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On the F.Smarandache LCM Ratio Sequence

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Abstract In this paper, we use the elementary methods to study the F.Smarandache LCM ratio sequence, and obtain three interesting recurrence relations for it.

Keywords Elementary method, Smarandache LCM ratio sequences, recurrence relation.

§1. Introduction

Let $(x_1, x_2, ..., x_t)$ and $[x_1, x_2, ..., x_t]$ denote the greatest common divisor and the least common multiple of any positive integers $x_1, x_2, ..., x_t$ respectively. Let r be a positive integer with $r > 1$. For any positive integer n, let

$$T(r,n) = \frac{[n, n+1, ..., n+r-1]}{[1, 2, ..., r]},$$

then the sequences $SLR(r) = T(r,n)_\infty$ is called the F.Samarandache LCM ratio sequences of degree r. In reference [1], Murthy asked us to find a reduction formula for $T(r,n)$. Maohua Le [2] solved this open problem for $r = 3$ and 4. That is, he proved that

$$T(3,n) = \begin{cases} \frac{1}{6}n(n+1)(n+2), & \text{if } n \text{ is odd,} \\ \frac{1}{12}n(n+1)(n+2), & \text{if } n \text{ is even.} \end{cases}$$

$$T(4,n) = \begin{cases} \frac{1}{24}n(n+1)(n+2)(n+3), & \text{if } n \not\equiv 0 \pmod 3, \\ \frac{1}{72}n(n+1)(n+2)(n+3), & \text{if } n \equiv 0 \pmod 3. \end{cases}$$

Furthermore, Wang Ting [3] and [4] computing the value of $T(5,n)$ and $T(6,n)$. For example, he obtained the identity

$$T(5,n) = \begin{cases} \frac{1}{1440}n(n+1)(n+2)(n+3)(n+4), & \text{if } n \equiv 0,\ 8 \pmod{12}, \\ \frac{1}{120}n(n+1)(n+2)(n+3)(n+4), & \text{if } n \equiv 1,\ 7 \pmod{12}, \\ \frac{1}{720}n(n+1)(n+2)(n+3)(n+4), & \text{if } n \equiv 2,\ 6 \pmod{12}, \\ \frac{1}{360}n(n+1)(n+2)(n+3)(n+4), & \text{if } n \equiv 3,\ 5,\ 9,\ 11 \pmod{12}, \\ \frac{1}{480}n(n+1)(n+2)(n+3)(n+4), & \text{if } n \equiv 4 \pmod{12}, \\ \frac{1}{240}n(n+1)(n+2)(n+3)(n+4), & \text{if } n \equiv 10 \pmod{12}. \end{cases}$$

In this paper, we study the recurrence relations between $T(r+1,n)$ and $T(r,n)$, and get three interesting recurrence formulas for it. That is, we shall prove the following conclusions:

Theorem 1. For any natural number n and r, we have the recurrence formula:

$$T(r+1,n) = \frac{n+r}{r+1} \cdot \frac{([1,2,...,r],r+1)}{([n,n+1,...,n+r-1],n+r)} \cdot T(r,n).$$

Especially, if both $r+1$ and $n+r$ are primes, then we can get a simple formula

$$T(r+1,n) = \frac{n+r}{r+1} \cdot T(r,n).$$

Theorem 2. For each natural number n and r, we also have another recurrence formula:

$$T(r,n+1) = \frac{n+r}{n} \cdot \frac{(n,[n+1,...,n+r])}{([n,n+1,...,n+r-1],n+r)} \cdot T(r,n).$$

Especially, if both n and $n+r$ are primes with $r < n$, then we can also get a simple formula

$$T(r,n+1) = \frac{n+r}{n} \cdot T(r,n);$$

If both n and $n+r$ are primes with $r \geq n$, then we have

$$T(r,n+1) = (n+r) \cdot T(r,n).$$

Theorem 3. For each natural number n and r, we have

$$\begin{aligned} T(r+1,n+1) \quad &= \frac{n+r}{n} \cdot \frac{n+r+1}{r+1} \cdot \frac{([1,2,...,r],r+1)}{([n+1,...,n+r],n+r+1)} \\ &\cdot \frac{(n,[n+1,...,n+r])}{([n,n+1,...,n+r-1],n+r)} \cdot T(r,n). \end{aligned}$$

§2. Some Lemmas

To complete the proof of the above theorems, we need the following several Lemmas.

