Graphs & Expressions for Higher-Loop Effective Quantum Action

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Abstract

We present the Feynman graphs and the corresponding expressions, up to 4th loop order, for a generic effective quantum field theory action. Whereas there are 2 graphs in the 2-loop order, and 8 graphs in the 3-loop order, we obtain 43 irreducible graphs in the 4-loop order. These results are obtained using Mathematica programming, where the underlying code is capable of generating graphs and expressions to any desired loop order. We explain the associated programming strategy.

1 Introduction

The effective action framework[1], [2], [3], [4], [5], [6], [7], [8] for computing the higher quantum contributions of field theory is essential in high-energy physics, where fundamental symmetries need to be preserved, and where a unified scheme for computing various higher-order quantum processes need be employed. Whereas the pertinent two-loop expression can be derived[4], [5] easily by analytic manipulations, higher-loop expressions would depend on the use of effective Feynman graphic rules. The 2-loop expression consists of two terms corresponding to two irreducible Feynman graphs. The irreducible Feynman graphs that are needed at the 3-loop order are also known[5], by simple guess-work hand depiction, their number is eight, and the corresponding expressions can be written. However, all higher-loop graphs are unobtainable by hand depictions. The real problem lies in the fact that the number of equivalent graphs can get very large. It is rather difficult to know what are all the possible graphs, what graphs are equivalent, and in many cases, it is difficult to determine the correct combinatoric factor since all the relevant symmetries of the graph may not be clear. Our purpose in this article is to show that computer-aided programming can be very useful in this regard, and shall present the essential steps for how the programming strategy is made. We shall present results up to 4 loops.

The first step in trying to produce the effective Feynman graphs, that are needed at a specific loop order, is to know what are the possible vertices (multi-leg objects) that are needed to produce that number of loops. The second step is to take a specific set of vertices and try all possible ways of joining their legs. The latter process consists of either joining two legs on the same vertex (forming a ring), or joining legs on different vertices (forming a link). In our computational code, we have a function **Diagrets**(L). This gives the various sets of vertices (each with a specific number of legs) that can produce diagrams of the specific L-loop order. The next step is to take a specific *diagret*, and subject it to the function **Contract**(d). This function would take the list of leg numbers that are specified by the diagret list d, then tries to examine all possible graphs. Only irreducible graphs are retained (in the sense of the effective action framework). Disconnected, or *disjointed* graphs, are discarded. Graphs of *equivalent topologies* are recognized and their contributions get combined in a single one. As a byproduct of this treatment, the correct combinatoric factor is always produced.

In the following section, we shall begin by treating the simple case of two loops, show the results of a corresponding computing session, producing the known graphs, and writing the resulting analytic expressions. Subsequent, more extensive, sections would treat the cases of 3 and 4 loops.

2 Two-Loop Contributions

When the command **Diagrets**(2) is executed, we obtain the following result:

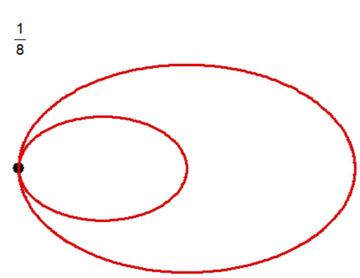
$$\texttt{Diagrets(2)} \Rightarrow \{\{4\},\{3,3\}\} \tag{1}$$

The resulting list tells us that the graphs which are pertinent to the 2-loop effective action are two. One consists of a single quartic vertex, decribed by $\{4\}$. The other consists of two cubic vertices, described by $\{3,3\}$. Now proceeding to the second step of obtaining the corresponding graphs, we apply the command **Contract** to each of the foregoing list items. This gives some analytic expressions with correct combinatoric factors. Subjecting these expressions to the command **ToFeynShow**, the associated graphs are produced.

