-tangent and we draw the six first p-tangent.

1 The p-Arm Formula

First we introduce the generalization to each basis $u(z)$ of the well known Taylor formula which is written in the basis $u(z) = z$ for each basis of the space $\mathbb{C}[(u(z)-z_0)^p]$ =span$\{1, (u(z)-z_0)^p, (u(z)-z_0)^{2p}, ...\}$

Theorem 1. *$\forall u(z) \in \mathcal{C}(\mathbb{C})$ if $\exists z \in \mathbb{C}$ such that $u(z) = z_0 \in \mathbb{C}$ then $\forall f(z) \in \mathbb{C}[(u(z)-z_0)^p]$*

$$f(z) = \sum_{k=0}^{\infty} \frac{1}{(pk)!} \left[\lim_{z \to u^{-1}(z_0)} \left(\frac{\partial z}{\partial u} \frac{\partial}{\partial z} \right)^{pk} f(z) \right] \big(u(z) - z_0\big)^{pk} \tag{1.6}$$

Proof :

It's enough to show this formula on the basis $\left\{(u(z)-z_0)^{pr}\right\}_{r\in\mathbb{N}}$.

If $k < r$:

$$\begin{aligned}
\frac{1}{(pk)!} \lim_{z \to u^{-1}(z_0)} \left(\frac{\partial z}{\partial u} \frac{\partial}{\partial z} \right)^{pk} \big(u(z)-z_0\big)^{pr} &= \frac{1}{(pk)!} \lim_{z \to u^{-1}(z_0)} \partial_{u(z)}^{pk} \big(u(z)-z_0\big)^{pr} \\
&= \frac{1}{(pk)!} \lim_{z \to u^{-1}(z_0)} \frac{(pr)!}{(p(r-k))!} \big(u(z)-z_0\big)^{p(r-k)} \\
\frac{1}{(pk)!} \lim_{z \to u^{-1}(z_0)} \left(\frac{\partial z}{\partial u} \frac{\partial}{\partial z} \right)^{pk} \big(u(z)-z_0\big)^{pr} &= 0 \qquad (1.7)
\end{aligned}$$

If $k > r$:

$$\begin{aligned}
\frac{1}{(pk)!} \lim_{z \to u^{-1}(z_0)} \left(\frac{\partial z}{\partial u} \frac{\partial}{\partial z} \right)^{pk} \big(u(z)-z_0\big)^{pr} &= \frac{1}{(pk)!} \lim_{z \to u^{-1}(z_0)} \left(\frac{\partial}{\partial u(z)} \right)^{pk} \big(u(z)-z_0\big)^{pr} \\
&= \frac{1}{(pk)!} \lim_{z \to u^{-1}(z_0)} \partial_{u(z)}^{p(k-r)} (pr)! \\
\frac{1}{(pk)!} \lim_{z \to u^{-1}(z_0)} \left(\frac{\partial z}{\partial u} \frac{\partial}{\partial z} \right)^{pk} \big(u(z)-z_0\big)^{pr} &= 0 \qquad (1.8)
\end{aligned}$$

If $k = r$:

$$\begin{aligned}
\frac{1}{(pk)!} \lim_{z \to u^{-1}(z_0)} \left(\frac{\partial z}{\partial u} \frac{\partial}{\partial z} \right)^{pk} \big(u(z)-z_0\big)^{pr} &= \lim_{z \to u^{-1}(z_0)} \frac{(pr)!}{(pk)!} \\
\frac{1}{(pk)!} \lim_{z \to u^{-1}(z_0)} \left(\frac{\partial z}{\partial u} \frac{\partial}{\partial z} \right)^{pk} \big(u(z)-z_0\big)^{pr} &= 1 \qquad (1.9)
\end{aligned}$$

So we can see that :

$$\frac{1}{(pk)!} \lim_{z \to u^{-1}(z_0)} \left(\frac{\partial z}{\partial u} \frac{\partial}{\partial z} \right)^{pk} \big(u(z)-z_0\big)^{pr} = \delta_{k,r} \tag{1.10}$$

♦

2 The Shifted p-Arm Formula

If you have a function $f \in \mathbb{C}[(u(z-z_0)^{-p}] \oplus \mathbb{C}[(u(z)-z_0)^p]$, you can know it if the coefficients on the negative basis are zeros before the infinity.

