Prove Beal's Conjecture by Fermat's Last Theorem

Zhang Tianshu
Zhanjiang city, Guangdong province, P.R.China.
Email: tianshu_zhang507@aliyun.com

Abstract

In this article, we will prove the Beal's conjecture by certain usual mathematical fundamentals with the aid of proven Fermat's last theorem, and finally reach a conclusion that the Beal's conjecture is tenable.

Keywords

Beal's conjecture, Inequality, Indefinite equation, Fermat's last theorem, Mathematical fundamentals, Odd-even attribute of A, B and C.

The proof

The Beal's Conjecture states that if $A^X+B^Y=C^Z$, where A, B, C, X, Y and Z are all positive integers, and X, Y and Z are greater than 2, then A, B and C must have a common prime factor.

We regard limits of values of above-mentioned A, B, C, X, Y and Z as known requirements, hereinafter.

First, we must remove following two kinds from $A^X+B^Y=C^Z$ under the known requirements.

1. If A, B and C are all positive odd numbers, then A^X+B^Y is an even number, yet C^Z is an odd number, evidently there is only $A^X+B^Y\neq C^Z$ under the known requirements according to an odd number ≠ an even number.

2. If any two of A, B and C are positive even numbers, yet another is a positive odd number, then when A^X+B^Y is an even number, C^Z is an odd number, yet when A^X+B^Y is an odd number, C^Z is an even number, so there is only $A^X+B^Y\neq C^Z$ under the known requirements according to an odd number $\neq$ an even number.

Thus, we reserve merely two kinds of indefinite equation $A^X+B^Y=C^Z$ under the known requirements plus each qualification as listed below.

1. A, B and C are all positive even numbers.

2. A, B and C are two positive odd numbers and a positive even number.

For indefinite equation $A^X+B^Y=C^Z$ under the known requirements plus aforementioned each qualification, in fact, it has many sets of solutions of positive integers. Let us instance following four concrete equations to explain such a viewpoint.

When A, B and C are all positive even numbers, if let A=B=C=2, X=Y=3, and Z=4, then indefinite equation $A^X+B^Y=C^Z$ is exactly equality $2^3+2^3=2^4$. Evidently $A^X+B^Y=C^Z$ has a set of solutions of positive integers (2, 2, 2) here, and A, B and C have common even prime factor 2.

In addition, if let A=B=162, C=54, X=Y=3, and Z=4, then, indefinite equation $A^X+B^Y=C^Z$ is exactly equality $162^3+162^3=54^4$. Evidently $A^X+B^Y=C^Z$ has a set of solutions of positive integers (162, 162, 54) here, and A, B and C have two common prime factors, i.e. even 2 and odd 3.

When A, B and C are two positive odd numbers and a positive even number, if let A=C=3, B=6, X=Y=3, and Z=5, then, indefinite equation

$A^X+B^Y=C^Z$ is exactly equality $3^3+6^3=3^5$. Evidently $A^X+B^Y=C^Z$ has a set of solutions of positive integers (3, 6, 3) here, and A, B and C have common prime factor 3.

In addition, if let A=B=7, C=98, X=6, Y=7, and Z=3, then, indefinite equation $A^X+B^Y=C^Z$ is exactly equality $7^6+7^7=98^3$. Evidently $A^X+B^Y=C^Z$ has a set of solutions of positive integers (7, 7, 98) here, and A, B and C have common prime factor 7.

Thus it can seen that by above-mentioned four concrete examples, we have proved that indefinite equation $A^X+B^Y=C^Z$ under the known requirements plus aforementioned each qualification can exist, but A, B and C have at least one common prime factor.

If we can prove that there is only $A^X+B^Y\neq C^Z$ under the known requirements plus the qualification that A, B and C have not any common prime factor, then, we precisely proven that there is only $A^X+B^Y=C^Z$ under the known requirements plus the qualification that A, B and C must have a common prime factor.

Since when A, B and C are all positive even numbers, A, B and C have common prime factor 2, therefore, for these circumstances that A, B and C have not any common prime factor, they can only occur under the prerequisite that A, B and C are two positive odd numbers and a positive even number.