With the composite command

$$\{4\}\texttt{//Contract//ToFeynShow} \tag{2}$$

we obtain the following graph, with the correct combinatoric factor shown in the upper-left corner:



The above shows how the four legs of the quartic vertex join to form two loops (rings). It should be clear here how the combinatoric factor corresponds to the symmetries of each of the two loops and the symmetry of the diagram as a whole ($\frac{1}{8} = \left(\frac{1}{2}\right)^3$). From the the effective Feynman rules[4], [5], (iW_{ij}^{-1}) for each effective propagator, (iW_{ijkl}) for the quartic vertex, and an overall factor of $(-i)$, we can write the corresponding well-known expression

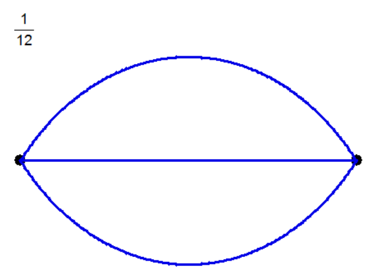
$$-\frac{1}{8}W_{ij}^{-1}W_{kl}^{-1}W_{ijkl} \tag{3}$$

Needless to remind you that W_{ij} denotes the bilinear kernel of a generic classical action or the 2nd functional derivative with respect to the fields, and $W_{ijk\cdots}$ represent the higher derivatives. The W's are functions of the effective field[4], [5].

Now with the command

$$\{3,3\}\texttt{//Contract//ToFeynShow} \tag{4}$$

we obtain the following graph,



The above shows how the legs of two cubic vertices would join together to form an irreducible 2-loop graph. Notice that we can have the possibility where only one line joins the two vertices, while the remaining legs form a ring on each vertex. Such a possibility does not correspond to an irreducible graph, and is *discarded* by the program. In fact, the program would give us a remark, "`1 removed`", before showing the above graph. Again, the correct combinatoric factor $\frac{1}{12} = \frac{1}{2} \times \frac{1}{3!}$, corresponding to the symmetries of the diagram, is produced. Again, the well-known effective action expression can be written:

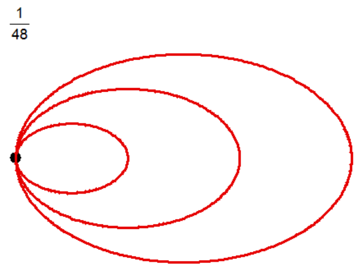
$$\frac{1}{12}W_{il}^{-1}W_{jm}^{-1}W_{kn}^{-1}W_{ijk}W_{lmn} \tag{5}$$

3 Three-Loop Contributions

In the 3-loop case the *diagrets* produced are given by the list:

$$\{\{6\},\{3,5\},\{4,4\},\{3,3,4\},\{3,3,3,3\}\} \tag{6}$$

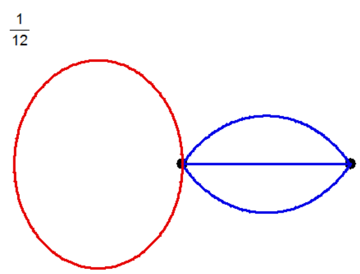
Corresponding to $\{6\}$ we obtain the following graph:



It should not be difficult to check that the correct combinatoric factor is produced, and to write the corresponding effective action expression

$$-i\frac{1}{48}W^{-1}_{ij}W^{-1}_{kl}W^{-1}_{mn}W_{ijklmn} \tag{7}$$

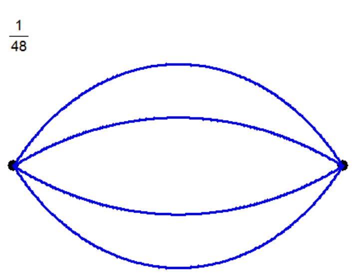
Corresponding to $\{3,5\}$ we obtain the following two-vertex graph:

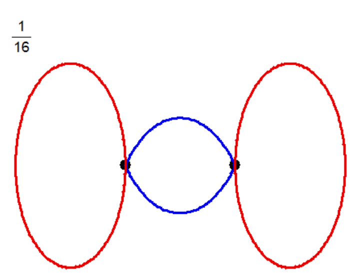


Notice that while processing the above, the program reports that one possible disjoint graph is removed (Can you figure out which one?). The corresponding 3-loop effective action expression would be:

$$i\frac{1}{12}W^{-1}_{il}W^{-1}_{jm}W^{-1}_{kn}W^{-1}_{rs}W_{ijk}W_{lmnrs} \tag{8}$$

Corresponding to $\{4,4\}$ the program gives us the following two graphs (reporting that a disjoint one is removed):