Theorem 2. *$\forall u(z) \in \mathcal{C}(\mathbb{C})$ if $\exists z \in \mathbb{C}$ such that $u(z) = z_0 \in \mathbb{C}$ then*
$\forall f(z) \in \mathbb{C}[(u(z)-z_0)^p] \oplus \mathbb{C}[(u(z)-z_0)^{-p}]$

$$f(z) = \sum_{k=-m_p(u,f)}^{\infty} \frac{1}{(p(k+m_p(u,f)))!}\left[\lim_{z\to u^{-1}(z_0)} \left(\frac{\partial z}{\partial u}\frac{\partial}{\partial z}\right)^{p(k+m_p(u,f))} (u(z)-z_0)^{pm_p(u,f)} f(z)\right]\left(u(z)-z_0\right)^{pk} \tag{2.11}$$

where the integer $m_p(u,f) \in \mathbb{N}$ is given by :

$$m_p(u,f) = \lim_{z\to u^{-1}(z_0)} -\frac{\ln(f(z))}{p\ln(u(z)-z_0)} < \infty \tag{2.12}$$

<u>Proof :</u>
Let $f(z)$ has the decomposition

$$f(z) = \sum_{k=-m_p(u,f)}^{\infty} \alpha_{pk}(u(z)-z_0)^{pk} = \sum_{k=0}^{\infty} \alpha_{p(k-m_p(u,f))}(u(z)-z_0)^{pk}(u(z)-z_0)^{-pm_p(u,f)} \tag{2.13}$$

where $\alpha_{pk} =< f, (u(z)-z_0)^{pk} >$. Pratically, we determine m_p in calculating

$$\begin{aligned}
\lim_{z\to u^{-1}(z_0)} -\frac{\ln(f(z))}{p\ln(u(z)-z_0)} &= \lim_{z\to u^{-1}(z_0)} -\frac{\ln(\sum_{k=-m(u,f)}^{\infty}\alpha_k(u(z)-z_0)^k)}{p\ln(u(z)-z_0)} \\
&= \lim_{z\to u^{-1}(z_0)} -\frac{\ln(\alpha_{-m(u,f)}(u(z)-z_0)^{-m(u,f)})}{p\ln(u(z)-z_0)} \\
\lim_{z\to u^{-1}(z_0)} -\frac{\ln(f(z))}{p\ln(u(z)-z_0)} &= m(u,f)
\end{aligned} \tag{2.14}$$

Inserting (2.13) in (1.6), we deduce

$$(u(z)-z_0)^{pm_p(u,f)}f(z) = \sum_{k=0}^{\infty}\frac{1}{(pk)!}\left[\lim_{z\to u^{-1}(z_0)}\left(\frac{\partial z}{\partial u}\frac{\partial}{\partial z}\right)^{pk}(u(z)-z_0)^{pm_p(u,f)}f(z)\right]\left(u(z)-z_0\right)^{pk} \tag{2.15}$$

from which we deduce (2.11) in changing $k' = k - m_p(u,f)$.

♦

Remark 1. *If you consider the shifted p-Arm formula (2.11) for $p = 2, u(z) = e^{iz}$ and $z_0 = 0$, you will check that :*

$$\cos^2(z) = \frac{e^{2iz} + 2 + e^{-2iz}}{4} \tag{2.16}$$

with $m_2(e^{iz}, \cos^2) = 1$.

The shifted p-Arm formula gives rise to a new mathematical function which make one the limit in the formula (2.11).

3 The p-exponential

Definition 1. *We define the generalised exponential function :*

$$e_{p,\mu}{}^{x} = \sum_{k=0}^{\infty} \frac{x^{kp+\mu}}{(kp+\mu)!} \tag{3.17}$$

for $p, \mu \in \mathbb{N}^*$.

In the rest of this paper, we will call $e_{p,0}{}^{x} = e_{p}{}^{x}$ the "p-exponential".

Now because we want see what are these new function, we draw the 6 first real p-exponentials :

