The SU_5 Structure of 14-Dimensional Unification

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Abstract

In a 14-dimensional gravidynamic unification model, the spacetime as well as the internal symmetries of 2 lepton-quark generations would be consolidated in a 64-component Weyl fermion. Alternatively, the latter fermionic multiplet can describe 8 charged leptons, with 8 associated neutrinos, and the corresponding antiparticles. In such a framework, the dynamics of vector bosons, as well as of Higgs scalars, would be generated at the quantum level via unified couplings to a vector, an antisymmetric tensor of 3rd rank, and an antisymmetric tensor of 5th rank. We exhibit the complete SU_5 structure of the latter couplings. The underlying SU_5 would contain a color SU_3 symmetry, in the case of the leptons and quarks, or a family SU_3 symmetry, in the alternative model of purely leptonic unification. This work begins by writing the Lorentz algebra of 14 dimensional spacetime in terms of its 4-dimensional Lorentz subalgebra, and an internal O_{10} factor. The latter is expressed via its U_5 subalgebra. The fermionic 64-plet is expressed in terms of 32 Weyl fermions in 4 dimensions. Likewise, the pertinent vector and the tensors are expressed in terms of vectors and scalars in 4 dimensions. The emerging picture regarding the fundamental fermions, and their interactions, would lead to aspects that are describable by the O_{10} and SU_5 unification models, whether the grand unified model of leptons and quarks, or the purely leptonic unification model.

1 Introduction

An O_{10} unification algebra, via its 16-component fundamental multiplet, would be able to consolidate the symmetries and describe the interactions of fundamental fermions in either of two ways. It is possible to describe the symmetries and associated interactions of a single lepton-quark generation[1], [2], particles and antiparticles. Alternatively[3], it is possible to describe the symmetries of 4 charged leptons, with 4 associated neutrinos, and their antiparticles. In the first case one encounters a color SU_3 symmetry, while in the second case, a family SU_3 symmetry is encountered. In either case the SU_3 is a subalgebra of SU_5, the latter being the maximal subalgebra of O_{10}.

In the framework of a higher-dimensional gravidynamic unification, we can embed the O_{10} symmetry within a 14-spacetime scheme. The underlying $O_{1,13}$ symmetry would consolidate O_{10} with the 4-spacetime Lorentz symmetry. Both the spin and the internal quantum numbers would become properties of a single spinorial representation. This scheme is similar to the embedding of an O_{14} internal symmetry in an 18-spacetime gravidynamic framework[4], [5], [6], where the fermionic content describes 4 generations of leptons and quarks.

Whereas in an 18-spacetime scheme, the symmetries of all fermions are consolidated in a 256-component Majorana-Weyl spinor that can describe 64 Weyl particles in 4-spacetime, we can only deal with a Weyl fermion in 14-spacetime, without a Majorana condition, and where the fundamental Weyl spinor has 64 components, affording to describe only 32 Weyl particles in 4-spacetime. Whereas the O_{14} internal symmetry of 18-spacetime, via its SU_7 maximal subalgebra, would account for both, a color SU_3 and a family SU_3, the O_{10} internal symmetry of 14-spacetime, via its SU_5 maximal subalgebra, would account to a single SU_3, the latter being either a color symmetry or a family symmetry. Hence, in a 14-spacetime gravidynamic model, we can either describe a lepton-quark model with color or, alternatively, a purely leptonic model with family structure. In fact, the 32 Weyl fermions emerging from 14-spacetime can either describe two (rather than one) lepton-quark generations, particles and antiparticles. Alternatively, they can describe 8 (rather than 4) charged leptons, with associated 8 neutrinos, and their antiparticles.

Our purpose in this article is to treat the $\mathrm{O}_{1,13}$ algebra of 14-spacetime in the same manner[6] used to treat the $\mathrm{O}_{1,14}$ algebra of 18-spacetime. We shall explore, in detail, the decomposition of the $\mathrm{O}_{1,13}$ symmetry through the 4-spacetime $\mathrm{O}_{1,3}$ symmetry and the internal O_{10} symmetry factor. The structure of the latter will take shape via the SU_5 subalgebra.

In the following section, we shall begin by constructing the $\mathrm{O}_{1,13}$ algebra in terms of $\mathrm{O}_{1,3}$ and U_5 structural elements. This will be followed by algebraic techniques for representing an $\mathrm{O}_{1,13}$ vector as well as antisymmetric tensors of ranks 3 and 5. In fact, a vector and antisymmetric tensors of ranks 3 and 5 are the only possibilites for bosons that can couple to a Weyl fermion in 14-spacetime, apart from the pertinent graviton field. These bosons would have components that can describe vectors and scalars in 4-spacetime, and their dynamics would be generated by quantum-loop contributions.

In a subsequent section, we shall construct an $\mathrm{O}_{1,13}$ algebraic representation of the 64-component Weyl spinor, and its Dirac conjugate, in terms of $\mathrm{O}_{1,3}$ Weyl spinors that carry SU_5 tensorial indices. This will be followed by the composition of the vector and the tensors from fermionic bilinears. This leads the way to the construction of the couplings of the fundamental fermions to the vector and the tensors.

2 The Algebra $\mathrm{O}_{1,13}$ in Terms of $\mathrm{O}_{1,3}$ and U_5

The generators of the Lorentz algebra $\mathrm{O}_{1,13}$ in 14-spacetime may be decomposed into the following set of generators:

$$\left\{J_{\mu\nu}, J_a{}^b, Q_{ab}, Q^{ab}, H_{\mu a}, H_\mu{}^a\right\} \tag{1}$$

Here, the symbols $(\mu, \nu, \lambda, \cdots)$ are used to represent vectorial indices in 4-spacetime, the symbols $(a, b, c, \cdots)$ to represent the indices of complex SU_5 vectors. In the above, the $J_{\mu\nu}$, being antisymmetric in (μ,ν), are the generators of the 4-spacetime Lorentz algebra. The $J_a{}^b$ are the generators of U_5, the trace part of which is a U_1 generator, while the traceless part gives the generators of SU_5. The conjugate generators Q_{ab} and Q^{ab}, being antisymmetric in (a,b), are in the coset of O_{14} over U_5. The conjugate generators $H_{\mu a}$ and $H_\mu{}^a$ are in the coset of $O_{1,13}$ over $O_{1,3}$ and O_{10}.

We begin by writing the commutators of the $O_{1,3}$ generators in the form

$$[J_{\mu\nu}, J_{\lambda\rho}] = (\eta_{\nu\lambda}J_{\mu\rho} - \eta_{\mu\lambda}J_{\nu\rho} + \eta_{\mu\rho}J_{\nu\lambda} - \eta_{\nu\rho}J_{\mu\lambda}) \tag{2}$$

Here $\eta_{\mu\nu}$ is the constant metric of 4-spacetime.

Whereas the generators $J_{\mu\nu}$ commute with the generators of O_{10}, namely, $J_a{}^b$, Q_{ab}, and Q^{ab}, we have

$$[J_{\mu\nu}, H_{\lambda a}] = (\eta_{\nu\lambda}H_{\mu a} - \eta_{\mu\lambda}H_{\nu a}) \tag{3}$$

$$[J_{\mu\nu}, H_\lambda{}^a] = (\eta_{\nu\lambda}H_\mu{}^a - \eta_{\mu\lambda}H_\nu{}^a) \tag{4}$$

The generators $J_a{}^b$ of the U_5 algebra satisfy the commutators:

$$\left[J_a{}^b, J_c{}^d\right] = \left(\delta_c{}^b J_a{}^d - \delta_a{}^d J_c{}^b\right) \tag{5}$$

The commutators of $J_a{}^b$ with the Q's are

$$\left[J_a{}^b, Q_{cd}\right] = \left(\delta_c{}^b Q_{ad} - \delta_d{}^b Q_{ac}\right) \tag{6}$$

$$\left[J_a{}^b, Q^{cd}\right] = -\left(\delta_a{}^c Q^{bd} - \delta_a{}^d Q^{bc}\right) \tag{7}$$

The commutators of $J_a{}^b$ with the H's are

$$\left[J_a{}^b, H_{\mu c}\right] = \delta_c{}^b H_{\mu a} \tag{8}$$

$$\left[J_a{}^b, H_\mu{}^c\right] = -\delta_a{}^c H_\mu{}^b \tag{9}$$

The commutators of the Q's among themselves are

$$[Q_{ab}, Q_{cd}] = 0 \tag{10}$$

$$\left[Q_{ab}, Q^{cd}\right] = \left(\delta_b{}^c J_a{}^d - \delta_a{}^c J_b{}^d + \delta_a{}^d J_b{}^c - \delta_b{}^d J_a{}^c\right) \tag{11}$$

$$\left[Q^{ab}, Q^{cd}\right] = 0 \tag{12}$$

The commutators of the Q's with the H's are

$$[Q_{ab}, H_{\mu c}] = 0 \tag{13}$$

$$[Q_{ab}, H_\mu{}^c] = (\delta_b{}^c H_{\mu a} - \delta_a{}^c H_{\mu b}) \tag{14}$$

$$\left[Q^{ab}, H_{\mu c}\right] = \left(\delta_c{}^b H_\mu^a - \delta_c{}^a H_\mu{}^b\right) \tag{15}$$

$$\left[Q^{ab}, H_\mu{}^c\right] = 0 \tag{16}$$

Finally, the commutators of the H's among themselves are

$$[H_{\mu a}, H_{\nu b}] = -\eta_{\mu\nu} Q_{ab} \tag{17}$$

$$\left[H_{\mu a}, H_{\nu}{}^{b}\right] = -\eta_{\mu\nu} J_{a}{}^{b} - \delta_{a}{}^{b} J_{\mu\nu} \tag{18}$$

$$\left[H_{\mu}{}^{a}, H_{\nu}{}^{b}\right] = -\eta_{\mu\nu} Q^{ab} \tag{19}$$

We can verify that all Jacobi identiries involving all generators are satisfied, and that all generators commute with the following quadratic (Casimir) operator:

$$\frac{1}{2} J_{\mu\nu} J_{\nu\mu} + J_{a}{}^{b} J_{b}{}^{a} + \frac{1}{2} Q_{ab} Q^{ba} + \frac{1}{2} Q^{ab} Q_{ba} - H_{\mu a} H_{\mu}{}^{a} - H_{\mu}{}^{a} H_{\mu a} \tag{20}$$