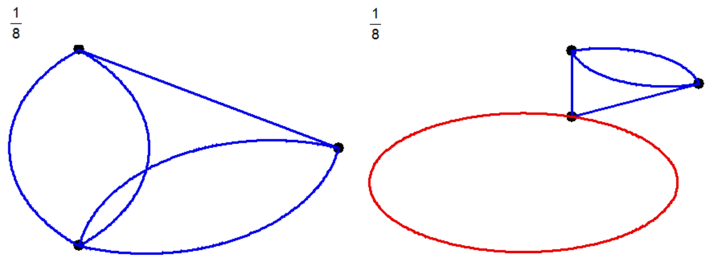


The following would be the respective effective action expressions:

$$i\frac{1}{48}W^{-1}_{im}W^{-1}_{jn}W^{-1}_{kr}W^{-1}_{ls}W_{ijkl}W_{mnrs} \tag{9}$$

$$i\frac{1}{16}W_{im}^{-1}W_{jn}^{-1}W_{kl}^{-1}W_{rs}^{-1}W_{ijkl}W_{mnrs} \tag{10}$$

Corresponding to $\{3,3,4\}$ the program gives the following two, 3-vertex graphs (reporting that 7 disjoint graphs are discarded!):

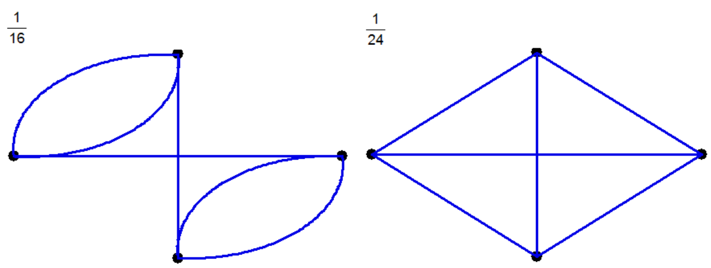


The following would be the respective effective action expressions:

$$-i\frac{1}{8}W_{il}^{-1}W_{jr}^{-1}W_{ks}^{-1}W_{mt}^{-1}W_{nu}^{-1}W_{ijk}W_{lmn}W_{rstu} \tag{11}$$

$$-i\frac{1}{8}W_{il}^{-1}W_{jm}^{-1}W_{kr}^{-1}W_{ns}^{-1}W_{tu}^{-1}W_{ijk}W_{lmn}W_{rstu} \tag{12}$$

Corresponding to the last diagret $\{3,3,3,3\}$ of the 3-loop case, the program gives us the following two 4-vertex graphs (reporting that 37 disjoint graphs were removed, and noting that out of a remaining 7 many were topologically equivalent, and only two remained!):



The following would be the respective effective action expressions:

$$i\frac{1}{16}W_{il}^{-1}W_{jm}^{-1}W_{ru}^{-1}W_{sv}^{-1}W_{kt}^{-1}W_{nw}^{-1}W_{ijk}W_{lmn}W_{rst}W_{uvw} \tag{13}$$

$$i\frac{1}{24}W_{il}^{-1}W_{jr}^{-1}W_{ku}^{-1}W_{ms}^{-1}W_{nv}^{-1}W_{tw}^{-1}W_{ijk}W_{lmn}W_{rst}W_{uvw} \tag{14}$$

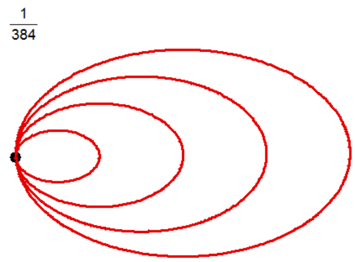
4 Four-Loop Contributions

In the 4-loop case, the diagrets produced are given by the list

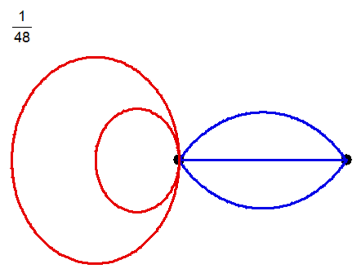
$$\left\{ \begin{array}{l} \{8\},\{3,7\},\{4,6\},\{5,5\},\{3,3,6\},\{3,4,5\},\{4,4,4\}, \\ \{3,3,3,5\},\{3,3,4,4\},\{3,3,3,3,4\},\{3,3,3,3,3,3\} \end{array} \right\} \tag{15}$$

In this section, we shall give the corresponding effective Feynman graphs produced by our program, without writing the associated effective action expressions. The reader should be able to do that following the work of the preceding sections.

