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Principle extremum of full action in electrodynamic

Annotation

Here we are going to formulate and prove variational extremum principle for electrodynamics, asserting that there exists a functional that depends on powers. This functional always has a single extremum, and the necessary and sufficient conditions of this extremum existence are represented by Maxwell equations. This principle is realized also in the case when the system contains magnetic charges and magnetic currents. Besides, this principle is valid also if there are heat losses in the system. The method for solving the Maxwell equations system by gradient descent to extremum is indicated.

Contents

Introduction

In his previous works [1, 2] the author has presented the principle extremum of full action, permitting to construct a functional for various physical systems, and, which is most important, for dissipative systems. Herewith the action is determined as an integral from kinetic, potential and heat energies. The principle has been applied in implicit form [3, 4] to electrical engineering, electromechanics, electrodynamics. In this paper the author is using a stricter extension of this principle for powers [2] applied to electrodynamics.

It is known [5], that Maxwell equations are deducted from the least action principle. For this purpose it is necessary to introduce the concept of vector magnetic potential and formulate a certain functional with respect to such potential and to scalar electrical potential, and this functional will be called action. Then by varying the action with respect to vector magnetic potential and to scalar electrical potential the conditions of this functional's minimum may be found. Further (after certain reductions) it is shown that this condition (with regard to the potentials) is equivalent to equations system with respect to electric and magnetic intensities. The obtained equation system corresponds **only to four** of Maxwell equations. It is evident, since the vector magnetic potential and electric scalar potential provide only four varying functions. But such partial result permits authors to conclude that **all** Maxwell equations (with respect to the intensities) are the consequences of least action principle as the above determined functional. But all Maxwell equations do not follow from this functional !

Furthermore, the Maxwell equations are dealing with currents in a medium with a certain electroconductivity. As a consequence, there are heat losses, i.e. energy dissipation. It means that, for the sake of the least action principle in addition to electromagnetic energy, the thermal energy should be also included in the functional; but this energy is **not** a part of Lagrangian. Therefore the Lagrange formalism is in principle not applicable to Maxwell equations.

Thus, the above conclusion, which has some cognitive value, does not demonstrate a triumph of the least action principle. And, all the more, this functional cannot be used for direct solution of technical problems (using the above described method of descent along the functional) So it turns out that the Lagrange formalism is insufficient for the deduction of Maxwell equations.

The matter becomes complicated also because for symmetrical form of Maxwell equations (figuring magnetic and magnetic current), an electromagnetic field cannot be described by vector potential that is continuous in all the space Therefore the symmetrical Maxwell equations cannot be deducted from variational least action principle, where the action is an integral of difference between kinetic and potential energies.

1. The power balance of electromagnetic field

The equation of electromagnetic field power balance in differential form is well known [6]. It has the following form

$$P_{\Pi} + P_{EH} + P_Q + P_C = 0, \tag{1}$$

where

P_{Π} - the density of power flow through a certain surface ,

P_{EH} - the density of electromagnetic power of an electromagnetic field,

P_Q - the density of heat loss power,

P_C - the density of outside current sources power.

Also

$$P_\Pi = \text{div}[E \times H] \tag{2}$$

or, according to a known formula of vector analysis,

$$P_\Pi = E \cdot \text{rot}(H) - H \cdot \text{rot}(E), \tag{3}$$

$$P_{EH} = \mu H \frac{dH}{dt} + \varepsilon E \frac{dE}{dt}, \tag{4}$$

$$P_Q = J_1 E, \tag{5}$$

$$P_C = J_2 E, \tag{6}$$

where

ε - absolute permittivity,
μ - absolute magnetic permeability,

J_1 - the density of conduction current,

J_2 - the current density of outside current source.

