Scientia Magna
Vol. 2 (2006), No. 2, 20-23

A new limit theorem involving the Smarandache LCM sequence

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Abstract The main purpose of this paper is using the elementary method to study the LCM Sequence, and give an asymptotic formula about this sequence.

Keywords Smarandache LCM Sequence, limitation.

§1. Introduction and results

For any positive integer n, we define $L(n)$ is the Least Common Multiply (LCM) of the natural number from 1 through n. That is

$$L(n) = [1, 2, \cdots, n].$$

The Smarandache Least Common Multiply Sequence is defined by:

$$\text{SLS} \longrightarrow L(1),\ L(2),\ L(3),\ \cdots,\ L(n),\ \cdots.$$

The first few numbers are: $1,\ 2,\ 6,\ 12,\ 60,\ 60, 420,\ 840,\ 2520,\ 2520,\ \cdots.$

About some simple arithmetical properties of $L(n)$, there are many results in elementary number theory text books. For example, for any positive integers a, b and c, we have

$$[a,b] = \frac{ab}{(a,b)} \quad \text{and} \quad [a,b,c] = \frac{abc\cdot(a,b,c)}{(a,b)(b,c)(c,a)},$$

where $(a_1, a_2, \cdots, a_k)$ denotes the Greatest Common Divisor of a_1, a_2, $\cdots$, a_{k-1} and a_k. But about the deeply arithmetical properties of L(n), it seems that none had studied it before, but it is a very important arithmetical function in elementary number theory. The main purpose of this paper is using the elementary methods to study a limit problem involving $L(n)$, and give an interesting limit theorem for it. That is, we shall prove the following:

Theorem. For any positive integer n, we have the asymptotic formula

$$\left(\frac{L(n^2)}{\prod\limits_{p\le n^2} p}\right)^{\frac{1}{n}} = e + O\left(\exp\left(-c\frac{(\ln n)^{\frac{3}{5}}}{(\ln\ln n)^{\frac{1}{5}}}\right)\right),$$

where $\prod\limits_{p\le n^2}$ denotes the production over all prime $p \le n^2$.

From this Theorem we may immediately deduce the following:

Corollary. Under the notations of above, we have

$$\lim_{n\to\infty}\left(\frac{L(n^2)}{\prod\limits_{p\le n^2} p}\right)^{\frac{1}{n}} = e,$$

where L$(n^2) = [1, 2, \cdots, n^2]$, p is a prime.

§2. Proof of the theorem

In this section, we shall complete the proof of this theorem. First we need the following simple Lemma.

Lemma. For $x > 0$, we have the asymptotic formula

$$\theta(x) = \sum_{p\le x} \ln p = x + O\left(x\exp\left(\frac{-c(\ln x)^{\frac{3}{5}}}{(\ln\ln x)^{\frac{1}{5}}}\right)\right),$$

where $c > 0$ is a constant, $\sum\limits_{p\le x}$ denotes the summation over all prime $p \le x$.

Proof. In fact, this is the different form of the famous prime theorem. Its proof can be found in reference [2].

Now we use this Lemma to prove our Theorem.

Let

$$L(n^2) = [1,\ 2,\ \cdots,\ n^2] = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}, \tag{1}$$

be the factorization of $L(n^2)$ into prime powers, then $\alpha_i = \alpha(p_i)$ is the highest power of p_i in the factorization of 1, 2, 3, $\cdots$, n^2. Since

$$\left(\frac{L(n^2)}{\prod\limits_{p\le n^2} p}\right)^{\frac{1}{n}} = \exp\left(\frac{1}{n}\ln\frac{L(n^2)}{\prod\limits_{p\le n^2} p}\right) = \exp\left(\frac{1}{n}\left(\ln L(n^2) - \ln\prod_{p\le n^2} p\right)\right),$$

while

$$\begin{aligned}
\ln L(n^2) - \ln\prod_{p\le n^2} p &= \ln\left(p_1^{\alpha_1} p_2^{\alpha_2}\cdots p_n^{\alpha_n}\right) - \ln\prod_{p\le n^2} p \\
&= \sum_{p\le n^2}\alpha(p)\ln p - \sum_{p\le n^2}\ln p \\
&= \sum_{p\le n^2}(\alpha(p)-1)\ln p \\
&= \sum_{p\le n^{\frac{2}{3}}}(\alpha(p)-1)\ln p + \sum_{n^{\frac{2}{3}}<p\le n}(\alpha(p)-1)\ln p \\
&\quad + \sum_{n<p\le n^2}(\alpha(p)-1)\ln p.
\end{aligned} \tag{2}$$