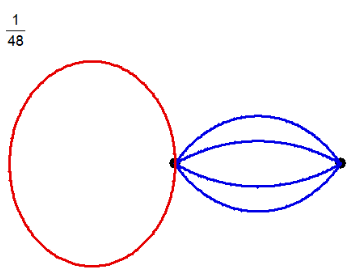
Corresponding to $\{8\}$, the program gives the following single-vertex graph,

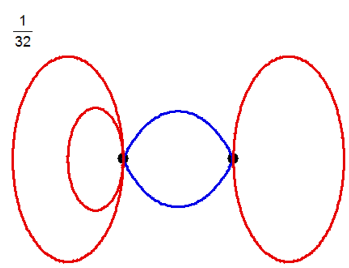


Corresponding to $\{3, 7\}$, the program gives the following 2-vertex graph (reporting the removal of one disjoint graph),

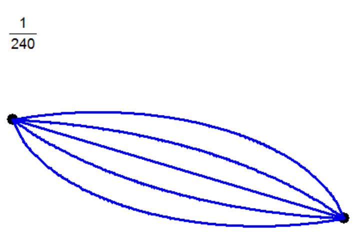


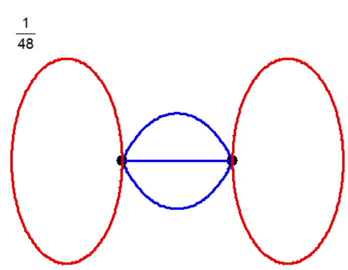
Corresponding to $\{4, 6\}$, the program gives the following two, 2-vertex graphs (reporting the removal of one disjoint graph),



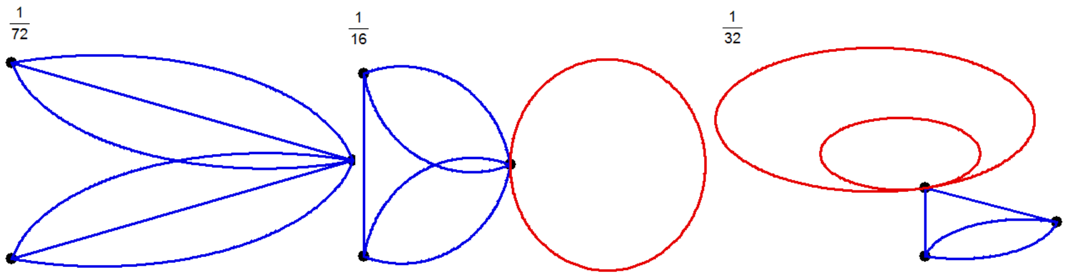


Corresponding to $\{5, 5\}$, the program gives the following two, 2-vertex graphs (reporting the removal of one disjoint graph),

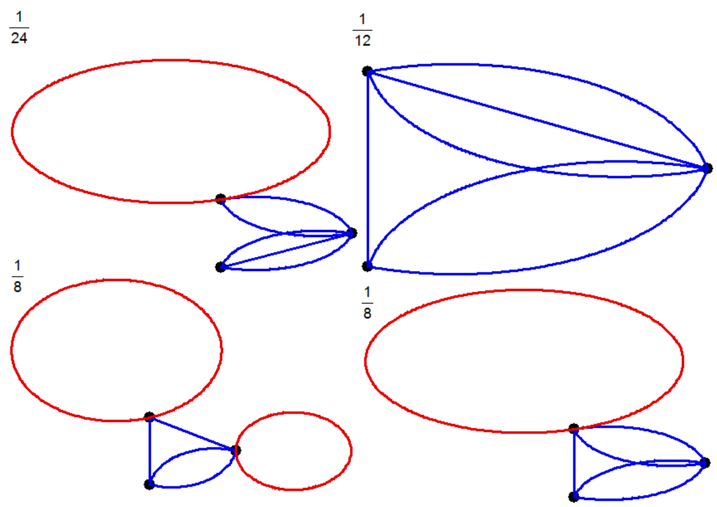




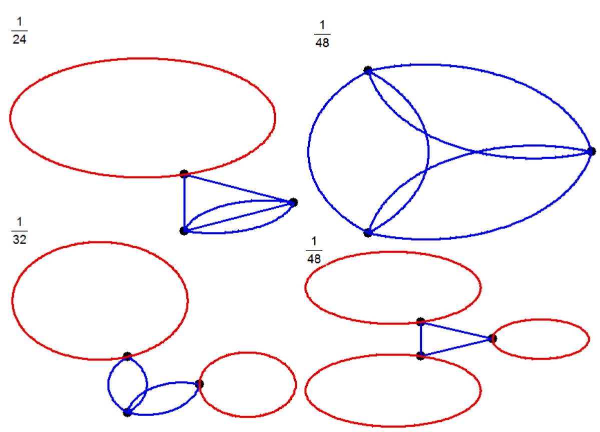
Corresponding to $\{3, 3, 6\}$, the program gives the following three, 3-vertex graphs (reporting the removal of 7 disjoint graphs),