Here and further the three-component vectors

$$H, \ \frac{dH}{dt}, \ E, \ \frac{dE}{dt}, \ J_1, \ J_2, \ \text{rot}(H), \ \text{rot}(E)$$

are considered vectors in the sense of vector algebra. So the operations of multiplication for them may be written in simplified form. For instance, a product of vectors $E \cdot \text{rot}(H)$ is a product of column-vector E by row-vector $\text{rot}(H)$.

Let us denote

$$J = J_1 + J_2, \tag{7}$$

$$P_J = P_Q + P_C. \tag{8}$$

$$J = \text{grad}(K), \tag{9}$$

гдe K – scalar potential. From (5-9) that <u>electric current power</u>

$$P_J = E \cdot \text{grad}(K). \tag{10}$$

The charges in scalar potential field have got potential energy. The appropriate power

$$P_\rho = K\rho / \varepsilon, \tag{11}$$

where ρ - distribution density of summary (free and outside) charges.

Let us assume now, that there exist magnetic charges with density distribution σ and magnetic currents

$$M = \mathrm{grad}(L), \tag{12}$$

where L is a scalar parameter. Then by symmetry we should assume that there exists <u>magnetic current power</u>

$$P_M = H \cdot \mathrm{grad}(L), \tag{13}$$

potential energy of magnetworkic charges and the appropriate energy

$$P_\sigma = L\sigma/\mu, \tag{14}$$

where σ - distribution density of magnetic charges.

Let us denote also <u>the summary currents power</u> (electric and magnetic)

$$P_{JM} = P_J + P_M = E \cdot \mathrm{grad}(K) + H \cdot \mathrm{grad}(L). \tag{15}$$

and <u>the summary charges power</u> (electric and magnetic)

$$P_{\rho\sigma} = P_\rho + P_\sigma = K\rho/\varepsilon + L\sigma/\mu. \tag{16}$$

Then the equation of power balance of electromagnetic field takes the form

$$P_\Pi + P_{EH} + P_{JM} + P_{\rho\sigma} = 0, \tag{17}$$

where the components are determined as (3, 4, 15, 16) accordingly.

2. Building the functional for Maxwell equations

2.1. Maxwell equations

It is known that Heaviside was the first who introduced magnetic charges and magnetic currents to Maxwell's electrodynamics. At present there have been found some materials whose properties may be interpreted as properties of materials in which there are monopoles and magnetic [8]. Therefore we shall further consider a symmetrical Maxwell equations system with magnetic charges and currents – see, for instance, [3, 4].

Denote:

E - electric field intensity,

H - magnetic field intensity,

μ - magnetic permeability ,

ε - dielectric permittivity,

φ - electric scalar potential,

ϑ - electrical conductivity,

ρ - electric charge density

σ **-** magnetic charge density,

Let us take into consideration a Maxwell equations system in Cartesian coordinate system of the following form:

1.	$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} - \varepsilon \frac{\partial E_x}{\partial t} - \frac{dK}{dx} = 0$	
2.	$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} - \varepsilon \frac{\partial E_y}{\partial t} - \frac{dK}{dy} = 0$	
3.	$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} - \varepsilon \frac{\partial E_z}{\partial t} - \frac{dK}{dz} = 0$	
4.	$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} + \mu \frac{\partial H_x}{\partial t} + \frac{dL}{dx} = 0$	
5.	$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} + \mu \frac{\partial H_y}{\partial t} + \frac{dL}{dy} = 0$	
6.	$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} + \mu \frac{\partial H_z}{\partial t} + \frac{dL}{dz} = 0$	
7.	$-\frac{\partial E_x}{\partial x} - \frac{\partial E_y}{\partial y} - \frac{\partial E_z}{\partial z} + \frac{\rho}{\varepsilon} = 0$	
8.	$\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} - \frac{\sigma}{\mu} = 0$	