In (1), it is clear that if $n < p_i \leq n^2$, then $\alpha(p_i) = 1$. If $n^{\frac{2}{3}} < p_i \leq n$, we have $\alpha(p_i) = 2$. (In fact if $\alpha(p_i) \geq 3$, then $p_i^3 > n$. This contradiction with $p_i \leq n$). If $p_i \leq n^{\frac{2}{3}}$, then $\alpha(p_i) \geq 3$. So from these and above Lemma we have

$$\sum_{n^{\frac{2}{3}} < p \leq n} (\alpha(p) - 1) \ln p = \sum_{n^{\frac{2}{3}} < p \leq n} (2 - 1) \ln p = \sum_{n^{\frac{2}{3}} < p \leq n} \ln p, \tag{3}$$

$$\sum_{n < p \leq n^2} (\alpha(p) - 1) \ln p = \sum_{n < p \leq n^2} (1 - 1) \ln p = 0, \tag{4}$$

$$\sum_{p \leq n^{\frac{2}{3}}} (\alpha(p) - 1) \ln p = O\left(\ln^2 n \sum_{p \leq n^{\frac{2}{3}}} 1\right) = O\left(\ln^2 n \frac{n^{\frac{2}{3}}}{\ln n}\right) = O\left(n^{\frac{2}{3}} \ln n\right). \tag{5}$$

Now combining (2), (3), (4) and (5) we may immediately get

$$\begin{aligned}
\ln L(n^2) - \ln \prod_{p \leq n^2} p &= O\left(n^{\frac{2}{3}} \ln n\right) + \sum_{n^{\frac{2}{3}} < p \leq n} \ln p \\
&= O\left(n^{\frac{2}{3}} \ln n\right) + \sum_{p \leq n} \ln p - \sum_{p \leq n^{\frac{2}{3}}} \ln p \\
&= O\left(n^{\frac{2}{3}} \ln n\right) + n + O\left(n \exp\left(\frac{-c(\ln n)^{\frac{3}{5}}}{(\ln \ln n)^{\frac{1}{5}}}\right)\right) \\
&\quad - n^{\frac{2}{3}} - O\left(n^{\frac{2}{3}} \exp\left(\frac{-c(\ln n^{\frac{2}{3}})^{\frac{3}{5}}}{(\ln \ln n^{\frac{2}{3}})^{\frac{1}{5}}}\right)\right) \\
&= n + O\left(n \exp\left(\frac{-c(\ln n)^{\frac{3}{5}}}{(\ln \ln n)^{\frac{1}{5}}}\right)\right).
\end{aligned}$$

That is,

$$\begin{aligned}
\left(\frac{L(n^2)}{\prod\limits_{p \leq n^2} p}\right)^{\frac{1}{n}} &= \exp\left(\frac{1}{n}\left(\ln L(n^2) - \ln \prod_{p \leq n^2} p\right)\right) \\
&= \exp\left[\frac{1}{n}\left[n + O\left(n \exp\left(\frac{-c(\ln n)^{\frac{3}{5}}}{(\ln \ln n)^{\frac{1}{5}}}\right)\right)\right]\right] \\
&= \exp\left[1 + O\left(\exp\left(\frac{-c(\ln n)^{\frac{3}{5}}}{(\ln \ln n)^{\frac{1}{5}}}\right)\right)\right] \\
&= e \cdot \exp\left[O\left(\exp\left(\frac{-c(\ln n)^{\frac{3}{5}}}{(\ln \ln n)^{\frac{1}{5}}}\right)\right)\right] \\
&= e\left[1 + O\left(\exp\left(\frac{-c(\ln n)^{\frac{3}{5}}}{(\ln \ln n)^{\frac{1}{5}}}\right)\right)\right] \\
&= e + O\left(\exp\left(\frac{-c(\ln n)^{\frac{3}{5}}}{(\ln \ln n)^{\frac{1}{5}}}\right)\right).
\end{aligned}$$

This completes the proof of Theorem.

The Corollary follows from Theorem with $n \to \infty$.

References

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