3 The Representation of an $\mathrm{O}_{1,13}$ Vector

In order to represent a vector in 14-spacetime, let us introduce the operators (K_{μ}, K_{a}, K^{a}). The commutators of these with the Lorentz generators $J_{\mu\nu}$ are

$$[J_{\mu\nu}, K_{\lambda}] = (\eta_{\nu\lambda} K_{\mu} - \eta_{\mu\lambda} K_{\nu}) \tag{21}$$

$$[J_{\mu\nu}, K_{a}] = 0 \tag{22}$$

$$[J_{\mu\nu}, K^{a}] = 0 \tag{23}$$

The commutators of the U_5 generators $J_{a}{}^{b}$ are

$$\left[J_{a}{}^{b}, K_{\lambda}\right] = 0 \tag{24}$$

$$\left[J_{a}{}^{b}, K_{c}\right] = \delta_{c}{}^{b} K_{a} \tag{25}$$

$$\left[J_{a}{}^{b}, K^{c}\right] = -\delta_{a}{}^{c} K^{b} \tag{26}$$

The commutators of Q_{ab} are

$$[Q_{ab}, K_{\lambda}] = 0 \tag{27}$$

$$[Q_{ab}, K_{c}] = 0 \tag{28}$$

$$[Q_{ab}, K^{c}] = (\delta_{b}{}^{c} K_{a} - \delta_{a}{}^{c} K_{b}) \tag{29}$$

The commutators of Q^{ab} are

$$\left[Q^{ab}, K_{\lambda}\right] = 0 \tag{30}$$

$$\left[Q^{ab}, K_{c}\right] = \left(\delta_{c}{}^{b} K^{a} - \delta_{c}{}^{a} K^{b}\right) \tag{31}$$

$$\left[Q^{ab}, K^{c}\right] = 0 \tag{32}$$

The commutators of $H_{\mu a}$ are

$$[H_{\mu a}, K_{\lambda}] = -\eta_{\mu\lambda} K_{a} \tag{33}$$

$$[H_{\mu a}, K_{c}] = 0 \tag{34}$$

$$[H_{\mu a}, K^{c}] = \delta_{a}{}^{c} K_{\mu} \tag{35}$$

And finally, the commutators of $H_{\mu}{}^{a}$ are

$$[H_{\mu}{}^{a}, K_{\lambda}] = -\eta_{\mu\lambda} K^{a} \tag{36}$$

$$[H_\mu{}^a, K_c] = \delta_c{}^a K_\mu \tag{37}$$

$$[H_\mu{}^a, K^c] = 0 \tag{38}$$

We can verify that all the Jacobi identities involving any two of the $O_{1,13}$ generators $J_{\mu\nu}$, $J_a{}^b$, Q_{ab}, Q^{ab}, $H_{\mu a}$ or $H_\mu{}^a$, with either of the operators K_μ, K_a, or K^a, are satisfied. Moreover any of the $O_{1,13}$ generators can be shown to commute with the following quadratic operator:

$$K_\mu K_\mu + K_a K^a + K^a K_a \tag{39}$$

We proceed now to the introduction of the multiplet that can be associated with the above operator representation, and to the construction of the infinitesimal $O_{1,13}$ transformations that act on it. The desired multiplet with components $\{B_\mu, B_a, B^a\}$ can be introduced by the vector module

$$\mathcal{B} = B_\mu K_\mu + B_a K^a + B^a K_a \tag{40}$$

Introducing the $O_{1,13}$ parameter module,

$$\mathcal{W} = \frac{1}{2}\Omega_{\mu\nu} J_{\mu\nu} + \Omega_a{}^b J_b{}^a + \frac{1}{2}\Omega_{ab} Q^{ab} + \frac{1}{2}\Omega^{ab} Q_{ab} + \Omega_{\mu a} H_\mu{}^a + \Omega_\mu{}^a H_{\mu a} \tag{41}$$

we can compute the commutator $[\mathcal{W}, \mathcal{B}]$. The latter gives a vector module whose components would define the needed infinitesimal transformations. We obtain

$$\delta B_\mu = \Omega_{\mu\nu} B_\nu + \Omega_{\mu a} B^a + \Omega_\mu{}^a B_a \tag{42}$$

$$\delta B_a = -\Omega_a{}^b B_b + \Omega_{ab} B^b - \Omega_{\mu a} B_\mu \tag{43}$$

$$\delta B^a = \Omega_b{}^a B^b + \Omega^{ab} B_b - \Omega_\mu{}^a B_\mu \tag{44}$$

We can verify that, for any two vector modules $\mathcal{A}$ and $\mathcal{B}$, the above infinitesimal transformations, acting in a like manner on the components of both, would leave invariant the following bilinear form:

$$\mathcal{A} \cdot \mathcal{B} = A_\mu B_\mu + A_a B^a + A^a B_a \tag{45}$$

4 The Antisymmetric Tensor Representation of Rank 3

An $O_{1,13}$ antisymmetric tensor representation of rank 3 would have the following $O_{1,3}$ and U_5 covariant components:

$$\{K_{\mu\nu\lambda}, K_{\mu\nu a}, K_{\mu\nu}{}^a, K_{\mu ab}, K_{\mu a}{}^b, K_\mu{}^{ab}, K_{abc}, K_{ab}{}^c, K_a{}^{bc}, K^{abc}\} \tag{46}$$

The symmetries of the above components with respect to their spacetime and SU_5 indices should be clear. Notice that the tensor $K_{\mu\nu\lambda}$ could be traded for a single-index counterpart using the 4-dimensional epsilon symbol $\epsilon_{\mu\nu\lambda\rho}$. However, it is convenient to leave it in this form, at this stage. Likewise, the 3-index SU_5 tensors can be traded for 2-index tensors using the pertinent epsilon symbol. Again, it is more convenient to leave them as such. The replacements can be made later after the couplings are constructed.

In order to be able to write out the infinitesimal transformations of an associated multiplet, we proceed now to the elaborate task of writing down the commutators of the above component operators with the generators of the $O_{1,13}$ algebra.

For the commutators of $J_{\mu\nu}$, we have

$$[J_{\mu\nu}, K_{\lambda\rho\sigma}] = (\eta_{\nu\lambda}K_{\mu\rho\sigma} + \eta_{\nu\rho}K_{\mu\sigma\lambda} + \eta_{\nu\sigma}K_{\mu\lambda\rho}) - (\mu \leftrightarrow \nu) \tag{47}$$

$$[J_{\mu\nu}, K_{\lambda\rho a}] = (\eta_{\nu\lambda}K_{\mu\rho a} - \eta_{\nu\rho}K_{\mu\lambda a}) - (\mu \leftrightarrow \nu) \tag{48}$$

$$[J_{\mu\nu}, K_{\lambda\rho}{}^{a}] = (\eta_{\nu\lambda}K_{\mu\rho}{}^{a} - \eta_{\nu\rho}K_{\mu\lambda}{}^{a}) - (\mu \leftrightarrow \nu) \tag{49}$$

$$[J_{\mu\nu}, K_{\lambda ab}] = \eta_{\nu\lambda}K_{\mu ab} - \eta_{\mu\lambda}K_{\nu ab} \tag{50}$$

$$\left[J_{\mu\nu}, K_{\lambda a}{}^{b}\right] = \eta_{\nu\lambda}K_{\mu a}{}^{b} - \eta_{\mu\lambda}K_{\nu a}{}^{b} \tag{51}$$

$$\left[J_{\mu\nu}, K_{\lambda}{}^{ab}\right] = \eta_{\nu\lambda}K_{\mu}{}^{ab} - \eta_{\mu\lambda}K_{\nu}{}^{ab} \tag{52}$$

The commutators of $J_{\mu\nu}$ with K_{abc}, $K_{ab}{}^{c}$, $K_{a}{}^{bc}$, and K^{abc} are vanishing.

Whereas $J_{a}{}^{b}$ commutes with $K_{\mu\nu\lambda}$, its commutators with the other K's are

$$\left[J_{a}{}^{b}, K_{\mu\nu c}\right] = \delta_{c}{}^{b}K_{\mu\nu a} \tag{53}$$

$$\left[J_{a}{}^{b}, K_{\mu\nu}{}^{c}\right] = -\delta_{a}{}^{c}K_{\mu\nu}{}^{b} \tag{54}$$

$$\left[J_{a}{}^{b}, K_{\mu cd}\right] = \delta_{c}{}^{b}K_{\mu ad} - \delta_{d}{}^{b}K_{\mu ac} \tag{55}$$

$$\left[J_{a}{}^{b}, K_{\mu c}{}^{d}\right] = \delta_{c}{}^{b}K_{\mu a}{}^{d} - \delta_{a}{}^{d}K_{\mu c}{}^{b} \tag{56}$$

$$\left[J_{a}{}^{b}, K_{\mu}{}^{cd}\right] = -\delta_{a}{}^{c}K_{\mu}{}^{bd} + \delta_{a}{}^{d}K_{\mu}{}^{bc} \tag{57}$$

$$\left[J_{a}{}^{b}, K_{cde}\right] = \delta_{c}{}^{b}K_{ade} + \delta_{d}{}^{b}K_{aec} + \delta_{e}{}^{b}K_{acd} \tag{58}$$

$$\left[J_{a}{}^{b}, K_{cd}{}^{e}\right] = \left(\delta_{c}{}^{b}K_{a,d}{}^{e} - \delta_{d}{}^{b}K_{ac}{}^{e}\right) - \left(\delta_{a}{}^{e}K_{cd}{}^{b}\right) \tag{59}$$

$$\left[J_{a}{}^{b}, K_{c}{}^{de}\right] = \left(\delta_{c}{}^{b}K_{a}{}^{de}\right) - \left(\delta_{a}{}^{d}K_{c}{}^{be} - \delta_{a}{}^{e}K_{c}{}^{bd}\right) \tag{60}$$

$$\left[J_{a}{}^{b}, K^{cde}\right] = -\left(\delta_{a}{}^{c}K^{bde} + \delta_{a}{}^{d}K^{bec} + \delta_{a}{}^{e}K^{bcd}\right) \tag{61}$$

