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A note on q-nanlogue of Sá́ndor's functions

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Abstract The additive analogues of Pseudo-Smarandache, Smarandache-simple functions and their duals have been recently studied by J. Sándor. In this note, we obtain q-analogues of Sándor's theorems [6].

Keywords q-gamma function; Pseudo-Smarandache function; Smarandache-simple function; Asymtotic formula.

Dedicated to Sun-Yi Park on 90th birthday

§1. Introduction

The additive analogues of Smarandache functions S and S_* have been introduced by Sándor [5] as follows:

$$S(x) = \min\{m \in N : x \leq m!\}, \qquad x \in (1, \infty),$$

and

$$S_*(x) = \max\{m \in N : m! \leq x\}, \qquad x \in [1, \infty),$$

He has studied many important properties of S_* relating to continuity, differentiability and Riemann integrability and also p roved the following theorems:

Theorem 1.1.

$$S_* \sim \frac{\log x}{\log \log x} \qquad (x \to \infty).$$

Theorem 1.2. The series

$$\sum_{n=1}^{\infty} \frac{1}{n(S_*(n))^{\alpha}},$$

is convergent for $\alpha > 1$ and divergent for $\alpha \leq 1$.

In [1], Adiga and Kim have obtained generalizations of Theorems 1.1 and 1.2 by the use of Euler's gamma function. Recently Adiga-Kim-Somashekara-Fathima [2] have established a q-analogues of these results on employing analogues of Pseudo-Smarandache, Smarandache-simple functions and their duals as follows:

$$Z(x) = \min\left\{m \in N : x \leq \frac{m(m+1)}{2}\right\}, \qquad x \in (0, \infty),$$

$$Z_*(x) = \max\left\{m \in N : \frac{m(m+1)}{2} \leq x\right\}, \qquad x \in [1, \infty),$$

$$P(x) = \min\{m \in N : p^x \leq m!\}, \quad p > 1, \quad x \in (0, \infty),$$

$$P_*(x) = \max\{m \in N : m! \leq p^x\}, \quad p > 1, \quad x \in [1, \infty).$$

He has also proved the following theorems:

Theorem 1.3.

$$Z_* \sim \frac{1}{2}\sqrt{8x+1} \qquad (x \to \infty).$$

Theorem 1.4. The series

$$\sum_{n=1}^{\infty} \frac{1}{(Z_*(n))^\alpha},$$

is convergent for $\alpha > 2$ and divergent for $\alpha \leq 2$. The series

$$\sum_{n=1}^{\infty} \frac{1}{n(Z_*(n))^\alpha},$$

is convergent for all $\alpha > 0$.

Theorem 1.5.

$$\log P_*(x) \sim \log x \quad (x \to \infty),$$

Theorem 1.6. The series

$$\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{\log\log n}{\log P_*(n)}\right)^\alpha$$

is convergent for all $\alpha > 1$ and divergent for $\alpha \leq 1$.

The main purpose of this note is to obtain q-analogues of Sándor's Theorems 1.3 and 1.5. In what follows, we make u se of the following notations and definitions. F. H. Jackson defined a q-analogues of the gamma function which extends the q-factorial

$$(n!)_q = 1(1+q)(1+q+q^2)\cdots(1+q+q^2+\cdots+q^{n-1}), \quad \text{cf} \quad [3],$$

which becomes the ordinary factorial as $q \to 1$. He defined the q-analogue of the gamma function as

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \quad 0 < q < 1,$$

and

$$\Gamma_q(x) = \frac{(q^{-1};q^{-1})_\infty}{(q^{-x};q^{-1})_\infty}(1-q)^{1-x}q^{\binom{x}{2}}, q > 1,$$

where

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n).$$

It is well known that $\Gamma_q(x) \to \Gamma(x)$ as $q \to 1$, where $\Gamma(x)$ is the ordinary gamma function.

§2. Main Theorems

We now defined the q-analogues of Z and Z_* as follows:

$$Z_q(x) = \min\left\{\frac{1-q^m}{1-q} : x \leq \frac{\Gamma_q(m+2)}{2\Gamma_q(m)}\right\}, \quad m \in N, \quad x \in (0,\infty),$$

and

$$Z_q^*(x) = \max\left\{\frac{1-q^m}{1-q} : \frac{\Gamma_q(m+2)}{2\Gamma_q(m)} \leq x\right\}, \quad m \in N, \quad x \in \left[\frac{\Gamma_q(m+2)}{2\Gamma_q(1)}, \infty\right),$$

where $0 < q < 1$. Clearly, $Z_q(x) \to Z(x)$ and $Z_q^*(x) \to Z_*(x)$ as $q \to 1^-$. From the definitions of Z_q and Z_q^*, it is clear that

$$Z_q(x) = \left\{\begin{array}{ll} 1, & \text{if } x \in \left(0, \frac{\Gamma_q(3)}{2\Gamma_q(1)}\right] \\ \frac{1-q^m}{1-q}, & \text{if} x \in \left(\frac{\Gamma_q(m+1)}{2\Gamma_q(m-1)}, \frac{\Gamma_q(m+2)}{2\Gamma_q(m)}\right], m \geq 2, \end{array}\right] \tag{1}$$

and

$$Z_q^* = \frac{1-q^m}{1-q} \quad \text{if} \quad x \in \left[\frac{\Gamma_q(m+2)}{2\Gamma_q(m)}, \frac{\Gamma_q(m+3)}{2\Gamma_q(m+1)}\right). \tag{2}$$

Since

$$\frac{1-q^{m-1}}{1-q} \leq \frac{1-q^m}{1-q} = \frac{1-q^{m-1}}{1-q} + q^{m-1} \leq \frac{1-q^{m-1}}{1-q} + 1,$$

(1) and (2) imply that for $x > \frac{\Gamma_q(3)}{2\Gamma_q(1)}$,

$$Z_q^* \leq Z_q \leq Z_q^* + 1.$$

Hence it suffices to study the function Z_q^*. We now prove our main theorems.