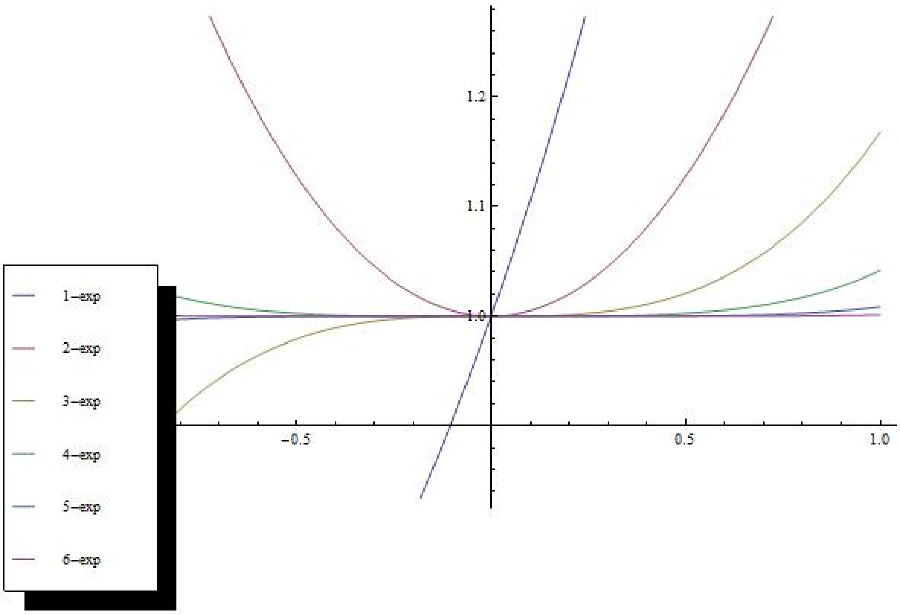

FIGURE 1 – The six first p-exponentials

We now explain why is this function interesting

Proposition 1. *The p-exponential is a function such that*

$$\frac{\partial^p e_p{}^{x}}{\partial x^p} = e_p{}^{x} \quad \textit{and} \quad \frac{\partial^l e_p{}^{x}}{\partial x^l} \neq e_p{}^{x} \tag{3.18}$$

for each $1 \leq l < p$.

Proof :

$$\begin{aligned}
\frac{\partial^p {e_p}^x}{\partial x^p} &= \frac{\partial^p}{\partial x^p} \sum_{k=0}^{\infty} \frac{x^{pk}}{(pk)!} \\
&= \sum_{k=1}^{\infty} \frac{(pk)!}{(pk-p)!} \frac{x^{pk}}{(pk)!} \\
&= \sum_{k=1}^{\infty} \frac{x^{pk-p}}{(pk-p)!} \\
&= \sum_{k=0}^{\infty} \frac{x^{pk}}{(pk)!} \\
\frac{\partial^p {e_p}^x}{\partial x^p} &= {e_p}^x \qquad (3.19)
\end{aligned}$$

The second part of (3.18) is trivial.

♦

Now, we calculate the derivative of the p-exponential

Proposition 2. *The derivate of the p-exponential is given by :*

$$\frac{\partial {e_p}^x}{\partial x} = {e_{p,p-1}}^x \qquad (3.20)$$

where $p \in \mathbb{N}^*$.

Proof :

$$\begin{aligned}
\frac{\partial {e_p}^x}{\partial x} &= \frac{\partial}{\partial x} \sum_{k=0}^{\infty} \frac{x^{pk}}{(pk)!} \\
&= \sum_{k=1}^{\infty} (pk) \frac{x^{pk-1}}{(pk)!} \\
&= \sum_{k=1}^{\infty} \frac{x^{pk-1}}{(pk-1)!} \\
&= \sum_{k=0}^{\infty} \frac{x^{pk+p-1}}{(pk+p-1)!} \\
\frac{\partial {e_p}^x}{\partial x} &= {e_{p,p-1}}^x \qquad (3.21)
\end{aligned}$$

♦

Remark 2. *Of course we have*

$$\frac{\partial e_p{}^{u(x)}}{\partial x} \quad = \quad \frac{\partial u}{\partial x} \; e_{p,p-1}{}^{u(x)} \tag{3.22}$$

Remark 3. *We see that because of (3.20), we have :*

$$\frac{\partial^k e_p{}^x}{\partial x^k} = e_{p,p-k}{}^x \tag{3.23}$$

for $1 \leq k \leq p$. So the derivation acts like a shift operator on the p-exponential.

Now we show an interesting relation which link the p-exponential with the traditional exponential.

Proposition 3. *The link between the p-exponential and the usual exponential is given by :*

$$\left(\sum_{\mu=0}^{p-1} \frac{\partial^\mu}{\partial x^\mu} \right) e_p{}^x \quad = \quad e^x \tag{3.24}$$

or equivalently :

$$\sum_{\mu=0}^{p-1} e_{p,\mu}{}^x \quad = \quad e^x \tag{3.25}$$

Proof :