The nonvanishing commutators of Q_{ab} with the K's are:

$$[Q_{ab}, K_{\mu\nu}{}^{c}] = \delta_{b}{}^{c}K_{\mu\nu a} - \delta_{a}{}^{c}K_{\mu\nu b} \tag{62}$$

$$\left[Q_{ab}, K_{\mu c}{}^{d}\right] = \delta_{a}{}^{d}K_{\mu bc} - \delta_{b}{}^{d}K_{\mu ac} \tag{63}$$

$$\left[Q_{ab}, K_{\mu}{}^{cd}\right] = -\left(\delta_{a}{}^{c}K_{\mu b}{}^{d} - \delta_{b}{}^{c}K_{\mu a}{}^{d} + \delta_{b}{}^{d}K_{\mu a}{}^{c} - \delta_{a}{}^{d}K_{\mu b}{}^{c}\right) \tag{64}$$

$$[Q_{ab}, K_{cd}{}^{e}] = (\delta_{b}{}^{e}K_{acd} - \delta_{a}{}^{e}K_{bcd}) \tag{65}$$

$$\left[Q_{ab}, K_{c}{}^{de}\right] = -\left(\delta_{b}{}^{d}K_{ac}{}^{e} - \delta_{a}{}^{d}K_{bc}{}^{e} + \delta_{a}{}^{e}K_{bc}{}^{d} - \delta_{b}{}^{e}K_{ac}{}^{d}\right) \tag{66}$$

$$\left[Q_{ab}, K^{cde}\right] = \left(\delta_{b}{}^{c}K_{a}{}^{de} + \delta_{b}{}^{d}K_{a}{}^{ec} + \delta_{b}{}^{e}K_{a}{}^{cd}\right) - (a \leftrightarrow b) \tag{67}$$

The nonvanishing commutators of Q^{ab} with the K's are:

$$\left[Q^{ab}, K_{\mu\nu c}\right] = \delta_{c}{}^{b}K_{\mu\nu}{}^{a} - \delta_{c}{}^{a}K_{\mu\nu}{}^{b} \tag{68}$$

$$\left[Q^{ab}, K_{\mu cd}\right] = \left(\delta_c{}^a K_{\mu d}{}^b - \delta_c{}^b K_{\mu d}{}^a + \delta_d{}^b K_{\mu c}{}^a - \delta_d{}^a K_{\mu c}{}^b\right) \tag{69}$$

$$\left[Q^{ab}, K_{\mu c}{}^d\right] = \delta_c{}^b K_\mu{}^{ad} - \delta_c{}^a K_\mu{}^{bd} \tag{70}$$

$$\left[Q^{ab}, K_{cde}\right] = \left(\delta_c{}^b K_{de}{}^a + \delta_d{}^b K_{ec}{}^a + \delta_e{}^b K_{cd}{}^a\right) - (a \leftrightarrow b) \tag{71}$$

$$\left[Q^{ab}, K_{cd}{}^e\right] = \left(-\delta_c{}^b K_d{}^{ae} + \delta_d{}^b K_c{}^{ae}\right) - (a \leftrightarrow b) \tag{72}$$

$$\left[Q^{ab}, K_c{}^{de}\right] = \delta_c{}^b K^{ade} - \delta_c{}^a K^{bde} \tag{73}$$

The nonvanishing commutators of $H_{\mu a}$ with the K's are:

$$[H_{\mu a}, K_{\nu\lambda\rho}] = -\left(\eta_{\mu\nu} K_{\lambda\rho a} + \eta_{\mu\rho} K_{\nu\lambda a} + \eta_{\mu\lambda} K_{\rho\nu a}\right) \tag{74}$$

$$[H_{\mu a}, K_{\nu\lambda b}] = \left(\eta_{\mu\nu} K_{\lambda ab} - \eta_{\mu\lambda} K_{\nu ab}\right) \tag{75}$$

$$\left[H_{\mu a}, K_{\nu\lambda}{}^b\right] = \left(\eta_{\mu\nu} K_{\lambda a}{}^b - \eta_{\mu\lambda} K_{\nu a}{}^b\right) + \left(\delta_a{}^b K_{\mu\nu\lambda}\right) \tag{76}$$

$$[H_{\mu a}, K_{\nu bc}] = -\eta_{\mu\nu} K_{abc} \tag{77}$$

$$[H_{\mu a}, K_{\nu b}{}^c] = -\eta_{\mu\nu} K_{ab}{}^c + \delta_a{}^c K_{\mu\nu b} \tag{78}$$

$$\left[H_{\mu a}, K_\nu{}^{bc}\right] = -\left(\eta_{\mu\nu} K_a{}^{bc}\right) - \left(\delta_a{}^b K_{\mu\nu}{}^c - \delta_a{}^c K_{\mu\nu}{}^b\right) \tag{79}$$

$$\left[H_{\mu a}, K_{bc}{}^d\right] = \delta_a{}^d K_{\mu bc} \tag{80}$$

$$\left[H_{\mu a}, K_b{}^{cd}\right] = -\delta_a{}^c K_{\mu b}{}^d + \delta_a{}^d K_{\mu b}{}^c \tag{81}$$

$$\left[H_{\mu a}, K^{bcd}\right] = \left(\delta_a{}^b K_\mu{}^{cd} + \delta_a{}^d K_\mu{}^{bc} + \delta_a{}^c K_\mu{}^{db}\right) \tag{82}$$

Finally, the nonvanishing commutators of $H_\mu{}^a$ with the K's are:

$$[H_\mu{}^a, K_{\nu\lambda\rho}] = -\left(\eta_{\mu\nu} K_{\lambda\rho}{}^a + \eta_{\mu\rho} K_{\nu\lambda}{}^a + \eta_{\mu\lambda} K_{\rho\nu}{}^a\right) \tag{83}$$

$$[H_\mu{}^a, K_{\nu\lambda b}] = -\left(\eta_{\mu\nu} K_{\lambda b}{}^a - \eta_{\mu\lambda} K_{\nu b}{}^a\right) + \left(\delta_b{}^a K_{\mu\nu\lambda}\right) \tag{84}$$

$$\left[H_\mu{}^a, K_{\nu\lambda}{}^b\right] = \left(\eta_{\mu\nu} K_\lambda{}^{ab} - \eta_{\mu\lambda} K_\nu{}^{ab}\right) \tag{85}$$

$$[H_\mu{}^a, K_{\nu bc}] = -\left(\eta_{\mu\nu} K_{bc}{}^a\right) - \left(\delta_b{}^a K_{\mu\nu c} - \delta_c{}^a K_{\mu\nu b}\right) \tag{86}$$

$$[H_\mu{}^a, K_{\nu b}{}^c] = \eta_{\mu\nu} K_b{}^{ac} - \delta_b{}^a K_{\mu\nu}{}^c \tag{87}$$

$$\left[H_\mu{}^a, K_\nu{}^{bc}\right] = -\eta_{\mu\nu} K^{abc} \tag{88}$$

$$[H_\mu{}^a, K_{bcd}] = \left(\delta_b{}^a K_{\mu cd} + \delta_d{}^a K_{\mu bc} + \delta_c{}^a K_{\mu db}\right) \tag{89}$$

$$\left[H_\mu{}^a, K_{bc}{}^d\right] = \left(\delta_b{}^a K_{\mu c}{}^d - \delta_c{}^a K_{\mu b}{}^d\right) \tag{90}$$

$$\left[H_\mu{}^a, K_b{}^{cd}\right] = \delta_b{}^a K_\mu{}^{cd} \tag{91}$$

We can verify that all the Jacobi identities involving any two of the $O_{1,13}$ generators $J_{\mu\nu}$, $J_a{}^b$, Q_{ab}, Q^{ab}, $H_{\mu a}$ or $H_\mu{}^a$, with either of the operators $K_{\mu\nu\lambda}$, $K_{\mu\nu a}$, $\cdots$, are satisfied.

Moreover any of the $O_{1,13}$ generators can be shown to commute with the following quadratic operator:

$$\left\{\begin{array}{l} \frac{1}{3!}K_{\mu\nu\lambda}K_{\mu\nu\lambda} \\ +\frac{1}{2}K_{\mu\nu a}K_{\mu\nu}{}^{a}+\frac{1}{2}K_{\mu\nu}{}^{a}K_{\mu\nu a} \\ +\frac{1}{2}K_{\mu ab}K_{\mu}{}^{ab}+\frac{1}{2}K_{\mu}{}^{ab}K_{\mu ab}-K_{\mu a}{}^{b}K_{\mu b}{}^{a} \\ +\frac{1}{3!}K_{abc}K^{abc}+\frac{1}{3!}K^{abc}K_{abc}+\frac{1}{2}K_{ab}{}^{c}K_{c}{}^{ab}+\frac{1}{2}K_{a}{}^{bc}K_{bc}{}^{a} \end{array}\right. \tag{92}$$

We proceed now to the introduction of the multiplet that can be associated with the above operator representation, and to the construction of the infinitesimal $O_{1,13}$ transformations that act on it. The desired multiplet with components $\{B_{\mu\nu\lambda}, B_{\mu\nu a}, \cdots\}$ can be introduced by the tensor module

$$\mathcal{B}=\left\{\begin{array}{l} \frac{1}{3!}B_{\mu\nu\lambda}K_{\mu\nu\lambda}+\frac{1}{2}B_{\mu\nu a}K_{\mu\nu}{}^{a}+\frac{1}{2}B_{\mu\nu}{}^{a}K_{\mu\nu a} \\ +\frac{1}{2}B_{\mu ab}K_{\mu}{}^{ab}+B_{\mu a}{}^{b}K_{\mu b}{}^{a}+\frac{1}{2}B_{\mu}{}^{ab}K_{\mu ab} \\ +\frac{1}{3!}B_{abc}K^{abc}+\frac{1}{2}B_{ab}{}^{c}K_{c}{}^{ab}+\frac{1}{2}B_{a}{}^{bc}K_{bc}{}^{a}+\frac{1}{3!}B^{abc}K_{abc} \end{array}\right\} \tag{93}$$

Introducing the $O_{1,13}$ parameter module,

$$\mathcal{W}=\frac{1}{2}\Omega_{\mu\nu}J_{\mu\nu}+\Omega_{a}{}^{b}J_{b}{}^{a}+\frac{1}{2}\Omega_{ab}Q^{ab}+\frac{1}{2}\Omega^{ab}Q_{ab}+\Omega_{\mu a}H_{\mu}{}^{a}+\Omega_{\mu}{}^{a}H_{\mu a} \tag{94}$$

we can compute the commutator $[\mathcal{W},\mathcal{B}]$. The latter gives a tensor module whose components would define the needed infinitesimal transformations.

