ON THE LORENTZ TRANSFORMATIONS

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Abstract

This paper examines Einstein's 1905 derivation of the Lorentz transformations using the average time of travel of a light pulse in opposite directions. It shows that the derivation has significant errors, produces a relationship between the coordinate values that does not exist, and that a more careful examination of his example implies very different characteristics of the velocity of light than those claimed. It also points out that more general derivations which do not address specific measurement conditions cannot be judged as to their applicability. The paper shows that Einstein's assertion of the independence of the shape of a spherical light pulse from the choice of the inertial reference frame of measurement is inconsistent with the known behavior of the photon. It further notes the impossibility of proving a general set of coordinate transformations, which would require a knowledge of all possible present and future coordinate measurement methods. It gives an example of a photon-emitting clock moving between two photon detectors, for which the transformations of the time of travel of the clock are consistent neither with the Lorentz nor the Galilean transformations.

Keywords: Special Relativity; Relativity; Lorentz transformations; Velocity of light; Coordinate transformations; Galilean transformations.

Einstein [1] attempts to derive a general set of coordinate transformations between a stationary and moving coordinate system by using the average time of travel of a light pulse from an emitter/detector to a mirror and back in the moving coordinate system. He expresses the average in terms of the time of travel of the light pulse as measured in the stationary system, and the x-coordinate of the moving system.

The coordinates in Einstein's moving system are ξ, τ, and in the stationary system x, t. The moving system travels at a constant velocity, v, along the x-axis of the stationary system. The emitter/detector is located at $\xi = 0$, and the mirror at $\xi = x'$. The light pulse is emitted at $x = \xi = 0$, $\tau = \tau_0$, travels to $\xi = x'$ at $\tau = \tau_1$, and arrives back at $\xi = 0$, at $\tau = \tau_2$.

His expression for the average time of travel in the moving coordinate system is

$$\tfrac{1}{2}\,[\tau_0(0,0,0,t) + \tau_2(0,0,0,t + x'/(c-v) + x'/(c+v))] = \tau_1(x',0,0,t + x'/(c-v)). \qquad (1)$$

But the expressions for the time values in the arguments of τ in this equation are already time coordinate transformations at $\xi = 0, \tau = \tau_0$, $\xi = x'$, $\tau = \tau_1$, and $\xi = 0$, $\tau = \tau_2$. That is, they represent the time values of the emission and arrival of the light pulse as measured in the stationary system at these points. They therefore raise the question of what sort of transformations he is in fact deriving with (1), and their applicability to the light pulse in his example. But let us put this logical issue aside for the moment, and simply look at the correctness of (1) and its use, based on his example.

Einstein converts (1) into a differential equation by treating x' as an arbitrary position coordinate along the ξ-axis in the moving coordinate system, and t as the time coordinate in the stationary coordinate system, and gets

$$\tfrac{1}{2}\,(1/(c-v) + 1/(c+v))\ \partial\tau/\partial t = \partial\tau/\partial x' + 1/(c+v)\ \partial\tau/\partial t\,, \text{ or}$$

$$\partial\tau/\partial x' = - c^2/(c^2-v^2)\ \partial\tau/\partial t \qquad (2)$$

This, however, is incorrect. x' is not an arbitrary ξ-axis coordinate value, but the magnitude of the distance from the emitter/detector to the mirror. If, for example, we place the mirror at $\xi = -x'$, then the time coordinates in the arguments for τ_1 and τ_2 become $(t - x'/(c-v))$, and $(t - x'/(c-v) - x'/(c+v))$. These are not only incorrect, but since t is the arbitrary but fixed time of emission of the initial light pulse, for any values of $|x'/(c-v)|$ and $|- x'/(c-v) - x'/(c+v)| > t$, then the time values in the arguments would be negative. Similarly, the time variable t in (1) is not the time coordinate in the stationary coordinate system, but rather the arbitrary initial time of emission, and must be correctly written as t_0. Therefore (1) must be correctly written as

$$\tfrac{1}{2}\left[\tau_0(0,0,0,t_0) + \tau_2(0,0,0,t_0+ |x'|/(c-v) + |x'|/(c+v))\right] = \tau_1(x',0,0,t_0 + |x'|/(c-v)) \qquad (3)$$