$$\begin{aligned}
\left(\sum_{\mu=0}^{p-1} \frac{\partial^\mu}{\partial x^\mu} \right) e_p{}^x \quad &= \quad e_p{}^x \; + \; \frac{\partial}{\partial x} \; e_p{}^x \; + \; ... \; + \; \frac{\partial^{p-1}}{\partial x^{p-1}} \; e_p{}^x \\
&= \quad \sum_{k=0}^{\infty} \frac{x^{pk}}{(pk)!} \; + \; \frac{\partial}{\partial x} \sum_{k=0}^{\infty} \frac{x^{pk}}{(pk)!} \; + \; ... + \; \frac{\partial^{p-1}}{\partial x^{p-1}} \sum_{k=0}^{\infty} \frac{x^{pk}}{(pk)!} \\
&= \quad \sum_{k=0}^{\infty} \frac{x^{pk}}{(pk)!} \; + \; \sum_{k=1}^{\infty} \frac{x^{pk-1}}{(pk-1)!} \; + \; ... + \; \sum_{k=1}^{\infty} \frac{x^{pk-p+1}}{(pk-p+1)!} \\
&= \quad \sum_{k=0}^{\infty} \frac{x^{pk}}{(pk)!} \; + \; \sum_{k=0}^{\infty} \frac{x^{pk+p-1}}{(pk+p-1)!} \; + \; ... + \; \sum_{k=0}^{\infty} \frac{x^{pk+1}}{(pk+1)!} \\
&= \quad e_p{}^x \; + \; e_{p,p-1}{}^x \; + \; ... + \; e_{p,1}{}^x \\
\left(\sum_{\mu=0}^{p-1} \frac{\partial^\mu}{\partial x^\mu} \right) e_p{}^x \quad &= \quad e^x
\end{aligned} \tag{3.26}$$

♦

Now we introduced the p-exponential, we can use it to solve somes differential equations. In fact, this is why I created it, the exponential solve the limit of the first order differential equation in the traditional Taylor formula whereas the p-exponential solve the limit of the pth order differential equation in (1.6).

Proposition 4. *Let the differential equation*

$$\frac{\partial^p u(x)}{\partial x^p} = u(x) \tag{3.27}$$

$\exists \alpha_1, ..., \alpha_p$ *such that the solution of (3.27) can be expressed as :*

$$u(x) = \sum_{k=1}^{p} \alpha_k \ \ {e_p}^{{\omega_p}^k x} \tag{3.28}$$

where $\omega_p = e^{\frac{2i\pi}{p}}$ *is the p-th root of unity.*

Proof :

$$\begin{aligned}
\frac{\partial^p u(x)}{\partial x^p} &= \sum_{k=1}^{p} \alpha_k \ \frac{\partial^p}{\partial x^p} \ {e_p}^{{\omega_p}^k x} \\
&= \sum_{k=1}^{p} \alpha_k \ \left(\frac{\partial({\omega_p}^k x)}{\partial x} \ \frac{\partial}{\partial({\omega_p}^k x)} \right)^p \ {e_p}^{{\omega_p}^k x} \\
&= \sum_{k=1}^{p} \alpha_k \ \omega_p^{pk} \ {e_p}^{{\omega_p}^k x} \\
\frac{\partial^p u(x)}{\partial x^p} &= u(x)
\end{aligned} \tag{3.29}$$

♦

Example :

As an example of (3.27), we solve the well-know case :

$$\frac{\partial^2 u(x)}{\partial x^2} = u(x) \tag{3.30}$$

The formula (3.28) gives the solution :

$$\begin{aligned}
u(x) &= \alpha_1 \ {e_2}^x + \alpha_2 \ {e_2}^{-x} \\
u(x) &= \alpha_1 \ \cosh(x) \ + \ \alpha_2 \ \cosh(-x)
\end{aligned} \tag{3.31}$$

where $\alpha_1, \alpha_2 \in \mathbb{C}$ depend on the initial conditions.

Now we define the p-cosinus and p-sinus functions

Definition 2. *The p-cosinus is the real part of the complex exponential given by*

$$\cos_p(x) = \frac{{e_p}^{ix} + {e_p}^{-ix}}{2} \tag{3.32}$$

We draw the 6 first p-cosinus

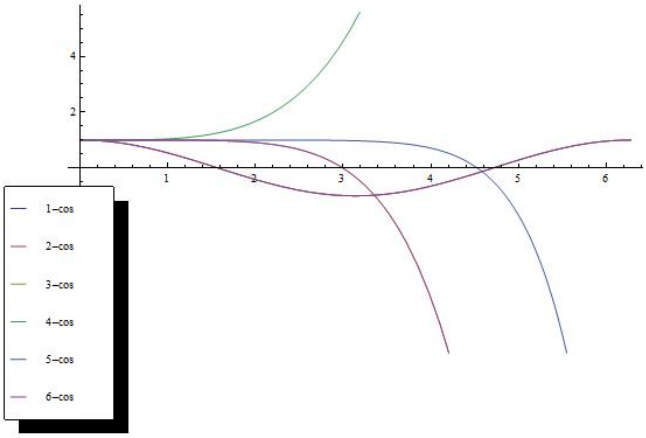

FIGURE 2 – The six first p-cosinus

Definition 3. *The p-sinus is the imaginary part of the complex exponential given by*