For $B_{\mu\nu\lambda}$, we obtain

$$\delta B_{\mu\nu\lambda}=\left\{\begin{array}{l} \Omega_{\lambda\rho}B_{\mu\nu\rho}-\Omega_{\mu\rho}B_{\lambda\nu\rho}+\Omega_{\nu\rho}B_{\lambda\mu\rho} \\ +\Omega_{\nu a}B_{\lambda\mu}{}^{a}-\Omega_{\mu a}B_{\lambda\nu}{}^{a}+\Omega_{\lambda a}B_{\mu\nu}{}^{a} \\ +\Omega_{\lambda}{}^{a}B_{\mu\nu a}-\Omega_{\mu}{}^{a}B_{\lambda\nu a}+\Omega_{\nu}{}^{a}B_{\lambda\mu a} \end{array}\right\} \tag{95}$$

For $B_{\mu\nu a}$, we obtain

$$\delta B_{\mu\nu a}=\left\{\begin{array}{l} -\Omega_{\lambda a}B_{\lambda\mu\nu}-\Omega_{\lambda\mu}B_{\lambda\nu a}+\Omega_{\lambda\nu}B_{\lambda\mu a} \\ +\Omega_{\nu b}B_{\mu a}{}^{b}+\Omega_{ab}B_{\mu\nu}{}^{b}-\Omega_{\mu b}B_{\nu a}{}^{b} \\ -\Omega_{a}{}^{b}B_{\mu\nu b}+\Omega_{\mu}{}^{b}B_{\nu ab}-\Omega_{\nu}{}^{b}B_{\mu ab} \end{array}\right\} \tag{96}$$

For $B_{\mu\nu}{}^{a}$, we obtain

$$\delta B_{\mu\nu}{}^{a} = \left\{ \begin{array}{l} \Omega^{ab} B_{\mu\nu b} - \Omega_{\nu b} B_{\mu}{}^{ab} + \Omega_{\mu b} B_{\nu}{}^{ab} \\ \\ +\Omega_{\lambda\nu} B_{\lambda\mu}{}^{a} - \Omega_{\lambda\mu} B_{\lambda\nu}{}^{a} + \Omega_{b}{}^{a} B_{\mu\nu}{}^{b} \\ \\ -\Omega_{\lambda}{}^{a} B_{\lambda\mu\nu} + \Omega_{\mu}{}^{b} B_{\nu b}{}^{a} - \Omega_{\nu}{}^{b} B_{\mu b}{}^{a} \end{array} \right\} \tag{97}$$

For $B_{\mu ab}$, we obtain

$$\delta B_{\mu ab} = \left\{ \begin{array}{l} \Omega_{\mu\nu} B_{\nu ab} - \Omega_{\nu a} B_{\mu\nu b} + \Omega_{\nu b} B_{\mu\nu a} \\ \\ +\Omega_{\mu c} B_{ab}{}^{c} - \Omega_{bc} B_{\mu a}{}^{c} + \Omega_{ac} B_{\mu b}{}^{c} \\ \\ +\Omega_{a}{}^{c} B_{\mu bc} - \Omega_{b}{}^{c} B_{\mu ac} + \Omega_{\mu}{}^{c} B_{abc} \end{array} \right\} \tag{98}$$

For $B_{\mu a}{}^{b}$, we obtain

$$\delta B_{\mu a}{}^{b} = \left\{ \begin{array}{l} -\Omega^{bc} B_{\mu ac} - \Omega_{\mu c} B_{a}{}^{bc} + \Omega_{ac} B_{\mu}{}^{bc} \\ \\ +\Omega_{\nu a} B_{\mu\nu}{}^{b} + \Omega_{\mu\nu} B_{\nu a}{}^{b} - \Omega_{a}{}^{c} B_{\mu c}{}^{b} \\ \\ +\Omega_{c}{}^{b} B_{\mu a}{}^{c} + \Omega_{\mu}{}^{c} B_{ac}{}^{b} - \Omega_{\nu}{}^{b} B_{\mu\nu a} \end{array} \right\} \tag{99}$$

For $B_{\mu}{}^{ab}$, we obtain

$$\delta B_{\mu}{}^{ab} = \left\{ \begin{array}{l} \Omega_{\mu c} B^{abc} + \Omega_{\mu\nu} B_{\nu}{}^{ab} + \Omega^{bc} B^{a}_{\mu c} \\ \\ -\Omega^{ac} B_{\mu c}{}^{b} - \Omega_{c}{}^{a} B_{\mu}{}^{bc} + \Omega_{c}{}^{b} B_{\mu}{}^{ac} \\ \\ \Omega_{\mu}{}^{c} B_{c}{}^{ab} - \Omega_{\nu}{}^{a} B_{\mu\nu}{}^{b} + \Omega_{\nu}{}^{b} B_{\mu\nu}{}^{a} \end{array} \right\} \tag{100}$$

For B_{abc}, we obtain

$$\delta B_{abc} = \left\{ \begin{array}{l} -\Omega_{\mu a} B_{\mu bc} + \Omega_{\mu b} B_{\mu ac} - \Omega_{\mu c} B_{\mu ab} \\ \\ +\Omega_{cd} B_{ab}{}^{d} - \Omega_{bd} B_{ac}{}^{d} + \Omega_{ad} B_{bc}{}^{d} \\ \\ -\Omega_{a}{}^{d} B_{bcd} + \Omega_{b}{}^{d} B_{acd} - \Omega_{c}{}^{d} B_{abd} \end{array} \right\} \tag{101}$$

For $B_{ab}{}^{c}$, we obtain

$$\delta B_{ab}{}^{c} = \left\{ \begin{array}{l} \Omega^{cd} B_{abd} - \Omega_{bd} B_{a}{}^{cd} + \Omega_{ad} B_{b}{}^{cd} \\ \\ -\Omega_{\mu b} B_{\mu a}{}^{c} + \Omega_{\mu a} B_{\mu b}{}^{c} + \Omega_{a}{}^{d} B_{bd}{}^{c} \\ \\ -\Omega_{b}{}^{d} B_{ad}{}^{c} + \Omega_{d}{}^{c} B_{ab}{}^{d} - \Omega_{\mu}{}^{c} B_{\mu ab} \end{array} \right\} \tag{102}$$

For $B_a{}^{bc}$, we obtain

$$\delta B_a{}^{bc} = \left\{ \begin{array}{l} \Omega_{ad} B^{bcd} - \Omega_{\mu a} B_\mu{}^{bc} - \Omega^{cd} B^c_{ad} \\ \\ +\Omega^{bd} B_{ad}{}^c - \Omega_a{}^d B_d{}^{bc} - \Omega_d{}^b B_a{}^{cd} \\ \\ +\Omega_d{}^c B_a{}^{bd} - \Omega_\mu{}^b B_{\mu a}{}^c + \Omega_\mu{}^c B_{\mu a}{}^b \end{array} \right\} \tag{103}$$

Finally, for B^{abc}, we obtain

$$\delta B^{abc} = \left\{ \begin{array}{l} \Omega^{cd} B_d{}^{ab} - \Omega^{bd} B_d{}^{ac} + \Omega^{ad} B_d{}^{bc} \\ \\ +\Omega_d{}^a B^{bcd} - \Omega_d{}^b B^{acd} + \Omega_d{}^c B^{abd} \\ \\ -\Omega_\mu{}^a B_\mu{}^{bc} + \Omega_\mu{}^b B_\mu{}^{ac} - \Omega_\mu{}^c B_\mu{}^{ab} \end{array} \right\} \tag{104}$$

We can verify that, for any two tensor modules $\mathcal{A}$ and $\mathcal{B}$, the above infinitesimal transformations, acting in a like manner on the components of both, would leave invariant the following bilinear form:

$$\mathcal{A} \cdot \mathcal{B} = \left\{ \begin{array}{l} \frac{1}{3!} A_{\mu\nu\lambda} B_{\mu\nu\lambda} + \frac{1}{2} A_{\mu\nu a} B_{\mu\nu}{}^a + \frac{1}{2} A_{\mu\nu}{}^a B_{\mu\nu a} \\ \\ +\frac{1}{2} A_{\mu ab} B_\mu{}^{ab} - A_{\mu a}{}^b B_{\mu b}{}^a + \frac{1}{2} A_\mu{}^{ab} B_{\mu ab} \\ \\ +\frac{1}{3!} A_{abc} B^{abc} + \frac{1}{2} A_{ab}{}^c B_c{}^{ab} + \frac{1}{2} A_a{}^{bc} B_{bc}{}^a + \frac{1}{3!} A^{abc} B_{abc} \end{array} \right\} \tag{105}$$

5 The Antisymmetric Tensor Representation of Rank 5

An $O_{1,13}$ antisymmetric tensor representation of rank 5 would have the following $O_{1,3}$ and U_5 covariant components:

$$\left\{ \begin{array}{l} K_{\mu\nu\lambda\rho a}, K_{\mu\nu\lambda\rho}{}^a, \\ \\ K_{\mu\nu\lambda ab}, K_{\mu\nu\lambda a}{}^b, K_{\mu\nu\lambda}{}^{ab}, \\ \\ K_{\mu\nu abc}, K_{\mu\nu ab}{}^c, K_{\mu\nu a}{}^{bc}, K_{\mu\nu}{}^{abc}, \\ \\ K_{\mu abcd}, K_{\mu abc}{}^d, K_{\mu ab}{}^{cd}, K_{\mu a}{}^{bcd}, K_\mu{}^{abcd}, \\ \\ K_{abcde}, K_{abcd}{}^e, K_{abc}{}^{de}, K_{ab}{}^{cde}, K_a{}^{bcde}, K^{abcde} \end{array} \right\} \tag{106}$$

The symmetries of the above tensorial components with respect to spacetime and SU_5 indices should be clear. Notice that tensors with 4 vectorial indices $(\mu\nu\lambda\rho)$ could be traded for zero-vectorial-index counterparts using the 4-dimensional epsilon symbol $\epsilon_{\mu\nu\lambda\rho}$, and tensors with 3 vectorial indices $(\mu\nu\lambda)$ could be traded for a single-vectorial-index counterparts, again, using the epsilon symbol. However, it is more convenient to leave them in the present form, at this stage. We now proceed to the elaborate task of writing down the commutators of the above component operators with the generators of the $O_{1,13}$ algebra.

We have treated the above tensorial multiplet of 5th rank exactly in the same manner by which we have treated the 3rd rank counterpart. However, the associated formalism is too extensive to be presented here. Interested readers can request the pertinent material if they think it would be useful for their work.

We now move to the treatment of the spinorial fermionic multiplet, being the most essential part of our development.

6 The Dirac-Weyl Spinorial Representation of $O_{1,13}$

A Dirac spinor in 14-spacetime has $2^7 = 128$ components. A Weyl (chiral) constraint would reduce this to 64 components. We cannot have an additional Majorana constraint for fermions in 14-spacetime. In order to construct a corresponding multiplet of Weyl spinors in 4-spacetime, with components that are described by U_5 tensors, we introduce the following set of operators:

$$\left\{R, L_a, R_{ab}, L^{ab}, R^a, L\right\} \tag{107}$$

The above objects are alternately right-handed and left-handed Weyl spinors of the 4-spacetime. As before, the symbols $(a, b, c, \ldots)$ do pertain to SU_5, with the 2-index objects antisymmetric. We now write the commutators of the above operators with the generators of the $O_{1,13}$ algebra.