Since $t = t_0$ is an initial condition, it merely defines an arbitrary term on both sides of (1) and (3), which we can extract as follows.

$$\tfrac{1}{2}\left[\tau_0(0,0,0,0) + \tau_2(0,0,0,|x'|/(c-v) + |x'|/(c+v))\right] + \tau'(t_0) = \tau_1(x',0,0,|x'|/(c-v)) + \tau'(t_0) \qquad (4)$$

If the clocks in both systems are synchronized, then $\tau' = t_0$. So in fact both (1) and (3) are independent of t_0, and we have more correctly

$$\tfrac{1}{2}\left[\tau_0(0,0,0,0) + \tau_2(0,0,0,|x'|/(c-v) + |x'|/(c+v))\right] = \tau_1(x',0,0,|x'|/(c-v)), \qquad (5)$$

By treating the initial condition, t_0, as the time coordinate t itself, and the magnitude of the distance to the mirror as the ξ-coordinate, Einstein creates a relationship between the moving and stationary coordinates in (2) that does not exist.

But let us examine more carefully Einstein's time coordinate transformations,

$$|x'|/(c-v), \qquad (6)$$

and

$$|x'|/(c+v), \qquad (7)$$

which represent the time of travel of the light pulse in either direction over the distance $|x'|$ in the moving system, as measured in the stationary system.

These do not appear to me to be correct. In the stationary system, the distance travelled by the pulse from the emitter to the mirror is not $|x'|$, but rather $|x'| + v(\tau_1-\tau_0)$, where $(\tau_1-\tau_0)$ is the time of travel of the pulse between the emitter and mirror in the moving system. That is, $v(\tau_1-\tau_0)$ is the distance travelled by the mirror in the stationary system during the time of travel of the pulse. $(\tau_1-\tau_0)$ is the same as the time of travel of an identical pulse between a fixed emitter and mirror over the same distance in the stationary system. If we designate this as (t_1-t_0), and we take $\tau_0 = t_0 = 0$ for convenience, then $\tau_1 = t_1$ and we have $|x'| + vt_1$.

Since $|x'| = ct_1$, we then have

$$|x'| + vt_1 = ct_1 + vt_1 = (c+v)t_1, \qquad (8)$$

and

$$|x'| - vt_1 = ct_1 - vt_1 = (c-v)t_1 \qquad (9)$$

for the distance traveled by the light pulse in the two directions. Therefore, instead of being merely an adjustment for the apparent velocity of light, these represent an actual difference in the distance travelled by the light pulse in the same time interval in the two coordinate systems.

In other words, these express a physical difference in the velocity of light as measured in the two systems. So by using the apparently realistic behavior of the photon in Einstein's example, we are put in the somewhat perplexing position of having the light pulse physically travel at a velocity of c between the detector and emitter in the moving coordinate system, but at a velocity of $(c \pm v)$ in the stationary system. This suggests that the physical (not apparent) velocity of the photon is independent of the motion of the emitter and detector relative to each other, but is not independent of the motion of both emitter and detector relative to a third coordinate system.

Einstein also claims that the shape of a spherical light pulse remains unchanged when viewed in a coordinate system moving at a constant velocity relative to the source. This does not seem to me to be correct. The light pulse appears spherical if the emitting coordinate system is moving relative to that of the detectors, but not the reverse. We can see this using Einstein's example, with the emitter at the origin of the moving system, and detectors placed along the x-axis of the stationary system.