If A, B and C have not any common prime factor, then any two of them have not any common prime factor either. Because on the supposition that any two of them have a common prime factor, namely A^X+B^Y or C^Z-A^X or C^Z-B^Y have the prime factor, yet another has not it, then, this will lead to $A^X+B^Y \neq C^Z$ or $C^Z-A^X \neq B^Y$ or $C^Z-B^Y \neq A^X$ according to the unique factorization theorem for a positive integer.

Such being the case, provided we can prove that there is only inequality $A^X+B^Y \neq C^Z$ under the known requirements plus the qualification that A, B and C have not any common prime factor, then the Beal's conjecture is surely tenable, otherwise it will be negated.

Unquestionably, following two inequalities together can wholly replace $A^X+B^Y \neq C^Z$ under the known requirements plus the qualification that A, B and C have not any common prime factor.

1. $A^X+B^Y \neq 2^Z G^Z$ under the known requirements plus the qualification that A, B and 2G have not any common prime factor, where 2G=C.

2. $A^X+2^Y D^Y \neq C^Z$ under the known requirements plus the qualification that A, 2D and C have not any common prime factor, where 2D=B.

We again divide $A^X+B^Y \neq 2^Z G^Z$ into two kinds, i.e. (1) $A^X+B^Y \neq 2^Z$, when G=1, and (2) $A^X+B^Y \neq 2^Z G^Z$, where G has at least an odd prime factor >1.

Likewise divide $A^X+2^Y D^Y \neq C^Z$ into two kinds, i.e. (3) $A^X+2^Y \neq C^Z$, when D=1, and (4) $A^X+2^Y D^Y \neq C^Z$, where D has at least an odd prime factor >1.

We will prove that aforesaid four inequalities under the known requirements plus their qualifications are on the existence.

On purpose of the citation for convenience, let us first prove $E^P+F^V\neq 2^M$, where E and F are two positive odd numbers without any common prime divisor, and P, V and M are integers >2. Since E and F have not any common prime factor, so there is $E^P\neq F^V$ according to the unique factorization theorem for a positive integer, then let $F^V>E^P$.

In other words, let us Prove that indefinite equation $E^P+F^V=2^M$ has not a set of solutions of positive integers, where P, V and M are integers >2.

We know that when P is an integer >2, indefinite equation $E^P+1^P=2^P$ has not a set of solutions of positive integers according to proven Fermat's last theorem [REFERENCES], thus E is not a positive integer.

In the light of the same reason, when V is an integer >2, indefinite equation $F^V-1^V=2^V$ has not a set of solutions of positive integers, so F is not a positive integer either.

Next, two sides of equal-sign of $E^P+1^P=2^P$ added respectively to two sides of equal-sign of $F^V-1^V=2^V$ make $E^P+F^V=2^P+2^V$.

For indefinite equation $E^P+F^V=2^P+2^V$, when P=V, $2^P+2^V=2^{P+1}$, so $E^P+F^V=2^{P+1}$. Let P+1=M, there is $E^P+F^V=2^M$, but E and F at here are not two positive integers according to preceding two conclusions. If enable E and F into two positive odd numbers, then, there is only $E^P+F^V\neq 2^M$.

However, when P≠V, $2^P+2^V\neq 2^M$, then $E^P+F^V=2^P+2^V\neq 2^M$, i.e. $E^P+F^V\neq 2^M$, where E and F at here are not two positive integers according to preceding two conclusions. If let E and F turn into two positive odd numbers, then, whether multiply E^P+F^V by a corresponding no positive integer such as μ, or E^P added to a corresponding no positive integer such as ζ, and F^V added to a corresponding no positive integer such as ξ, so whether must multiply 2^P+2^V by μ, or 2^P+2^V must add to ζ + ξ on another side of the equality. Then, a result on another side can only be (2^P+2^V) μ or $2^P+2^V+\zeta+\xi$, and either result $\neq 2^M$, thus when E and F are two positive odd numbers, there is still $E^P+F^V\neq 2^M$.

In a word, we have proven $E^P+F^V\neq 2^M$, where E and F are two positive odd numbers without any common prime divisor, and P, V and M are integers >2.

On the basis of proven $E^P+F^V\neq 2^M$, we just set to prove aforementioned four inequalities, one by one, thereinafter.