For the commutators with $J_{\mu\nu}$, the generators of the 4-spacetime Lorentz algebra, all the foregoing operator elements would satisfy commutators like this:

$$[J_{\mu\nu}, R] = -\frac{1}{2}\gamma_{\mu\nu} R \tag{108}$$

In the above, $\gamma_{\mu\nu}$ is a member of the Dirac algebra, being equal to $\frac{1}{2}\left[\gamma_\mu, \gamma_\nu\right]$ in terms of the Dirac matrix operators γ_μ that satisfy $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$.

For the commutators with ${J_a}^b$, the generators of U_5, we have

$$\left[{J_a}^b, R\right] = -\frac{1}{2}{\delta_a}^b R \tag{109}$$

$$\left[{J_a}^b, L_c\right] = {\delta_c}^b L_a - \frac{1}{2}{\delta_a}^b L_c \tag{110}$$

$$\left[{J_a}^b, R_{cd}\right] = \left({\delta_c}^b R_{ad} - {\delta_d}^b R_{ac}\right) - \frac{1}{2}{\delta_a}^b R_{cd} \tag{111}$$

$$\left[{J_a}^b, L^{cd}\right] = -\left({\delta_a}^c L^{bd} - {\delta_a}^d L^{bc}\right) + \frac{1}{2}{\delta_a}^b L^{cd} \tag{112}$$

$$\left[{J_a}^b, R^c\right] = -{\delta_a}^c R^b + \frac{1}{2}{\delta_a}^b R^c \tag{113}$$

$$\left[J_a{}^b, L\right] = \frac{1}{2}\delta_a{}^b L \tag{114}$$

For the commutators with Q_{ab}, we have

$$[Q_{ab}, R] = R_{ab} \tag{115}$$

$$[Q_{ab}, L_c] = \frac{1}{2}\epsilon_{abcde} L^{de} \tag{116}$$

$$[Q_{ab}, R_{cd}] = \epsilon_{abcde} R^e \tag{117}$$

$$\left[Q_{ab}, L^{cd}\right] = \left(\delta_a{}^c \delta_b{}^d - \delta_b{}^c \delta_a{}^d\right) L \tag{118}$$

$$[Q_{ab}, R^c] = 0 \tag{119}$$

$$[Q_{ab}, L] = 0 \tag{120}$$

For the commutators with Q^{ab}, we have

$$\left[Q^{ab}, R\right] = 0 \tag{121}$$

$$\left[Q^{ab}, L_c\right] = 0 \tag{122}$$

$$\left[Q^{ab}, R_{cd}\right] = -\left(\delta_c{}^a \delta_d{}^b - \delta_c{}^b \delta_d{}^a\right) R \tag{123}$$

$$\left[Q^{ab}, L^{cd}\right] = -\epsilon^{abcde} L_e \tag{124}$$

$$\left[Q^{ab}, R^c\right] = -\frac{1}{2}\epsilon^{abcde} R_{de} \tag{125}$$

$$\left[Q^{ab}, L\right] = -L^{ab} \tag{126}$$

For the commutators with $H_{\mu a}$, we have

$$[H_{\mu a}, R] = -\frac{1}{\sqrt{2}}\gamma_\mu L_a \tag{127}$$

$$[H_{\mu a}, L_b] = \frac{1}{\sqrt{2}}\gamma_\mu R_{ab} \tag{128}$$

$$[H_{\mu a}, R_{bc}] = -\frac{1}{2\sqrt{2}}\epsilon_{abcde}\gamma_\mu L^{de} \tag{129}$$

$$\left[H_{\mu a}, L^{bc}\right] = -\frac{1}{\sqrt{2}}\gamma_\mu \left(\delta_a{}^b R^c - \delta_a{}^c R^b\right) \tag{130}$$

$$\left[H_{\mu a}, R^b\right] = -\frac{1}{\sqrt{2}}\delta_a{}^b \gamma_\mu L \tag{131}$$

$$[H_{\mu a}, L] = 0 \tag{132}$$

For the commutators with $H_\mu{}^a$, we have

$$[H_\mu{}^a, R] = 0 \tag{133}$$

$$\left[H_\mu{}^a, L_b\right] = \frac{1}{\sqrt{2}}\delta_b{}^a\gamma_\mu R \tag{134}$$

$$\left[H_\mu{}^a, R_{bc}\right] = -\frac{1}{\sqrt{2}}\gamma_\mu\left(\delta_b{}^a L_c - \delta_c{}^a L_b\right) \tag{135}$$

$$\left[H_\mu{}^a, L^{bc}\right] = \frac{1}{2\sqrt{2}}\epsilon^{abcde}\gamma_\mu R_{de} \tag{136}$$

$$\left[H_\mu{}^a, R^b\right] = \frac{1}{\sqrt{2}}\gamma_\mu L^{ab} \tag{137}$$

$$\left[H_\mu{}^a, L\right] = \frac{1}{\sqrt{2}}\gamma_\mu R^a \tag{138}$$

We can verify that all the Jacobi identities involving any two of the $\mathrm{O}_{1,13}$ generators $J_{\mu\nu}$, $J_a{}^b$, Q_{ab}, Q^{ab}, $H_{\mu a}$, or $H_\mu{}^a$, with any of the operators R, L_a, R_{ab}, L^{ab}, R^a,or L, are satisfied.

7 The Dirac Conjugate Spinorial Representation of $\mathrm{O}_{1,13}$

In order to be able to write Lagrangian terms for fermionic fields we must introduce the conjugate spinorial representation. This can be done with the operator set

$$\left\{\bar{R}, \bar{L}^a, \bar{R}^{ab}, \bar{L}_{ab}, \bar{R}_a, \bar{L}\right\} \tag{139}$$

All these are Dirac conjugate spinors. We now write the commutators of all the $\mathrm{O}_{1,13}$ generators with the elements of the above set.

First, all the above set of operators, being all Dirac conjugate spinors, would have commutators with $J_{\mu\nu}$, the 4-spacetime Lorentz generators, that are like this:

$$\left[J_{\mu\nu}, \bar{R}\right] = \frac{1}{2}\bar{R}\gamma_{\mu\nu} \tag{140}$$

For the commutators with $J_a{}^b$, the U_5 generators, we have

$$\left[J_a{}^b, \bar{R}\right] = \frac{1}{2}\delta_a{}^b\bar{R} \tag{141}$$

$$\left[J_a{}^b, \bar{L}^c\right] = -\delta_a{}^c\bar{L}^b + \frac{1}{2}\delta_a{}^b\bar{L}^c \tag{142}$$

$$\left[J_a{}^b, \bar{R}^{cd}\right] = -\left(\delta_a{}^c\bar{R}^{bd} - \delta_a{}^d\bar{R}^{bc}\right) + \frac{1}{2}\delta_a{}^b\bar{R}^{cd} \tag{143}$$

$$\left[J_a{}^b, \bar{L}_{cd}\right] = \left(\delta_c{}^b\bar{L}_{ad} - \delta_d{}^b\bar{L}_{ac}\right) - \frac{1}{2}\delta_a{}^b\bar{L}_{cd} \tag{144}$$

$$\left[J_a{}^b, \bar{R}_c\right] = \delta_c{}^b\bar{R}_a - \frac{1}{2}\delta_a{}^b\bar{R}_c \tag{145}$$

$$\left[J_a{}^b, \bar{L}\right] = -\frac{1}{2}\delta_a{}^b\bar{L} \tag{146}$$

For the commutators with Q_{ab}, we have

$$\left[Q_{ab}, \bar{R}\right] = 0 \tag{147}$$

$$\left[Q_{ab}, \bar{L}^c\right] = 0 \tag{148}$$

$$\left[Q_{ab}, \bar{R}^{cd}\right] = \left(\delta_a{}^c\delta_b{}^d - \delta_b{}^c\delta_a{}^d\right)\bar{R} \tag{149}$$

$$\left[Q_{ab}, \bar{L}_{cd}\right] = \epsilon_{abcde}\bar{L}^e \tag{150}$$

$$\left[Q_{ab}, \bar{R}_c\right] = \frac{1}{2}\epsilon_{abcde}\bar{R}^{de} \tag{151}$$

$$\left[Q_{ab}, \bar{L}\right] = \bar{L}_{ab} \tag{152}$$

For the commutators with Q^{ab}, we have

$$\left[Q^{ab}, \bar{R}\right] = -\bar{R}^{ab} \tag{153}$$

$$\left[Q^{ab}, \bar{L}^c\right] = -\frac{1}{2}\epsilon^{abcde}\bar{L}_{de} \tag{154}$$

$$\left[Q^{ab}, \bar{R}^{cd}\right] = -\epsilon^{abcde}\bar{R}_e \tag{155}$$

$$\left[Q^{ab}, \bar{L}_{cd}\right] = -\left(\delta_c{}^a\delta_d{}^b - \delta_c{}^b\delta_d{}^a\right)\bar{L} \tag{156}$$

$$\left[Q^{ab}, \bar{R}_c\right] = 0 \tag{157}$$

$$\left[Q^{ab}, \bar{L}\right] = 0 \tag{158}$$

For the commutators with $H_{\mu a}$, we have

$$\left[H_{\mu a}, \bar{R}\right] = 0 \tag{159}$$

$$\left[H_{\mu a}, \bar{L}^b\right] = -\frac{1}{\sqrt{2}}\delta_a{}^b\bar{R}\gamma_\mu \tag{160}$$

$$\left[H_{\mu a}, \bar{R}^{bc}\right] = -\frac{1}{\sqrt{2}}\left(\delta_a{}^b\bar{L}^c - \delta_a{}^c\bar{L}^b\right)\gamma_\mu \tag{161}$$

$$\left[H_{\mu a}, \bar{L}_{bc}\right] = -\frac{1}{2\sqrt{2}}\epsilon_{abcde}\bar{R}^{de}\gamma_\mu \tag{162}$$

$$\left[H_{\mu a}, \bar{R}_b\right] = \frac{1}{\sqrt{2}}\bar{L}_{ab}\gamma_\mu \tag{163}$$

$$\left[H_{\mu a}, \bar{L}\right] = -\frac{1}{\sqrt{2}}\bar{R}_a\gamma_\mu \tag{164}$$

For the commutators with $H_\mu{}^a$, we have

$$\left[H_\mu{}^a, \bar{R}\right] = \frac{1}{\sqrt{2}}\bar{L}^a\gamma_\mu \tag{165}$$

$$\left[H_\mu{}^a, \bar{L}^b\right] = \frac{1}{\sqrt{2}} \bar{R}^{ab} \gamma_\mu \tag{166}$$

$$\left[H_\mu{}^a, \bar{R}^{bc}\right] = \frac{1}{2\sqrt{2}} \epsilon^{abcde} \bar{L}_{de} \gamma_\mu \tag{167}$$

$$\left[H_\mu{}^a, \bar{L}_{bc}\right] = -\frac{1}{\sqrt{2}} \left(\delta_b{}^a \bar{R}_c - \delta_c{}^a \bar{R}_b\right) \gamma_\mu \tag{168}$$

$$\left[H_\mu{}^a, \bar{R}_b\right] = \frac{1}{\sqrt{2}} \delta_b{}^a \bar{L} \gamma_\mu \tag{169}$$

$$\left[H_\mu{}^a, \bar{L}\right] = 0 \tag{170}$$

We can verify that all the Jacobi identities involving any two of the $O_{1,13}$ generators $J_{\mu\nu}$, $J_a{}^b$, Q_{ab}, Q^{ab}, $H_{\mu a}$, or $H_\mu{}^a$, with anyone of the operators $\bar{R}$, $\bar{L}^a$, $\bar{R}^{ab}$, $\bar{L}_{ab}$, $\bar{R}_a$, or $\bar{L}$, are satisfied.