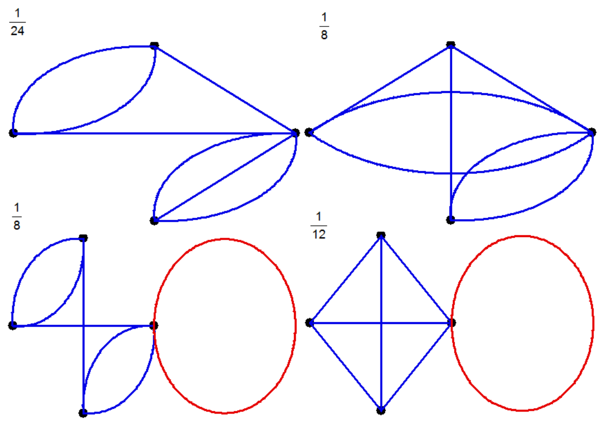
Corresponding to $\{3, 4, 5\}$, the program gives the following four, 3-vertex graphs (reporting the removal of 8 disjoint graphs),



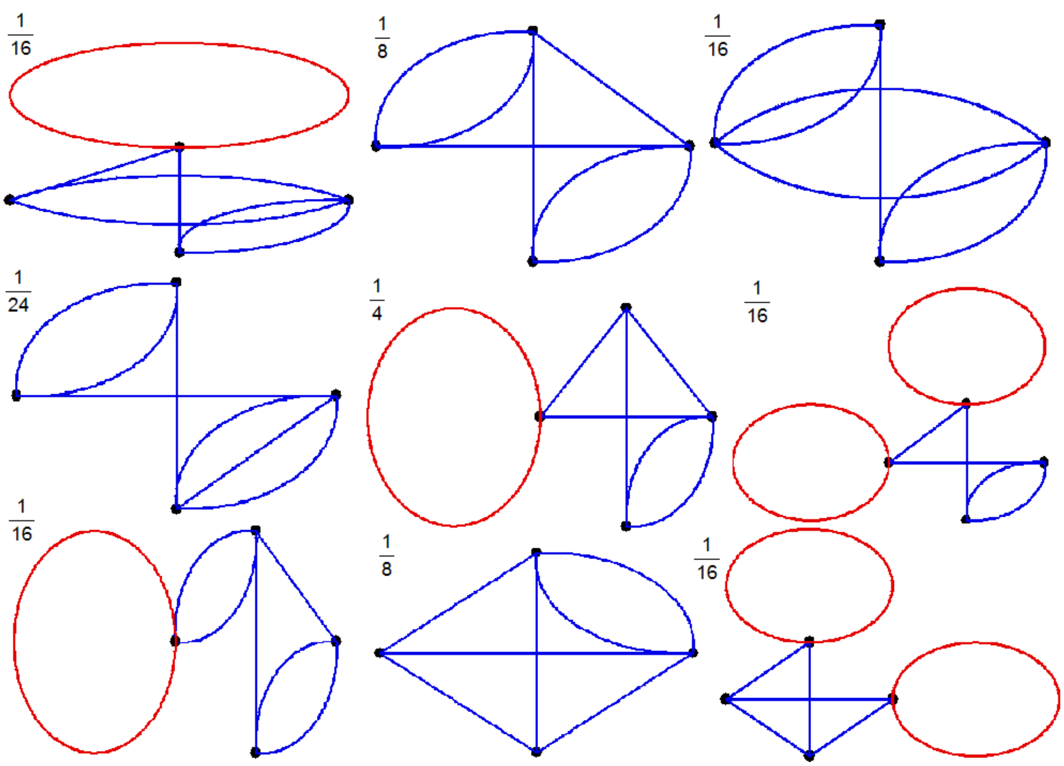
Corresponding to $\{4, 4, 4\}$, the program gives the following four, 3-vertex graphs (reporting the removal of 7 disjoint graphs, and noting the existence of several topologically equivalent graphs in the remaining 8 that have been combined to give only 4),