Firstly, let $A^X=E^P$, $B^Y=F^V$, and $2^Z=2^M$ for proven $E^P+F^V\neq 2^M$, we get $A^X+B^Y\neq 2^Z$ under the known requirements, where 2 is a value of C.

Secondly, let us successively prove $A^X+B^Y\neq 2^Z G^Z$ under the known requirements plus the qualification that A, B and 2G have not any common prime factor, where 2G=C, and G has at least an odd prime factor >1.

To begin with, multiply each term of proven $E^P+F^V\neq 2^M$ by G^M is $E^P G^M+F^V G^M\neq 2^M G^M$.

For any positive even number, either it is able to be expressed as A^X+B^Y, or it is unable. No doubt, $E^P G^M+F^V G^M$ is a positive even number.

If $E^P G^M+F^V G^M$ is able to be expressed as A^X+B^Y, then there is $A^X+B^Y \neq 2^M G^M$.

If $E^P G^M+F^V G^M$ is unable to be expressed as A^X+B^Y, then it has nothing to do with proving $A^X+B^Y \neq 2^M G^M$.

Under this case, there are still $E^P G^M+F^V G^M \neq A^X+B^Y$ and $E^P G^M+F^V G^M \neq 2^M G^M$, so let $E^P G^M+F^V G^M$ equals A^X+B^Y+2b or A^X+B^Y-2b, where b is a positive integer. Also use sign "±" to denote sign "+" and sign "-" hereinafter, then we get $A^X+B^Y \pm 2b \neq 2^M G^M$, i.e. $A^X+B^Y \neq 2^M G^M \pm 2b$.

Since 2b can express every positive even number, then $2^M G^M \pm 2b$ can express all positive even numbers except for $2^M G^M$.

For a positive even number, either it is able to be expressed as $2^K N^K$, or it is unable, where K is an integer >2, and N is a positive integer which has at least an odd prime factor >1.

On the one hand, where $2^M G^M \pm 2b=2^K N^K$, there is $A^X+B^Y \neq 2^K N^K$. On the other hand, where $2^M G^M \pm 2b \neq 2^K N^K$, $2^M G^M \pm 2b$ has nothing to do with proving $A^X+B^Y \neq 2^K N^K$.

That is to say, for $E^P G^M+F^V G^M \neq 2^M G^M$, if $E^P G^M+F^V G^M$ is unable to be expressed as A^X+B^Y, we can deduce $A^X+B^Y \neq 2^K N^K$ too, elsewhere.

Hereto, we have proven $A^X+B^Y \neq 2^M G^M$ or $A^X+B^Y \neq 2^K N^K$ on the existence.

Since either M or K is to express an integer >2, also either G or N is a positive integer which has at least an odd prime factor >1, therefore both can represent from each other.

Thirdly, we proceed to prove $A^X+2^Y\neq C^Z$ under the known requirements plus the qualification that A and C are two positive odd numbers without any common prime factor, where 2 is a value of B.

In the former passage, we have proven $E^P+F^V\neq 2^M$, where $F^V>E^P$, so let $F^V=C^Z$, then there is $E^P+C^Z\neq 2^M$.

Moreover, let $2^M>2^3$, then there is $2^M=2^{M-1}+2^{M-1}$.

So there is $E^P+C^Z> 2^{M-1}+2^{M-1}$ or $E^P+C^Z< 2^{M-1}+2^{M-1}$.

Namely, there is $C^Z-2^{M-1}>2^{M-1}-E^P$ or $C^Z-2^{M-1}<2^{M-1}-E^P$.

In addition, there is $A^X+E^P\neq 2^{M-1}$ according to proven $E^P+F^V\neq 2^M$.

Then, we deduce $2^{M-1}-E^P>A^X$ or $2^{M-1}-E^P<A^X$ from $A^X+E^P\neq 2^{M-1}$.

Therefore, there is $C^Z-2^{M-1}>2^{M-1}-E^P>A^X$ or $C^Z-2^{M-1}<2^{M-1}-E^P<A^X$.

Consequently, there is $C^Z-2^{M-1}>A^X$ or $C^Z-2^{M-1} < A^X$.