(1)

where

$$K = -\vartheta\varphi. \quad (2)$$

$$L = -\varsigma\phi, \quad (3)$$

Let us introduce also

j - electric current density,

m **-** magnetic current density

Then we get

$$\mathrm{j} = \mathrm{grad}(K), \quad (4)$$

$$\mathrm{m} = \mathrm{grad}(L). \quad (5)$$

In abbreviated form the system (1) looks as follows:

$$\mathrm{rot}H-\varepsilon\frac{dE}{dt}-\mathrm{grad}(K)=0\,, \tag{6}$$

$$\mathrm{rot}E+\mu\frac{dH}{dt}+\mathrm{grad}(L)=0\,, \tag{7}$$

$$\mathrm{div}E-\frac{\rho}{\varepsilon}=0, \tag{8}$$

$$\mathrm{div}H-\frac{\sigma}{\mu}=0. \tag{9}$$

Let us note some particular features of the equation system (1):

1. the existence of magnetic charges and currents is assumed,
2. instead of electric and magnetic currents we shall introduce scalar potentials and conductivities, not only electrical, but also magnetic ones.
3. it is assumed that the densities of electric and magnetic charges vary with time
4. these equations are extended also to physical systems containing microscopic bearers of electric and magnetic charges (as that we may consider, for instance, the ends of the permanent magnets).

The introduction of electric and magnetic potentials allows considering the system of 8 Maxwell equations as 8 unknown functions – 6 intensities и 2 scalar differentials. In the solution of these equations there appear the products $\vartheta\varphi$ and $\varsigma\phi$. A reader who doesn't accept the conception of magnetic resistancy ς of the environment and scalar magnetic potential ϕ, may notice that for $\varsigma=\infty,\ \ \phi=0$ the value of the product $\varsigma\phi$ is not determined, and may be taken equal to any value required by the problem's conditions. But here another paradox appears: the existence of magnetic current in the absence of magnetic conductivity and scalar magnetic potential. Nevertheless, accepting further the conception of magnetic resistancy and scalar magnetic potential we shall be able to solve some problems with definite physical meaning. We may note also that materials with high permeability μ, as, for example, soft iron, behave approximately as magnetic conductors.

2.2. Principle extremum of full action-2

The principle extremum of full action-2 [8] asserts that for this field of physics it is possible to construct a functional with respect to powers, and quasiextremals of this functional are the equations of real dynamic processes with respect to integral generalized coordinates q. In [1] some examples of such functionals in mechanics and electrical engineering are given. In these cases there exists a single independent variable - time on which all the generalized coordinates q depend.

For electrodynamics such functional should include the powers depending on generalized coordinates q, which, in their turn, depend not only on time, but on space coordinates.

For further discussion we shall merge the unknown functions of Maxwell equations into a vector of generalized coordinates

$$q = \lfloor E_x, E_y, E_z, H_x, H_y, H_z, K, L \rfloor . \tag{10}$$

This vector and all its components are functions of (x, y, z, t). We are considering an electromagnetic field in volume V, bounded by surface S. Full action-2 [1] in electrodynamics we shall submit in the form

$$\Phi = \int_0^T \left\{ \oint_V \Re(q(x,y,z,t)dV \right\} dt, \tag{11}$$

Having in mind the definition of Energian-2 in [1], we shall write Energian-2 in the following form:

$$\Re(q) = \left\{ P_{EH} - P_{\Pi} - P_{JM} - P_{\rho\sigma} \right\}. \tag{12}$$

Taking into account the formulas (1.3, 1.4, 1.17), we get:

$$\Re(q) = \left\{ \begin{array}{l} H \cdot \mathrm{rot}(E) - E \cdot \mathrm{rot}(H) + \mu H \dfrac{dH}{dt} + \varepsilon E \dfrac{dE}{dt} - \\ - \left(E \cdot \mathrm{grad}(K) + \dfrac{K\rho}{\varepsilon} \right) - \left(H \cdot \mathrm{grad}(L) + \dfrac{L\sigma}{\mu} \right) \end{array} \right\}. \tag{13}$$

Let us remind that the necessary conditions of extremum for a functional from functions of several independent variables - Ostrogradsky equations [7] have for each function the form

$$\frac{\partial f}{\partial q} - \sum_{a=x,y,z,t} \left[\frac{\partial}{\partial a} \left(\frac{\partial f}{\partial (dq/da)} \right) \right] = 0, \tag{14}$$

where f – the integrand, $q(x,y,z,t)$ – the variable function, a – independent variable. Further we shall denote the derivative, computed by this formula, by the symbol $\frac{\partial_o}{\partial v}$, contrary to ordinary partial derivative $\frac{\partial}{\partial v}$.