Having constructed the manifestly 4-spacetime Lorentz covariant, as well as the U_5 covariant, algebraic representations for a fundamental $O_{1,13}$ spinor, and its Dirac conjugate, we can verify that all generators $J_{\mu\nu}$, $J_a{}^b$, Q_{ab}, Q^{ab}, $H_{\mu a}$, or $H_\mu{}^a$, of $O_{1,13}$ do commute with the following quadratic operator:

$$\bar{L}L - \bar{R}_a \bar{R}^a - \frac{1}{2} \bar{L}_{ab} L^{ab} - \frac{1}{2} \bar{R}^{ab} R_{ab} - \bar{L}^a L_a + \bar{R} R \tag{171}$$

We now proceed to construct the spinorial multiplet modules, giving the infinitesimal transformations of the components, and the invariant bilinear.

8 Fundamental Spinorial Multiplet

We introduce an SU_5 covariant multiplet of Weyl spinors in 4-spacetime, represented by the following module:

$$\Psi = \bar{L}\xi + \bar{R}_a \chi^a + \frac{1}{2} \bar{L}_{ab} \xi^{ab} + \frac{1}{2} \bar{R}^{ab} \chi_{ab} + \bar{L}^a \xi_a + \bar{R} \chi \tag{172}$$

Notice that the ξ's are right-handed Weyl spinors, while the χ's are left-handed. We also introduce the Dirac conjugate module:

$$\bar{\Psi} = \bar{\xi} L + \bar{\chi}_a R^a + \frac{1}{2} \bar{\xi}_{ab} L^{ab} + \frac{1}{2} \bar{\chi}^{ab} R_{ab} + \bar{\xi}^a L_a + \bar{\chi} R \tag{173}$$

Now, with the $O_{1,13}$ parameter module,

$$\mathcal{W} = \frac{1}{2} \Omega_{\mu\nu} J_{\mu\nu} + \Omega_a{}^b J_b{}^a + \frac{1}{2} \Omega_{ab} Q^{ab} + \frac{1}{2} \Omega^{ab} Q_{ab} + \Omega_{\mu a} H_\mu{}^a + \Omega_\mu{}^a H_{\mu a} \tag{174}$$

we can compute the commutators $[\mathcal{W}, \Psi]$ and $\left[\mathcal{W}, \bar{\Psi}\right]$. These give the corresponding spinorial modules whose components define the $O_{1,13}$ infinitesimal transformations.

For the infinitesimal transformations of the Ψ components, we obtain

$$\delta\xi = \frac{1}{4}\Omega_{\mu\nu}\gamma_{\mu\nu}\xi - \frac{1}{2}\Omega_a{}^a\xi - \frac{1}{2}\Omega_{ab}\xi^{ab} + \frac{1}{\sqrt{2}}\Omega_{\mu a}\gamma_\mu\chi^a \tag{175}$$

$$\delta\chi^a = \begin{pmatrix} \frac{1}{4}\Omega_{\mu\nu}\gamma_{\mu\nu}\chi^a + \Omega_b{}^a\chi^b - \frac{1}{2}\Omega_b{}^b\chi^a \\ -\frac{1}{4}\epsilon^{abcde}\Omega_{bc}\chi_{de} + \frac{1}{\sqrt{2}}\Omega_{\mu b}\gamma_\mu\xi^{ab} - \frac{1}{\sqrt{2}}\Omega_\mu{}^a\gamma_\mu\xi \end{pmatrix} \tag{176}$$

$$\delta\xi^{ab} = \begin{pmatrix} \frac{1}{4}\Omega_{\mu\nu}\gamma_{\mu\nu}\xi^{ab} - \Omega_c{}^a\xi^{bc} + \Omega_c{}^b\xi^{ac} - \frac{1}{2}\Omega_c{}^c\xi^{ab} \\ -\frac{1}{2}\epsilon^{abcdef}\Omega_{cd}\xi_e + \Omega^{ab}\xi + \frac{1}{2\sqrt{2}}\epsilon^{abcde}\Omega_{\mu c}\gamma_\mu\chi_{de} \\ -\frac{1}{\sqrt{2}}\Omega_\mu{}^b\gamma_\mu\chi^a + \frac{1}{\sqrt{2}}\Omega_\mu{}^a\gamma_\mu\chi^b \end{pmatrix} \tag{177}$$

$$\delta\chi_{ab} = \begin{pmatrix} \frac{1}{4}\Omega_{\mu\nu}\gamma_{\mu\nu}\chi_{ab} + \Omega_a{}^c\chi_{bc} - \Omega_b{}^c\chi_{ac} + \frac{1}{2}\Omega_c{}^c\chi_{ab} \\ -\Omega_{ab}\chi + \frac{1}{2}\epsilon_{abcde}\Omega^{cd}\chi^e - \frac{1}{\sqrt{2}}\Omega_{\mu b}\gamma_\mu\xi_a \\ +\frac{1}{\sqrt{2}}\Omega_{\mu a}\gamma_\mu\xi_b - \frac{1}{2\sqrt{2}}\epsilon_{abcde}\Omega_\mu{}^c\gamma_\mu\xi^{de} \end{pmatrix} \tag{178}$$

$$\delta\xi_a = \begin{pmatrix} \frac{1}{4}\Omega_{\mu\nu}\gamma_{\mu\nu}\xi_a - \Omega_a{}^b\xi_b + \frac{1}{2}\Omega_b{}^b\xi_a \\ +\frac{1}{4}\epsilon_{abcde}\Omega^{bc}\xi^{de} + \frac{1}{\sqrt{2}}\Omega_{\mu a}\gamma_\mu\chi + \frac{1}{\sqrt{2}}\Omega_\mu{}^b\gamma_\mu\chi_{ab} \end{pmatrix} \tag{179}$$

$$\delta\chi = \frac{1}{4}\Omega_{\mu\nu}\gamma_{\mu\nu}\chi + \frac{1}{2}\Omega_a{}^a\chi + \frac{1}{2}\Omega^{ab}\chi_{ab} - \frac{1}{\sqrt{2}}\Omega_\mu{}^a\gamma_\mu\xi_a \tag{180}$$

For the infinitesimal transformations of the $\bar{\Psi}$ components, we obtain

$$\delta\bar{\xi} = -\frac{1}{4}\Omega_{\mu\nu}\bar{\xi}\gamma_{\mu\nu} + \frac{1}{2}\Omega_a{}^a\bar{\xi} + \frac{1}{2}\Omega^{ab}\bar{\xi}_{ab} - \frac{1}{\sqrt{2}}\Omega_\mu{}^a\bar{\chi}_a\gamma_\mu \tag{181}$$

$$\delta\bar{\chi}_a = \begin{pmatrix} -\frac{1}{4}\Omega_{\mu\nu}\bar{\chi}_a\gamma_{\mu\nu} - \Omega_a{}^b\bar{\chi}_b + \frac{1}{2}\Omega_b{}^b\bar{\chi}_a \\ +\frac{1}{4}\epsilon_{abcde}\Omega^{bc}\bar{\chi}^{de} + \frac{1}{\sqrt{2}}\Omega_{\mu a}\bar{\xi}\gamma_\mu + \frac{1}{\sqrt{2}}\Omega_\mu{}^b\bar{\xi}_{ab}\gamma_\mu \end{pmatrix} \tag{182}$$

$$\delta\bar{\xi}_{ab} = \begin{pmatrix} -\frac{1}{4}\Omega_{\mu\nu}\bar{\xi}_{ab}\gamma_{\mu\nu} + \Omega_a{}^c\bar{\xi}_{bc} - \Omega_b{}^c\bar{\xi}_{ac} + \frac{1}{2}\Omega_c{}^c\bar{\xi}_{ab} \\ -\Omega_{ab}\bar{\xi} + \frac{1}{2}\epsilon_{abcde}\Omega^{cd}\bar{\xi}^e - \frac{1}{\sqrt{2}}\Omega_{\mu b}\bar{\chi}_a\gamma_\mu \\ +\frac{1}{\sqrt{2}}\Omega_{\mu a}\bar{\chi}_b\gamma_\mu - \frac{1}{2\sqrt{2}}\epsilon_{abcde}\Omega_\mu{}^c\bar{\chi}^{de}\gamma_\mu \end{pmatrix} \tag{183}$$

$$\delta\bar{\chi}^{ab} = \begin{pmatrix} -\frac{1}{4}\Omega_{\mu\nu}\bar{\chi}^{ab}\gamma_{\mu\nu} - \Omega_c{}^a\bar{\chi}^{bc} + \Omega_c{}^b\bar{\chi}^{ac} - \frac{1}{2}\Omega_c{}^c\bar{\chi}^{ab} \\ -\frac{1}{2}\epsilon^{abcde}\Omega_{cd}\bar{\chi}_e + \Omega^{ab}\bar{\chi} + \frac{1}{2\sqrt{2}}\epsilon^{abcde}\Omega_{\mu c}\bar{\xi}_{de}\gamma_\mu \\ -\frac{1}{\sqrt{2}}\Omega_\mu{}^b\bar{\xi}^a\gamma_\mu + \frac{1}{\sqrt{2}}\Omega_\mu{}^a\bar{\xi}^b\gamma_\mu \end{pmatrix} \tag{184}$$

$$\delta\bar{\xi}^a = \begin{pmatrix} -\frac{1}{4}\Omega_{\mu\nu}\bar{\xi}^a\gamma_{\mu\nu} + \Omega_b{}^a\bar{\xi}^b - \frac{1}{2}\Omega_b{}^b\bar{\xi}^a \\ -\frac{1}{4}\epsilon^{abcde}\Omega_{bc}\bar{\xi}_{de} + \frac{1}{\sqrt{2}}\Omega_{\mu b}\bar{\chi}^{ab}\gamma_\mu - \frac{1}{\sqrt{2}}\Omega_\mu{}^a\bar{\chi}\gamma_\mu \end{pmatrix} \tag{185}$$

$$\delta\bar{\chi} = -\frac{1}{4}\Omega_{\mu\nu}\bar{\chi}\gamma_{\mu\nu} - \frac{1}{2}\Omega_a{}^a\bar{\chi} - \frac{1}{2}\Omega_{ab}\bar{\chi}^{ab} + \frac{1}{\sqrt{2}}\Omega_{\mu a}\bar{\xi}^a\gamma_\mu \tag{186}$$

