A Proof of Smarandache-Pătraşcu's Theorem using Barycentric Coordinates

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In this article we prove the Smarandache–Pătraşcu's Theorem in relation to the inscribed orthohomological triangles using the barycentric coordinates.

Will remind the following:

Definition

Two triangles ABC and $A_1B_1C_1$, where $A_1 \in BC$, $B_1 \in AC$, $C_1 \in AB$, are called inscribed ortho homological triangles if the perpendiculars in A_1, B_1, C_1 on BC, AC, AB respectively are concurrent.

Observation

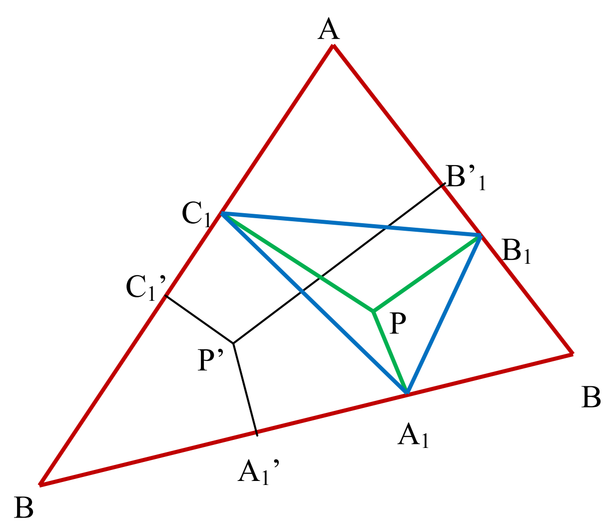
The concurrency point P of the perpendiculars on the triangle ABC's sides from above definition is the orthological center of triangles ABC and $A_1B_1C_1$.

Smarandache–Pătraşcu Theorem

If the triangles ABC and $A_1B_1C_1$ are orthohomological, then the pedal triangle $A_1^{'}B_1^{'}C_1^{'}$ of the second center of orthology of triangles ABC and $A_1B_1C_1$, and the triangle ABC are orthohomological triangles.

Proof

Let $P(\alpha, \beta, \gamma)$, $\alpha + \beta + \gamma = 1$, be the first orthologic center of triangles ABC and $A_1B_1C_1$ (See figure).



The perpendicular vectors on the sides are:

$$\mathcal{U}_{BC}^{\perp}\left(2a^2,-a^2-b^2+c^2,-a^2+b^2-c^2\right)$$

$$\mathcal{U}_{CA}^{\perp}=\left(-a^2-b^2+c^2,2b^2,a^2-b^2-c^2\right)$$

$$\mathcal{U}_{AB}^{\perp}=\left(-a^2+b^2-c^2,a^2-b^2-c^2,2c^2\right)$$

We know that:

$$\frac{\overrightarrow{A_1B}}{\overrightarrow{A_1C}}\cdot\frac{\overrightarrow{B_1C}}{\overrightarrow{B_1A}}\cdot\frac{\overrightarrow{C_1A}}{\overrightarrow{C_1B}}=-1 \qquad (1)$$

and we want to prove that:

$$\frac{\overrightarrow{A_1^{'}B}}{\overrightarrow{A_1^{'}C}}\cdot\frac{\overrightarrow{B_1^{'}C}}{\overrightarrow{B_1^{'}A}}\cdot\frac{\overrightarrow{C_1^{'}A}}{\overrightarrow{C_1^{'}B}}=-1 \qquad (2)$$

We will show that:

$$\frac{\overrightarrow{A_1B}}{\overrightarrow{A_1C}}\cdot\frac{\overrightarrow{B_1C}}{\overrightarrow{B_1A}}\cdot\frac{\overrightarrow{C_1A}}{\overrightarrow{C_1B}}\cdot\frac{\overrightarrow{A_1^{'}B}}{\overrightarrow{A_1^{'}C}}\cdot\frac{\overrightarrow{B_1^{'}C}}{\overrightarrow{B_1^{'}A}}\cdot\frac{\overrightarrow{C_1^{'}A}}{\overrightarrow{C_1^{'}B}}=1$$

implies the relation (2)

The equation of the line BC is $x=0$, and the equation of the line PA_1 is

$$\begin{vmatrix} 0 & y & z \\ \alpha & \beta & \gamma \\ 2a^2 & -a^2-b^2+c^2 & -a^2+b^2-c^2 \end{vmatrix}=0$$