Let us write quasiextremal of the functional (11) for each i-component q_i of vector q

$$\left\{ \begin{array}{l} \frac{\partial P_{JM}}{\partial q_i} - \sum_{a=x,y,z,t} \left[\frac{d}{da} \left(\frac{\partial P_{JM}}{\partial [dq_i / da]} \right) \right] \\ + \frac{\partial P_{\rho\sigma}}{\partial q_i} + \frac{\partial P_{\Pi}}{\partial q_i} + \frac{\partial P_{EH}}{\partial q_i} \end{array} \right\} = 0. \quad (15)$$

The first four terms here correspond to the Ostrogradsky equation (14), and two others are ordinary partial derivatives. Differentiating (28), we get:

- by variable $E = \lfloor E_x, E_y, E_z \rfloor$ - equation (6),
- by variable $H = \lfloor H_x, H_y, H_z \rfloor$ - equation (7),
- by variables K, L - equations (8, 9) accordingly.

3. Splitting the functional for Maxwell equations

Let us associate with the functional (2.11) the functional oa split full action-2

$$\Phi_2 = \int_0^T \left\{ \oint_z \left\{ \oint_y \left\{ \oint_x \Re_2(q', q'') dx \right\} dy \right\} dz \right\} dt, \quad (1)$$

Let us present the split energian in the form

$$\Re_2(q',q'')=\begin{Bmatrix} \frac{1}{2}\left(H'\cdot \mathrm{rot}(E')+E'\cdot \mathrm{rot}(H')\right) \\ -\frac{1}{2}\left(H''\cdot \mathrm{rot}(E'')+E''\cdot \mathrm{rot}(H'')\right)+ \\ \frac{\mu}{2}\left(H'\frac{dH''}{dt}-H''\frac{dH'}{dt}\right)-\frac{\varepsilon}{2}\left(E'\frac{dE''}{dt}-E''\frac{dE'}{dt}\right) \\ -\left(E'\cdot \mathrm{grad}(K')+\frac{K'\rho}{\varepsilon}\right)+\left(E''\cdot \mathrm{grad}(K'')+\frac{K''\rho}{\varepsilon}\right) \\ -\left(H'\cdot \mathrm{grad}(L')+\frac{L'\sigma}{\mu}\right)+\left(H''\cdot \mathrm{grad}(L'')+\frac{L''\sigma}{\mu}\right) \end{Bmatrix} \tag{2}$$

The extremals of integral (1) by functions q', q'' found from Ostrogradsky equation, are the necessary conditions of the existence of a <u>sole saddle line</u>. It's possible also to prove that these extremals are sufficient condition of this saddle line existence [3, 4]. The optimal functions q'_o, q''_o satisfy these extremals.

We may also see [3, 4], that optimal functions q'_o, q''_o, satisfy also the condition

$$q'_o = q''_o, \tag{3}$$

and the sum of these extremals forms the Maxwell equations system (2.1), where

$$q = q'_o + q''_o. \tag{4}$$

Thus, the quasiextremal (2.15) coincides with Maxwell equations system and is the necessary and sufficient condition of the existence of a single saddle line of the functional (2.11, 2.12). Consequently, the Maxwell equations system determines the single and always existing extremum of the functional (2.11, 2.12).

The existence of a single extremum of the functional allows to solve the Maxwell equations system by the method of gradient motion to extremum [3, 4].

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