Having written the infinitesimal transformations, we can use them to show that the following kinetic spinorial bilinear is invariant:

$$\bar{\Psi}(i\gamma\cdot\partial)\Psi \Rightarrow \begin{pmatrix} \bar{\xi}(i\gamma\cdot\partial)\xi - \bar{\chi}_a(i\gamma\cdot\partial)\chi^a - \frac{1}{2}\bar{\xi}_{ab}(i\gamma\cdot\partial)\xi^{ab} \\ +\frac{1}{2}\bar{\chi}^{ab}(i\gamma\cdot\partial)\chi_{ab} + \bar{\xi}^a(i\gamma\cdot\partial)\xi_a - \bar{\chi}(i\gamma\cdot\partial)\chi \end{pmatrix} \tag{187}$$

9 The Composition of a Vector Mutiplet from Fermionic Bilinears

Here we give the composition of an $O_{1,13}$ vector multiplet with components $\{V_\mu, V_a, V^a\}$ from the χ and ξ components of a fundamental spinorial multiplet:

$$V_\mu = \frac{1}{\sqrt{2}}\begin{pmatrix} \bar{\xi}\gamma_\mu\xi + \bar{\chi}_a\gamma_\mu\chi^a - \frac{1}{2}\bar{\xi}_{ab}\gamma_\mu\xi^{ab} \\ -\frac{1}{2}\bar{\chi}^{ab}\gamma_\mu\chi_{ab} + \bar{\xi}^a\gamma_\mu\xi_a + \bar{\chi}\gamma_\mu\chi \end{pmatrix} \tag{188}$$

$$V_a = -\left(\bar{\chi}_a\xi + \bar{\xi}_{ab}\chi^b - \frac{1}{4}\epsilon_{abcde}\bar{\chi}^{bc}\xi^{de} + \bar{\xi}^b\chi_{ab} + \bar{\chi}\xi_a\right) \tag{189}$$

$$V^a = \left(\bar{\xi}\chi^a - \bar{\chi}_b\xi^{ab} - \frac{1}{4}\epsilon^{abcde}\bar{\xi}_{bc}\chi_{de} - \bar{\chi}^{ab}\xi_b + \bar{\xi}^a\chi\right) \tag{190}$$

Using the $O_{1,13}$ infinitesimal transformations of the components on both sides, we can verify that the above expressions are identities, in the sense that they are constructed properly to be covariant with respect to the full $O_{1,13}$ algebra.

10 The $O_{1,13}$ Couplings of a Vector to a Weyl Fermion

Using the forgoing composition of an $O_{1,13}$ vector in terms of the components os a Weyl fermion, we can now construct the couplings. Starting with a bilinear invariant of two vector modules $\mathcal{V}$ and $\mathcal{W}$,

$$\mathcal{V}\cdot\mathcal{W} = V_\mu W_\mu + V_a W^a + V^a W_a \tag{191}$$

we then replace the components W_μ, W_a, and W^a, by their compositions in terms of the χ and the ξ fermionic fields, and obtain the following manifestly Lorentz invariant, as well as U_5 invariant, coupling terms. These will be given according to the associated bosonic component.

First for the couplings to the vector V_μ, we have

$$V_\mu \times \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{\xi}\gamma_\mu\xi + \bar{\chi}_a\gamma_\mu\chi^a - \frac{1}{2}\bar{\xi}_{ab}\gamma_\mu\xi^{ab} \\ -\frac{1}{2}\bar{\chi}^{ab}\gamma_\mu\chi_{ab} + \bar{\xi}^a\gamma_\mu\xi_a + \bar{\chi}\gamma_\mu\chi \end{pmatrix} \tag{192}$$

For the couplings of the scalar 5-plet V_a, we have

$$V_a \times \left(\bar{\xi}\chi^a - \bar{\chi}_b\xi^{ab} - \frac{1}{4}\epsilon^{abcde}\bar{\xi}_{bc}\chi_{de} - \bar{\chi}^{ab}\xi_b + \bar{\xi}^a\chi\right) \tag{193}$$

For the couplings of the conjugate scalar 5-plet V^a, we have

$$V^a \times \left(-\bar{\chi}_a\xi - \bar{\xi}_{ab}\chi^b + \frac{1}{4}\epsilon_{abcde}\bar{\chi}^{bc}\xi^{de} - \bar{\xi}^b\chi_{ab} - \bar{\chi}\xi_a\right) \tag{194}$$

11 The Composition of a 3rd Rank Tensor Multiplet from Fermionic Bilinears

Here we give the composition of an $O_{1,13}$ tensor multiplet of 3rd rank with components $V_{\mu\nu\lambda}$, $V_{\mu\nu a}$, ${V_{\mu\nu}}^a$, etc., from the χ and the ξ components of a fundamental spinorial multiplet:

$$V_{\mu\nu\lambda} = \frac{1}{2} \begin{pmatrix} \bar{\xi}\gamma_{\mu\nu\lambda}\xi + \bar{\chi}_a\gamma_{\mu\nu\lambda}\chi^a - \frac{1}{2}\bar{\xi}_{ab}\gamma_{\mu\nu\lambda}\xi^{ab} \\ -\frac{1}{2}\bar{\chi}^{ab}\gamma_{\mu\nu\lambda}\chi_{ab} + \bar{\xi}^a\gamma_{\mu\nu\lambda}\xi_a + \bar{\chi}\gamma_{\mu\nu\lambda}\chi \end{pmatrix} \tag{195}$$

$$V_{\mu\nu a} = -\frac{1}{\sqrt{2}} \begin{pmatrix} \bar{\chi}_a\gamma_{\mu\nu}\xi + \bar{\chi}\gamma_{\mu\nu}\xi_a + \bar{\xi}_{ab}\gamma_{\mu\nu}\chi^b \\ +\bar{\xi}^b\gamma_{\mu\nu}\chi_{ab} - \frac{1}{4}\epsilon_{abcde}\bar{\chi}^{bc}\gamma_{\mu\nu}\xi^{de} \end{pmatrix} \tag{196}$$

$${V_{\mu\nu}}^a = \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{\xi}^a\gamma_{\mu\nu}\chi + \bar{\xi}\gamma_{\mu\nu}\chi^a - \bar{\chi}^{ab}\gamma_{\mu\nu}\xi_b \\ -\bar{\chi}_b\gamma_{\mu\nu}\xi^{ab} - \frac{1}{4}\epsilon^{abcde}\bar{\xi}_{bc}\gamma_{\mu\nu}\chi_{de} \end{pmatrix} \tag{197}$$

$$V_{\mu ab} = \left(-\bar{\xi}_{ab}\gamma_\mu\xi + \bar{\chi}\gamma_\mu\chi_{ab} + \frac{1}{2}\epsilon_{abcde}\bar{\xi}^c\gamma_\mu\xi^{de} - \frac{1}{2}\epsilon_{abcde}\bar{\chi}^{cd}\gamma_\mu\chi^e\right) \tag{198}$$

$${V_{\mu a}}^b = \left\{ \begin{matrix} \left(\bar{\xi}^b\gamma_\mu\xi_a - \bar{\chi}_a\gamma_\mu\chi^b + \bar{\xi}_{ac}\gamma_\mu\xi^{bc} - \bar{\chi}^{bc}\gamma_\mu\chi_{ac}\right) \\ +\frac{1}{2}{\delta_a}^b \begin{pmatrix} \bar{\xi}\gamma_\mu\xi + \bar{\chi}_c\gamma_\mu\chi^c - \frac{1}{2}\bar{\xi}_{cd}\gamma_\mu\xi^{cd} \\ +\frac{1}{2}\bar{\chi}^{cd}\gamma_\mu\chi_{cd} - \bar{\xi}^c\gamma_\mu\xi_c - \bar{\chi}\gamma_\mu\chi \end{pmatrix} \end{matrix} \right\} \tag{199}$$

$${V_\mu}^{ab} = \left(-\bar{\xi}\gamma_\mu\xi^{ab} + \bar{\chi}^{ab}\gamma_\mu\chi + \frac{1}{2}\epsilon^{abcde}\bar{\xi}_{cd}\gamma_\mu\xi_e - \frac{1}{2}\epsilon^{abcde}\bar{\chi}_c\gamma_\mu\chi_{de}\right) \tag{200}$$

$$V_{abc} = \sqrt{2}\epsilon_{abcde}\left(\frac{1}{2}\bar{\chi}^{de}\xi - \frac{1}{2}\bar{\chi}\xi^{de} - \bar{\xi}^d\chi^e\right) \tag{201}$$

$$V_{ab}{}^{c} = \frac{1}{\sqrt{2}} \left\{ \begin{array}{l} \left(-\bar{\xi}_{ab}\chi^{c} + \bar{\xi}^{c}\chi_{ab} - \frac{1}{2}\epsilon_{abdef}\bar{\chi}^{cd}\xi^{ef}\right) \\ \\ -\delta_{a}{}^{c} \left(\begin{array}{l} \bar{\chi}_{b}\xi + \bar{\chi}\xi_{b} \\ \\ +\bar{\xi}_{bd}\chi^{d} - \bar{\xi}^{d}\chi_{bd} + \frac{1}{4}\epsilon_{bdefg}\bar{\chi}^{de}\xi^{fg} \end{array} \right) \end{array} \right\} - (a \leftrightarrow b) \tag{202}$$

$$V_{a}{}^{bc} = \frac{1}{\sqrt{2}} \left\{ \begin{array}{l} \left(-\bar{\chi}^{bc}\xi_{a} + \bar{\chi}_{a}\xi^{bc} + \frac{1}{2}\epsilon^{bcdef}\bar{\xi}_{ad}\chi_{ef}\right) \\ \\ -\delta_{a}{}^{b} \left(\begin{array}{l} \bar{\xi}\chi^{c} - \bar{\xi}^{c}\chi \\ \\ -\bar{\chi}_{d}\xi^{cd} + \bar{\chi}^{cd}\xi_{d} - \frac{1}{4}\epsilon^{cdefg}\bar{\xi}_{de}\chi_{fg} \end{array} \right) \end{array} \right\} - (b \leftrightarrow c) \tag{203}$$

$$V^{abc} = -\sqrt{2}\epsilon^{abcde}\left(\bar{\xi}\chi_{de} - \bar{\xi}_{de}\chi - \frac{1}{2}\bar{\chi}_{d}\xi_{e}\right) \tag{204}$$

Using the infinitesimal $O_{1,13}$ transformations of the components on both sides, we can verify that the above expressions are identities, in the sense that they are constructed properly to be covariant with respect to the full $O_{1,13}$ algebra.