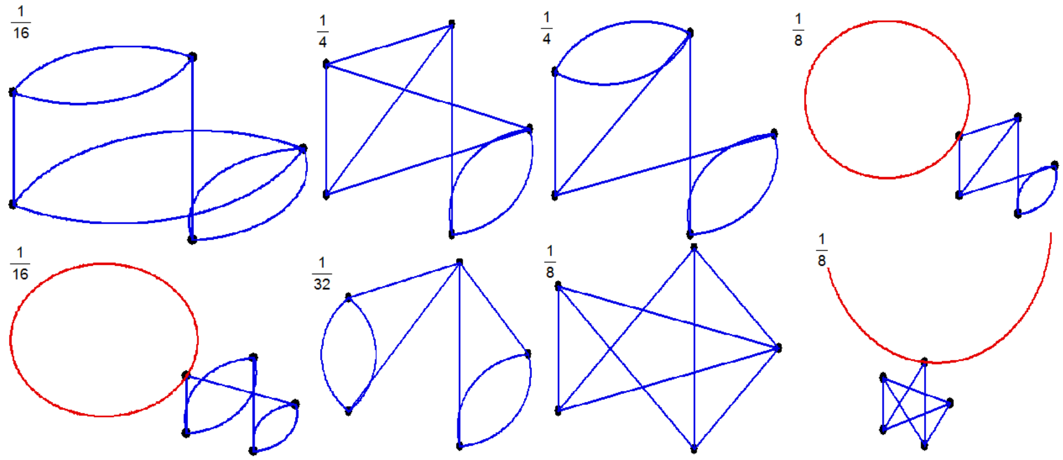
Corresponding to $\{3,3,3,5\}$, the program gives the following four, 4-vertex graphs (reporting the removal of 52 disjoint graphs, and noting the existence of several topologically equivalent graphs in the remaining 13 that have been combined to give 4),



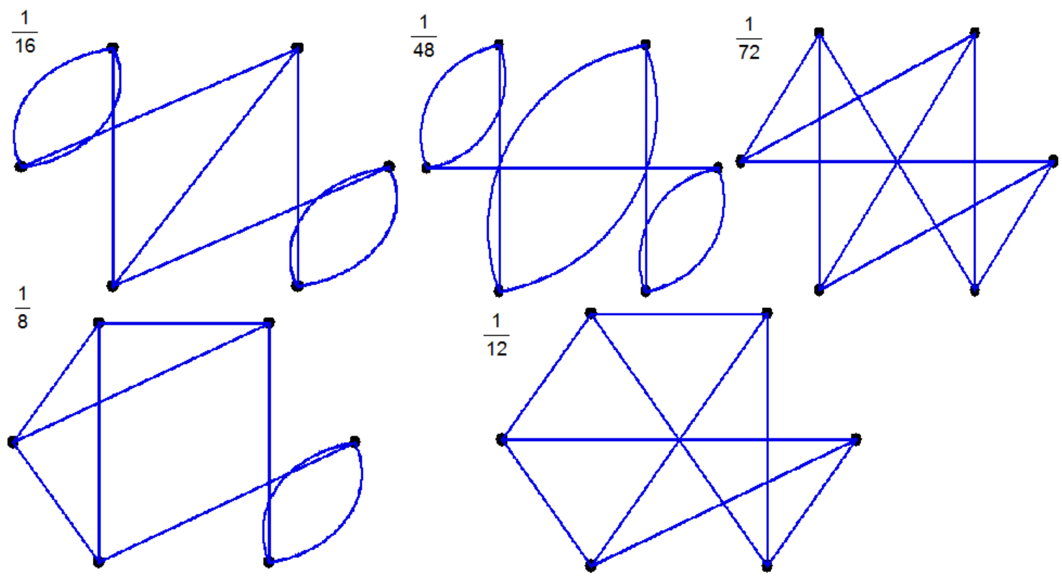
Corresponding to $\{3,3,4,4\}$, the program gives the following nine, 4-vertex graphs (reporting the removal of 54 disjoint graphs, and noting the existence of several topologically equivalent graphs in the remaining 18 that have been combined to give 9),