In a word, there is $C^Z-2^{M-1}\neq A^X$, i.e. $A^X+2^{M-1}\neq C^Z$.

For $A^X+2^{M-1}\neq C^Z$, let $2^{M-1}=2^Y$, we get $A^X+2^Y\neq C^Z$.

Fourthly, let us last prove $A^X+2^YD^Y\neq C^Z$ under the known requirements plus the qualification that A, 2D and C have not any common prime factor, where 2D=B, and D has at least an odd prime factor >1.

For the sake that distinguish between differing cases, we need to start using another inequality $H^U+2^Y\neq K^T$ in the light of proven inequality $A^X+2^Y\neq C^Z$, where H and K are two positive odd numbers without any common prime factor, and U, Y and T are integers>2.

Then, there is $K^T-H^U \neq 2^Y$. Like that, multiply each term of $K^T-H^U \neq 2^Y$ by D^Y is $K^TD^Y-H^UD^Y \neq 2^YD^Y$.

For any positive even number, either it is able to be expressed as C^Z-A^X, or it is unable. Undoubtedly, $K^TD^Y-H^UD^Y$ is a positive even number.

If $K^TD^Y-H^UD^Y$ is able to be expressed as C^Z-A^X, then there is $C^Z-A^X \neq 2^YD^Y$, i.e. $A^X+2^YD^Y \neq C^Z$.

If $K^TD^Y-H^UD^Y$ is unable to be expressed as C^Z-A^X, then $K^TD^Y-H^UD^Y$ at here has nothing to do with proving $A^X+2^YD^Y \neq C^Z$. Under this case, there are still $K^TD^Y-H^UD^Y \neq C^Z-A^X$ and $K^TD^Y-H^UD^Y \neq 2^YD^Y$.

Let $K^TD^Y-H^UD^Y$ equals $C^Z-A^X \pm 2d$, where d is a positive integer.

Then, there is $C^Z-A^X \pm 2d \neq 2^YD^Y$, i.e. $C^Z-A^X \neq 2^YD^Y \pm 2d$.

Since 2d can express every positive even number, then $2^YD^Y \pm 2d$ can express all positive even numbers except for 2^YD^Y.

For a positive even number, either it is able to be expressed as 2^SR^S, or it is unable, where S is an integer>2, and R is a positive integer which has at least an odd prime factor >1.

On the one hand, where $2^YD^Y \pm 2d=2^SR^S$, there is $C^Z-A^X \neq 2^SR^S$, i.e. $A^X+2^SR^S \neq C^Z$. On the other hand, where $2^YD^Y \pm 2d \neq 2^SR^S$, $2^YD^Y \pm 2d$ has nothing to do with proving $A^X+2^SR^S \neq C^Z$.

That is to say, for $K^TD^Y-H^UD^Y \neq 2^YD^Y$, if $K^TD^Y-H^UD^Y$ is unable to be expressed as C^Z-A^X, we can deduce $A^X+2^SR^S \neq C^Z$ too, elsewhere.

Thus far, we have proven $A^X+2^YD^Y \neq C^Z$ or $A^X+2^SR^S \neq C^Z$ on the existence. Since either Y or S is to express an integer >2, also either D or R is a positive integer which has at least an odd prime factor >1, therefore both can represent from each other.

To sun up, we have proven every kind of $A^X+B^Y \neq C^Z$ under the known requirements plus the qualification that A, B and C have not any common prime factor.

Previous, we have proven $A^X+B^Y=C^Z$ under the known requirements plus the qualification that A, B and C have at least a common prime factor, it has certain sets of solutions of positive integers.

Overall, after the compare between $A^X+B^Y=C^Z$ under the known requirements and $A^X+B^Y \neq C^Z$ under the known requirements, we have reached inevitably such a conclusion, namely an indispensable prerequisite of the existence of $A^X+B^Y=C^Z$ under the known requirements is that A, B and C must have a common prime factor.

The proof was thus brought to a close. As a consequence, the Beal conjecture is tenable.

REFERENCES: Modular Elliptic Curves and Fermat's Last Theorem, By Andrew Wiles, Annals of Mathematics, Second Series, Vol. 141, №.3, (May, 1995), pp. 443-551.