12 The $O_{1,13}$ Couplings of a 3rd Rank Tensor to a Weyl Fermion

Using the foregoing composition of a 3rd rank $O_{1,13}$ tensor in terms of the components of a Weyl fermion, we can now construct the couplings. Starting with the bilinear invariant $\mathcal{V} \cdot \mathcal{W}$ of two tensor modules $\mathcal{V}$ and $\mathcal{W}$, we would replace the components of $\mathcal{W}$ by their compositions in terms of the χ and ξ fermionic components. We obtain the manifestly 4-spacetime Lorentz invariant, and U_5 invariant, coupling terms. These will be given according to the associated bosonic field component.

Here we give the couplings of the fundamental fermions to the bosonic component $V_{\mu\nu\lambda}$,

$$\frac{1}{12}V_{\mu\nu\lambda} \times \left(\begin{array}{l} \bar{\xi}\gamma_{\mu\nu\lambda}\xi + \bar{\chi}\gamma_{\mu\nu\lambda}\chi - \frac{1}{2}\bar{\xi}_{ab}\gamma_{\mu\nu\lambda}\xi^{ab} \\ \\ +\bar{\chi}_{a}\gamma_{\mu\nu\lambda}\chi^{a} + \bar{\xi}^{a}\gamma_{\mu\nu\lambda}\xi_{a} - \frac{1}{2}\bar{\chi}^{ab}\gamma_{\mu\nu\lambda}\chi_{ab} \end{array} \right) \tag{205}$$

Notice that the tensor $V_{\mu\nu\lambda}$ can be traded for a vector using the epsilon symbol of 4-spacetime, and likewise, the Dirac matric $\gamma_{\mu\nu\lambda}$ can be traded for $\gamma_{\mu}\gamma_{5}$. This should be done in practical applications.

Here we give the couplings to the bosonic components $V_{\mu\nu a}$. The latter are a 5-plet of field-strength-like tensors. We have

$$\frac{1}{2\sqrt{2}}V_{\mu\nu a} \times \left(\begin{array}{l} \bar{\xi}\gamma_{\mu\nu}\chi^{a} - \bar{\chi}_{b}\gamma_{\mu\nu}\xi^{ab} + \bar{\xi}^{a}\gamma_{\mu\nu}\chi \\ \\ -\bar{\chi}^{ab}\gamma_{\mu\nu}\xi_{b} - \frac{1}{4}\epsilon^{abcde}\bar{\xi}_{bc}\gamma_{\mu\nu}\chi_{de} \end{array} \right) \tag{206}$$

Here we give the couplings to the conjugate 5-plet,

$$-\frac{1}{2\sqrt{2}}V_{\mu\nu}{}^{a}\times\begin{pmatrix}\bar{\chi}\gamma_{\mu\nu}\xi_a+\bar{\xi}_{ab}\gamma_{\mu\nu}\chi^b+\bar{\chi}_a\gamma_{\mu\nu}\xi\\ +\bar{\xi}^b\gamma_{\mu\nu}\chi_{ab}-\frac{1}{4}\epsilon_{abcde}\bar{\chi}^{bc}\gamma_{\mu\nu}\xi^{de}\end{pmatrix}\tag{207}$$

Here we give the couplings to the components $V_{\mu ab}$. These are a 10-plet of vector bosons that lie in the coset of O_{10} over U_5. We have

$$\frac{1}{2}V_{\mu ab}\times\left(-\bar{\xi}\gamma_\mu\xi^{ab}+\bar{\chi}^{ab}\gamma_\mu\chi+\frac{1}{2}\epsilon^{abcde}\bar{\xi}_{cd}\gamma_\mu\xi_e-\frac{1}{2}\epsilon^{abcde}\bar{\chi}_c\gamma_\mu\chi_{de}\right)\tag{208}$$

Here we give the couplings to the conjugate 10-plet $V_\mu{}^{ab}$,

$$\frac{1}{2}V_\mu{}^{ab}\times\left(\bar{\chi}\gamma_\mu\chi_{ab}-\bar{\xi}_{ab}\gamma_\mu\xi+\frac{1}{2}\epsilon_{abcde}\bar{\xi}^c\gamma_\mu\xi^{de}-\frac{1}{2}\epsilon_{abcde}\bar{\chi}^{cd}\gamma_\mu\chi^e\right)\tag{209}$$

Here we give the couplings to the components $V_{\mu a}{}^{b}$. These are the vector bosons of U_5. We have

$$\left\{\begin{array}{l}V_{\mu a}{}^{b}\times\left(\bar{\chi}_b\gamma_\mu\chi^a-\bar{\xi}^a\gamma_\mu\xi_b+\bar{\chi}^{ac}\gamma_\mu\chi_{bc}-\bar{\xi}_{bc}\gamma_\mu\xi^{ac}\right)\\ -\frac{1}{2}V_{\mu a}{}^{a}\times\begin{pmatrix}\bar{\xi}\gamma_\mu\xi-\bar{\chi}\gamma_\mu\chi-\bar{\xi}^b\gamma_\mu\xi_b+\bar{\chi}_b\gamma_\mu\chi^b\\ +\frac{1}{2}\bar{\chi}^{bc}\gamma_\mu\chi_{bc}-\frac{1}{2}\bar{\xi}_{bc}\gamma_\mu\xi^{bc}\end{pmatrix}\end{array}\right\}\tag{210}$$

Notice that we can decompose the U_5 tensor $V_{\mu a}{}^{b}$ into a trace part $V_{\mu a}{}^{a}$ and a traceless part, corresponding respectively to U_1 and SU_5.

Here we give the couplings to the components V_{abc},

$$-\frac{1}{\sqrt{2}}\frac{1}{3!}\epsilon^{abcde}V_{abc}\times\left(\bar{\xi}_{de}\chi-\bar{\xi}\chi_{de}+2\bar{\chi}_d\xi_e\right)\tag{211}$$

Notice that the 3-index antisymmetric SU_5 tensor V_{abc} can be traded for a 2-index antisymmetric tensor, using

$$\frac{1}{3!}\epsilon^{abcde}V_{abc}\Rightarrow V^{de}\tag{212}$$

These represent a conjugate 10-plet of scalars.

Here we give the couplings to the conjugate components V^{abc},

$$\frac{1}{\sqrt{2}}\frac{1}{3!}\epsilon_{abcde}V^{abc}\times\left(\bar{\chi}^{de}\xi-\bar{\chi}\xi^{de}-2\bar{\xi}^d\chi^e\right)\tag{213}$$

Also, we can make the replacement,

$$\frac{1}{3!}\epsilon_{abcde}V^{abc}\Rightarrow V_{de}\tag{214}$$

Here we give the couplings to the components ${V_{ab}}^c$,

$$\sqrt{2}\left\{\begin{array}{l} \frac{1}{2}{V_{ab}}^c \times \left(\bar{\chi}_c\xi^{ab} - \bar{\chi}^{ab}\xi_c + \frac{1}{2}\epsilon^{abdef}\bar{\xi}_{cf}\chi_{de}\right) \\ \\ +\frac{1}{2}{V_{ab}}^b \times \left(\bar{\xi}\chi^a - \bar{\xi}^a\chi + \bar{\chi}^{ac}\xi_c - \bar{\chi}_c\xi^{ac} - \frac{1}{4}\epsilon^{acdef}\bar{\xi}_{cd}\chi_{ef}\right) \end{array}\right\} \tag{215}$$

Finally, we give the couplings to the conjugate components ${V_a}^{bc}$,

$$\sqrt{2}\left\{\begin{array}{l} \frac{1}{2}{V_a}^{bc} \times \left(\bar{\xi}^a\chi_{bc} - \bar{\xi}_{bc}\chi^a - \frac{1}{2}\epsilon_{bcdef}\bar{\chi}^{ad}\xi^{ef}\right) \\ \\ +\frac{1}{2}{V_a}^{ab} \times \left(\bar{\chi}\xi_b - \bar{\chi}_b\xi + \bar{\xi}^c\chi_{bc} - \bar{\xi}_{bc}\chi^c - \frac{1}{4}\epsilon_{bcdef}\bar{\chi}^{cd}\xi^{ef}\right) \end{array}\right\} \tag{216}$$

13 Discussion

In this article, we have explored the structure of the fundamental spin-$\frac{1}{2}$ fermionic SU_5 multiplets that constitute a Weyl spinor in a 14-spacetime gravidynamic unification. We displayed the structure of the bosonic multiplets. These are spin-1 and spin-0 particles that could arise from the quantum contributions to the effective action. We have decomposed the pertinent fundamental couplings in the unified scheme.

At this point, the scheme could be decomposed further to display either the *color* SU_3 or the *family* SU_3 structure of the particles and the couplings. The subsequent steps can easily be done if we follow the work done[3] in this regard, in connection with O_{10} and U_5. The features and phenomenolgy discussed there, pertaining to quark color, or lepton family, with particular emphasis on the family structure of the W-like particles in the leptonic model, would all be applicable here. And we should remember that the 14-spacetime extension has duplicated the number of fermions with respect to O_{10}.

Again, we promise to return, in other articles, to the important problem of symmetry breaking in the effective action framework. We should give particular concern to the role that would be played by the scalar particle components in selecting the track of symmetry breaking via their mass-generating vacuum components.

As we have remarked in discussing the SU_3 family symmetry that appears in an SU_5 or O_{10} leptonic unification model[3], the important question is *whether the observed three generations of leptons would correspond to this triplet structure*, and whether the new (heavier) singlet leptons do exist. Now, in the 14-spacetime model treated in this paper, where we have a *duplication* of O_{10} fermions, it is important to contemplate the role played by the new triplet and the new singlet (electron-like and neutrino-like) particles, especially in connection with their *possible role in hadronic structure*, recalling that the spectrum of hadrons can very well be described by utilizing leptonic varieties of particles[7], [8] with integral electric charges. Whether the gauge and geometrical structure of the extra-dimensional theory could have a role in generating *solitonic*, or *magnetic-like*, interactions that can help understand hadronic physics, all pertains to interesting speculations, and should be the subject of further contemplation.

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