Corresponding to $\{3,3,3,3,4\}$, the program gives the following eight, 5-vertex graphs (reporting the removal of 411 disjoint graphs, and noting the existence of many of the topologically equivalent graphs in the remaining 84 that have been combined to give 8),

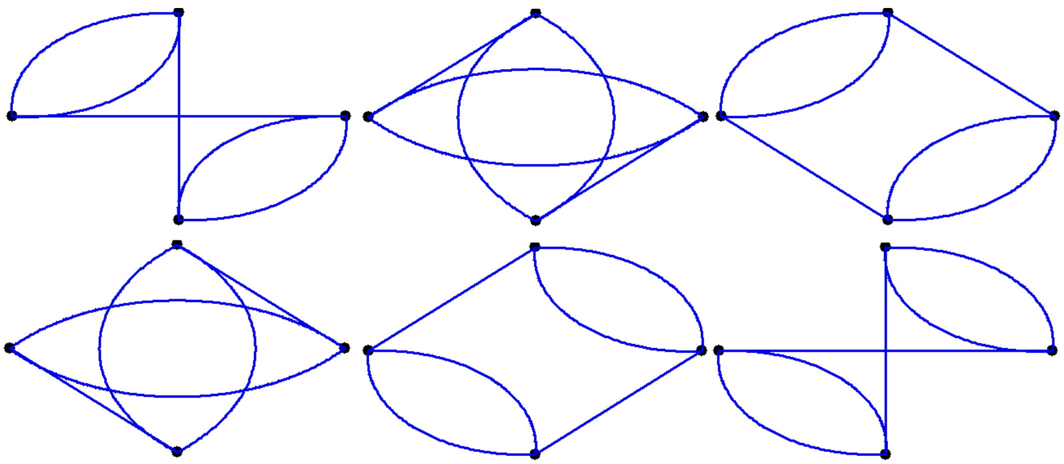


Corresponding to $\{3,3,3,3,3,3\}$, the program gives the following five, 6-vertex graphs (reporting the removal of 4166 disjoint graphs, and noting the existence of many of the topologically equivalent graphs in the remaining 550, that have been combined to give only 5),



5 Discussion

Many of the code-generated graphs depicted in the foregoing sections may look rather unnatural, and somehow entangled. This comes about because, once the locations of the vertices are specified (we choose to distribute them on a circle), the program selects from many possible equivalent graphs the one that corresponds to the first one, in some ordering of associated analytical expressions. However, the program also allows to examine various forms of the same graph in order to be able to select the most pleasing form. The various forms would correspond to permutations of the locations that are associated with the respective vertices. For example, the following six forms can be generated for one of the 3-loop graphs that corresponds to the diagret $\{3,3,3,3\}$:



In the above, the user might select either the 3rd form or the 5th form as the natural choice, since that does not involve crossings of the internal lines. Such examination can be done for all graphic contributions, and it is up to the user to select the most natural or pleasing form. However, for effective graphs with many vertices, this examination might take a long time on an ordinary computer, and may be tedious work, since one must deal with permutations that involve factorials. It is always possible for the user to change the locations of the vertices by hand on a piece of paper. However, we can do even better by creating a program that allows the user to take any entangled graph, and proceed to disentangle the vertices, by moving them with a mouse or whatever on the computer screen.

In any case, the 43 irreducible graphs that are given for the 4-loop effective action would represent a new result that will be useful in our future computations. The *Mathematica* code we have been using can be applied to any loop order, whenever there comes a need. In fact, we have obtained results up to 11 loops, but cannot be reported here. The work can be applied to any quantum field theory, and would be extended to embrace both bosonic and fermionic fields. However, it should be clear to the alert reader that not all the graphic contributions are needed in a particular field theory. For example, in ordinary non-gravitational field theories, one only needs graphs with effective vertices that have 3 and 4 legs only. However, quantum gravity requires the use of vertices with all numbers of legs.

Finally, we would encourage users of *Mathematica* and similar software programs to try to simulate the underlying code used in generating the graphs of the present work, and perhaps try to extend it to include bosonic and fermionic, neutral and charged fields (that might demand the use of lines with arrows).[1]

[1] **The source code that we have would require a version of *Mathematica* that can read *notebooks* (files with extension like xxx.nb) and *packages* (files with extension like xxx.m). If needed, the pertaining files**

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