Lemma 1. For any positive integers a and b, we have $(a,b)[a,b] = ab$.

Lemma 2. For any positive integers s and t with $s < t$, we have

$$(x_1,x_2,...,x_t) = ((x_1,...,x_s),(x_{s+1},...,x_t))$$

and

$$[x_1,x_2,...,x_t] = [[x_1,...,x_s],[x_{s+1},...,x_t]].$$

The proof of Lemma 1 and Lemma 2 can be found in [3].

§3. Proof of the theorems

In this section, we shall complete the proof of the theorems.

First we prove Theorem 1. According to the definition of $T(r,n)$, Lemma 1 and Lemma 2, we have:

$$\begin{aligned}
T(r+1,n) &= \frac{[n,n+1,...,n+r]}{[1,2,...,r+1]} \\
&= \frac{[[n,n+1,...,n+r-1],n+r]}{[[1,2,...,r],r+1]} \\
&= \frac{\frac{(n+r)[n,n+1,...,n+r-1]}{([n,n+1,...,n+r-1],n+r)}}{\frac{(r+1)[1,...,r]}{([1,2,...,r],r+1)}} \\
&= \frac{n+r}{r+1}\frac{[n,n+1,...,n+r-1]}{[1,2,...,r]}\cdot\frac{([1,2,...,r],r+1)}{([n,n+1,...,n+r-1],n+r)} \\
&= \frac{n+r}{r+1}\cdot\frac{([1,2,...,r],r+1)}{([n,n+1,...,n+r-1],n+r)}T(r,n).
\end{aligned}$$

It is easily to get

$$T(r+1,n) = \frac{n+r}{r+1}T(r,n)$$

if both $r+1$ and $n+r$ are primes. Because at this time

$$([1,2,...,r],r+1) = 1$$

and

$$([n,n+1,...,n+r-1],n+r) = 1.$$

This proves Theorem 1.

Now we prove Theorem 2. From the Lemmas and the definition of $T(r,n)$, we have

$$\begin{aligned}
T(r,n+1) &= \frac{[n+1,...,n+r]}{[1,2,...,r]} \\
&= \frac{[n,n+1,...,n+r](n,[n+1,...,n+r])}{n}\cdot\frac{1}{[1,2,...,r]} \\
&= \frac{(n,[n+1,...,n+r])}{n[1,2,...,r]}\cdot\frac{[n,n+1,...,n+r-1](n+r)}{([n,n+1,...,n+r-1],n+r)} \\
&= \frac{n+r}{n}\frac{(n,[n+1,...,n+r])}{([n,n+1,...,n+r-1],n+r)}\cdot\frac{[n,n+1,...,n+r-1]}{[1,2,...,r]} \\
&= \frac{n+r}{n}\cdot\frac{(n,[n+1,...,n+r])}{([n,n+1,...,n+r-1],n+r)}T(r,n).
\end{aligned}$$

If n and $n+r$ are primes with $n < r$, then we can also get a simple formula

$$T(r,n+1) = \frac{n+r}{n}T(r,n);$$

If n and $n+r$ are primes with $n \geq r$, this time note that $(n,[n+1,...,n+r]) = n$, we have

$$T(r,n+1) = (n+r)\cdot T(r,n).$$

This proves Theorem 2.

The proof of Theorem 3. Applying Theorem 1 and Theorem 2 we can easily get identity

$$T(r+1,n+1)=\frac{n+r+1}{r+1}\cdot\frac{([1,2,...,r],r+1)}{([n+1,...,n+r],n+r+1)}T(r,n+1)$$

$$=\frac{(n+r+1)(n+r)}{(r+1)n}\cdot\frac{([1,2,...,r],r+1)}{([n+1,...,n+r],n+r+1)}\cdot\frac{(n,[n+1,...,n+r])}{([n,...,n+r-1],n+r)}T(r,n).$$

This completes the proof of Theorem 3.

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