Theorem 2.1. If $0 < q < 1$, then

$$\frac{\sqrt{1+8xq}-(1+2q)}{2q^2} < Z_q^* \leq \frac{\sqrt{1+8xq}-1}{2q}, \quad x \geq \frac{\Gamma_q(3)}{2\Gamma_q(1)}.$$

Proof. If

$$\frac{\Gamma_q(k+2)}{2\Gamma_q(k)} \leq x < \frac{\Gamma_q(k+3)}{2\Gamma_q(k+1)}, \tag{3}$$

then

$$Z_q^* = \frac{1-q^k}{1-q}$$

and

$$(1-q^k)(1-q^{k+1}) - 2x(1-q)^2 \le 0 < (1-q^{k+1})(1-q^{k+2}) - 2x(1-q)^2. \qquad (4)$$

Consider the functions f and g defined by

$$f(y) = (1-y)(1-yq) - 2x(1-q)^2$$

and

$$g(y) = (1-yq)(1-yq^2) - 2x(1-q)^2.$$

Note that f is monotonically decreasing for $y \le \frac{1+q}{2q}$ and g is strictly decreasing for $y \le \frac{1+q}{2q^2}$. Also $f(y_1) = 0 = g(y_2)$ where

$$y_1 = \frac{(1+q) - (1-q)\sqrt{1+8xq}}{2q},$$

$$y_2 = \frac{(q+q^2) - q(1-q)\sqrt{1+8xq}}{2q^3}.$$

Since $y_1 \le \frac{1+q}{2q}$, $y_2 \le \frac{1+q}{2q^2}$ and $q^k < \frac{1+q}{2q} < \frac{1+q}{2q^2}$, from (4), it follows that

$$f(q^k) \le f(y_1) = 0 = g(y_2) < g(q^k).$$

Thus $y_1 < q^k < y_2$ and hence

$$\frac{1-y_2}{1-q} < \frac{1-q^k}{1-q} < \frac{1-y_1}{1-q}.$$

i.e.

$$\frac{\sqrt{1+8xq} - (1+2q)}{2q^2} < Z_q^* \le \frac{\sqrt{1+8xq} - 1}{2q}.$$

This completes the proof.

Remark. Letting $q - 1^-$ in the above theorem, we obtain Sándor's Theorem 1.3.

We define the q-analogues of P and P_* as follows:

$$P_q(x) = \min\{m \in N : p^x \le \Gamma_q(m+1)\}, \quad p > 1, \quad x \in (0, \infty),$$

and

$$P_q^*(x) = \max\{m \in N : \Gamma_q(m+1) \le p^x\}, \quad p > 1, \quad x \in [1, \infty),$$

where $0 < q < 1$. Clearly, $P_q(x) \to P(x)$ and $P_q^* \to P_*(x)$ as $q \to 1^-$. From the definitions of P_q and P_q^*, we have

$$P_q^*(x) \le P_q(x) \le P_q^*(x) + 1.$$

Hence it is enough to study the function P_q^*.

Theorem 2.2. If $0 < q < 1$, then

$$P_*(x) \sim \frac{x \log p}{\log\left(\frac{1}{1-q}\right)} \qquad (x \to \infty).$$

Proof. If $\Gamma_q(n+1) \leq p^x < \Gamma_q(n+2)$, then

$$P_q^*(x) = n$$

and

$$\log \Gamma_q(n+1) \leq \log p^x < \log \Gamma_q(n+2). \tag{5}$$

But by the q-analogue of Stirling's formula established by Moak [4], we have

$$\log \Gamma_q(n+1) \sim \left(n+\frac{1}{2}\right)\log\left(\frac{q^{n+1}}{q-1}\right) \sim n\log\left(\frac{1}{1-q}\right). \tag{6}$$

Dividing (5) throughout by $n\log\left(\frac{1}{1-q}\right)$, we obtain

$$\frac{\log \Gamma_q(n+1)}{n\log\left(\frac{1}{1-q}\right)} \leq \frac{x\log p}{P_q^*(x)\log\left(\frac{1}{1-q}\right)} < \frac{\log \Gamma_q(n+2)}{n\log\left(\frac{1}{1-q}\right)}. \tag{7}$$

Using (6) and (7), we deduce

$$\lim_{x\to\infty} \frac{x\log p}{P_q^*(x)\log\left(\frac{1}{1-q}\right)} = 1.$$

This completes the proof.

References

[1] C. Adiga and T. Kim, On a generalization of Sándor's function, Proc. Jangjeon Math. Soc., **5**(2002), 121-124.

[2] C. Adiga, T. Kim, D. D. Somashekara and N. Fathima, On a q-analogue of Sándor's function, J. Inequal. Pure and Appl. Math., **4**(2003), 1-5.

[3] T. Kim, Non-archimedean q-integrals with multiple Changhee q-Bernoulli polynomials, Russian J. Math. Phys., **10**(2003), 91-98.

[4] D. S. Moak, The q-analogue of Stirling's formula, Rocky Mountain J. Math., **14**(1984), 403-413.

[5] J. Sándor, On an additive analogue of the function S, Note Numb. Th. Discr. Math., **7**(2001), 91-95.

[6] J. Sándor, On an additive analogue of certain arithmetic function, J. Smarandache Notions, **14**(2004), 128-133.