$$\sin_p(x) = \frac{{e_p}^{ix} - {e_p}^{-ix}}{2i} \tag{3.33}$$

We draw the 6 first p-sinus

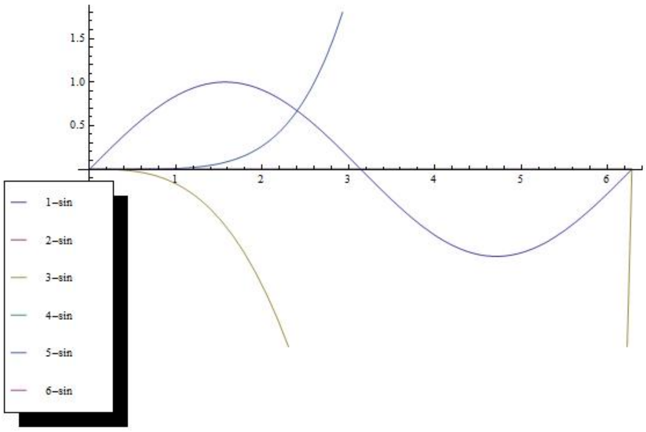

FIGURE 3 – The six first p-sinus

Definition 4. *The p-tangent is given by*

$$\tan_p(x) = \frac{\sin_p(x)}{\cos_p(x)} \tag{3.34}$$

We draw the 6 first p-tangent

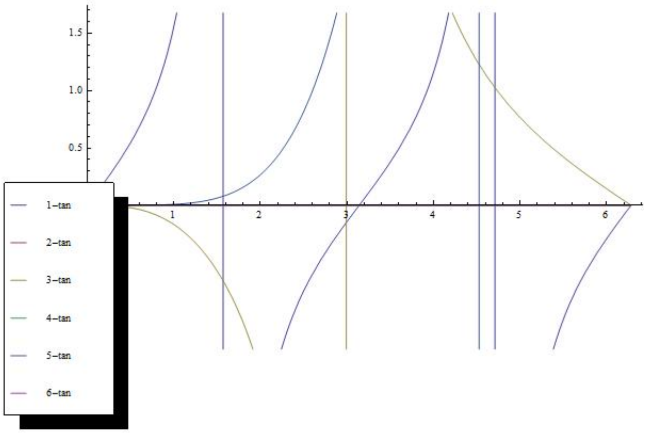

FIGURE 4 – The six first p-tangent

Discussion

Even if the limit of the sum of two elements seems to be

$$\lim_{x+y\to\infty} e_p^{x+y} \quad = \quad \frac{1}{p}\ e^x\ e^y \tag{3.35}$$

on the graph for $p \geq 2$, I didn't find a simple relation between the sum of arguments and the product of exponentials. In a same way, we don't have an equivalent of the Moivre formula which links the n-th power of the exponential with the multiplication with n of the argument. However this relation seems to exist on the graph if we consider it in the infinity limit :

$$\lim_{x\to\infty}\ e_p^{nx} \quad = \quad \frac{1}{p}\ (e^x)^n \tag{3.36}$$

for $p \geq 2$

In addition I also search for the value of the module of the p-exponential but it seems to not have a fixed valued on the graph. So on the graph, it seems to be :

$$\lim_{x\to\infty}\ \cos_p^2(x)\ +\ \sin_p^2(x)\ =\ \infty \tag{3.37}$$

for $p \geq 3$. There is an exception for $p = 2$ because $e_2 = \cosh$ and we have that :

$$|{e_2}^{ix}| \quad = \quad \cos(x) \tag{3.38}$$

For now, I didn't find yet the inverse function of the p-exponential or of the generalized exponential function. I tried finding an expression for the derivate of the "p-logarithm" :

$$\left.\frac{\partial \ln_p(x)}{\partial x}\right|_{x={e_p}^{x}} \quad = \quad \frac{1}{{e_{p,p-1}}^{x}} \tag{3.39}$$

but we need a relation between the p-exponential ${e_p}^{x}$ and its derivate ${e_{p,p-1}}^{x}$ other than the derivation relation itself.

Références

[1] Arm B. N., The Arm Theory

[2] Bologna M., Short Introduction to Fractional Calculus