It results that:

$$y\cdot\begin{vmatrix} \alpha & \gamma \\ 2a^2 & -a^2+b^2+c^2 \end{vmatrix}=z\cdot\begin{vmatrix} \alpha & \beta \\ 2a^2 & -a^2-b^2+c^2 \end{vmatrix}.$$

Because $y+z=1$, we find:

$$A_1\left(0,\ \frac{\alpha}{2a^2}\left(a^2+b^2-c^2\right)+\beta,\ \frac{\alpha}{2a^2}\left(a^2-b^2+c^2\right)+\gamma\right).$$

Similarly:

$$B_1\left(\frac{-\beta}{2b^2}\left(-a^2-b^2+c^2\right)+\alpha,\ 0,\ \frac{-\beta}{2b^2}\left(a^2-b^2-c^2\right)+\gamma\right),$$

$$C_1\left(\frac{-\gamma}{2c^2}\left(-a^2+b^2-c^2\right)+\alpha,\ \frac{-\gamma}{2c^2}\left(a^2-b^2-c^2\right)+\beta,\ 0\right)$$

We'll make the following notations:

$$-a^2+b^2-c^2=i;\quad -a^2-b^2+c^2=j,\quad a^2-b^2-c^2=k,$$

And we have:

$$A_1\left(0,\ \frac{-\alpha}{2a^2}j+\beta,\ \frac{-\alpha}{2a^2}i+\gamma\right)$$

$$B_1\left(\frac{-\beta}{2b^2}j+\alpha,\ 0,\ \frac{-\beta}{2b^2}k+\gamma\right)$$

$$C_1\left(\frac{-\gamma}{2c^2}i+\alpha,\ \frac{-\gamma}{2c^2}k+\beta,\ 0\right)$$

$$\frac{\overrightarrow{A_1B}}{\overrightarrow{A_1C}}=-\frac{\frac{-\alpha}{2a^2}i+\gamma}{\frac{-\alpha}{2a^2}j+\beta};$$

$$\frac{\overrightarrow{B_1C}}{\overrightarrow{B_1A}}=-\frac{\frac{-\beta}{2b^2}j+\alpha}{\frac{-\beta}{2b^2}k+\gamma};$$

$$\frac{\overrightarrow{C_1A}}{\overrightarrow{C_1B}}=-\frac{\frac{-\gamma}{2c^2}k+\beta}{\frac{-\gamma}{2c^2}i+\alpha}$$

If $P'(\alpha',\beta',\gamma')$ is the second center of orthology of the triangles ABC and $A_1B_1C_1$, and $A_1^{'}, B_1^{'}, C_1^{'}$ are the projections of P' on $BC,\ AC,\ AB$ respectively, similarly, we'll find:

$$\frac{\overrightarrow{A_1^{'}B}}{\overrightarrow{A_1^{'}C}}=-\frac{\frac{-\alpha'}{2a^2}i+\gamma'}{\frac{-\alpha'}{2a^2}i+\beta'};$$

$$\frac{\overrightarrow{B_1^{'}C}}{\overrightarrow{B_1^{'}A}}=-\frac{\frac{-\beta'}{2b^2}j+\alpha'}{\frac{-\beta'}{2b^2}k+\gamma'};$$

$$\frac{\overrightarrow{C_1^{'}A}}{\overrightarrow{C_1^{'}B}}=-\frac{\frac{-\gamma'}{2c^2}k+\beta'}{\frac{-\gamma'}{2c^2}i+\alpha'}$$

It is known the theorem [2]

Theorem

Given two isogonal conjugated points $P(\alpha,\beta,\gamma)$ and $P'(\alpha',\beta',\gamma')$ with respect to the triangle ABC ($BC=a,\ CA=b,\ AB=c$), then:

$$\frac{\alpha\alpha'}{a^2}=\frac{\beta\beta'}{b^2}=\frac{\gamma\gamma'}{c^2}.$$

On the other side:

$$\frac{\overrightarrow{A_1B}}{\overrightarrow{A_1C}}\cdot\frac{\overrightarrow{A_1'B}}{\overrightarrow{A_1'C}}=\frac{\left(\frac{-\alpha}{2a^2}i+\gamma\right)\left(\frac{-\alpha'}{2a^2}i+\gamma'\right)}{\left(\frac{-\alpha}{2a^2}j+\beta\right)\left(\frac{-\alpha'}{2a^2}j+\beta'\right)}=\frac{\frac{\alpha\alpha'}{4a^4}i^2-\frac{\alpha\gamma'}{2a^2}i-\frac{\alpha'\gamma}{2a^2}i+\gamma\gamma'}{\frac{\alpha\alpha'}{4a^4}j^2-\frac{\alpha\beta'}{2a^2}j-\frac{\alpha'\beta}{2a^2}j+\beta\beta'}=\frac{U_1}{V_1};$$

$$\frac{\overrightarrow{B_1C}}{\overrightarrow{B_1A}}\cdot\frac{\overrightarrow{B_1'C}}{\overrightarrow{B_1'A}}=\frac{\frac{\beta\beta'}{4b^4}j^2-\frac{\beta\alpha'}{2b^2}j-\frac{\alpha\beta'}{2b^2}j+\alpha\alpha'}{\frac{\beta\beta'}{4b^4}k^2-\frac{\beta\gamma'}{2b^2}k-\frac{\beta'\gamma}{2b^2}k+\gamma\gamma'}=\frac{U_2}{V_2},$$

$$\frac{\overrightarrow{C_1A}}{\overrightarrow{C_1B}}\cdot\frac{\overrightarrow{C_1'A}}{\overrightarrow{C_1'B}}=\frac{\frac{\gamma\gamma'}{4c^4}k^2-\frac{\gamma\beta'}{2c^2}k-\frac{\gamma'\beta}{2c^2}k+\beta\beta'}{\frac{\gamma\gamma'}{4c^4}i^2-\frac{\gamma\alpha'}{2c^2}i-\frac{\gamma'\alpha}{2c^2}i+\alpha\alpha'}=\frac{U_3}{V_3}$$

The only thing left to be proved is that:

$$\frac{U_1}{V_1}\cdot\frac{U_2}{V_2}\cdot\frac{U_3}{V_3}=1 \text{ if and only if } \frac{\frac{a^2}{c^2}U_1}{V_1}\cdot\frac{\frac{b^2}{a^2}U_2}{V_2}\cdot\frac{\frac{c^2}{b^2}U_3}{V_3}=1.$$

We show that

$$\frac{b^2}{a^2}U_2=V_1,\ \frac{c^2}{b^2}U_3=V_2,\ \frac{a^2}{c^2}U_1=V_3$$

$$\frac{b^2}{a^2}U_2=\frac{\beta\beta'}{4a^2b^2}j^2-\frac{\beta\alpha'}{2a^2}j-\frac{\alpha\beta'}{2a^2}j+\frac{b^2}{a^2}\alpha\alpha'=\frac{1}{4a^2}\frac{\alpha\alpha'}{a^2}j^2-\frac{\beta\alpha'}{2a^2}j-\frac{\alpha\beta'}{2a^2}j+\beta\beta'=V_1$$

$$\frac{c^2}{b^2}U_3=\frac{\gamma\gamma'}{4c^2b^2}k^2-\frac{\gamma\beta'}{2b^2}k-\frac{\gamma'\beta}{2b^2}k+\frac{c^2}{b^2}\beta\beta'=\frac{\beta\beta'}{4b^4}k^2-\frac{\gamma\beta'}{2b^2}k-\frac{\gamma'\beta}{2b^2}k+\gamma\gamma'=V_2$$

$$\frac{a^2}{c^2}U_1=\frac{\alpha\alpha'}{4a^2c^2}i^2-\frac{\alpha\gamma'}{2c^2}i-\frac{\alpha'\gamma}{2c^2}i+\frac{a^2}{c^2}\gamma\gamma'=\frac{\gamma\gamma'}{4c^4}i^2-\frac{\alpha\gamma'}{2c^2}i-\frac{\alpha'\gamma}{2c^2}i+\alpha\alpha'=V_3$$

References

[1] Ion Pătrașcu and Florentin Smarandache, *A Theorem about Simultaneous Orthological and Homological Triangles*, in arXiv:1004.0347v1, Cornell University, NY, USA, 2010.

[2] Claudiu Coandă, *Geometrie Analitică în coordonate baricentrice*, Editura Reprograph, Craiova, 2005.

[3] Mihai Dicu, *The Smarandache-Pătrașcu Theorem of Orthohomological Triangles*, http://www.scribd.com/doc/28311880/Smarandache-Patrascu-Theorem-of-Orthohomological-Triangles, 2010.