If we emit two photons at $\tau = 0$ in opposite directions from the origin along the ξ-axis in the moving system, then the velocity of the photons in the stationary coordinate system in which the detectors are located will be $\pm c$, the same as in the moving system, so the pulse will appear spherical. If, however, the detecting coordinate system is moving relative to the emitter, this will not be so. In this case, the movement of the detectors during the time of travel of the photon must be taken into account, so the measured (apparent) velocity of the two photons in the detecting coordinate system will be $(c \pm v)$, and the light pulse will never appear spherical,

Therefore, in principle, the emitting coordinate system defines a unique inertial reference frame relative to all other coordinate systems. That is, the magnitude and sign of the velocity of any other coordinate system moving relative to it can be determined in that moving coordinate system by measuring the degree and direction of distortion of a spherical light pulse from the emitting coordinate system.

Einstein's assertion of the universal applicability of a set of coordinate transformations also seems to me incorrect. In my view, the proof of such an assertion would be impossible, as it would assume a knowledge of all possible present and future coordinate measurement methods. I give as an example a photon-emitting clock moving at a constant velocity between two photon detectors. The time of travel of the clock as measured by the two detectors is consistent with neither the Lorentz nor the Galilean transformations.

The photon-emitting clock travels in a straight line between two fixed detectors, located at x=0 and x=x in the stationary coordinate system. In this case, the emitter is moving relative to the detectors, so the velocity of the photon is independent of the motion of the emitting clock relative to the detectors.

If the clock, traveling at velocity v along the x-axis, emits photons at a fixed rate, starting at t=0 and x=0, then when it arrives at x=x at t=t, it will have emitted N photons in an arbitrary direction. In its own coordinate system, then, the value of the time of arrival, τ, can be represented by N. To determine what the laboratory detectors will see, that is, the number of photons measured in the laboratory, we have to examine what happens to the photons emitted by the clock during the time of travel.

Depending on where the detectors are located, the results will differ. If the photons are emitted in the forward direction, then a detector stationed at the end of the path, at x=x, will have detected all N photons in the time of travel, t, because each photon emitted during the period of travel will move towards the detector more rapidly than the clock, and will have already arrived at the detector prior to, or along with the clock's arrival at x. Therefore for the front detector, $t=\tau$.

For photons emitted in the backward direction, however, this is not the case. There will be a threshold point, at some $x-\Delta x$, where the remaining time of travel of the clock will be less than the time of travel of emitted photons to the rear detector. That is, $\Delta x/v < (x-\Delta x)/c$. Equating the two, and solving for Δx, gives $\Delta x = vx/(c+v)$. During such a time period, the number of photons emitted will be $\Delta N = \Delta t\, dn/dt$, where Δt is the time of travel of the clock over the distance Δx, and dn/dt is the rate of photon emission of the clock in an arbitrary direction. These will not arrive at the detector during the time interval of travel. Therefore the number of photons detected at the rear detector in time t will be $N-\Delta N$. Since the rate of photon emission is constant over the length of travel, the ratio of $\Delta N/N$ will be the same as $\Delta x/x$, and we can then determine ΔN:

$$\Delta N/N = v/(c+v), \quad \Delta N = Nv/(c+v), \quad \text{so that}$$

$$N-\Delta N = N-Nv/(c+v), \text{ or } N(1-v/(c+v)).$$

Using t and τ instead of $N-\Delta N$ and N for the time interval expressed in the two coordinate systems, for the rear detector we then have

$$t = \tau\,(1-v/(c+v)). \tag{10}$$

If we assume a clock velocity of .9c, the rear transformation gives a value for t of approximately $.526\tau$, which indicates that a significant number of photons would not be observed at high velocities. If $v = c$, then $t = .5\tau$, and of course half the photons would be excluded, as the velocities would be the same. Thus the time measured by the moving clock during time τ as seen by the stationary observer would be the same as an equivalent stationary clock if measured by the front detector, that is $t = \tau$, but reduced by this factor when measured by the rear detector. The Lorentz transformation, $\tau = t\sqrt{(1-v^2/c^2)}$, with $v = .9c$, gives $t = 2.294\tau$ measured by either detector. The Galilean transformation gives $t = \tau$ for both detectors.

The above arguments do not, of course, preclude the possible appropriateness of the Lorentz transformations in some circumstances. But it seems to me difficult to see what those might be. More general derivations such as [2] cannot be judged as to their applicability without examining specific measurement conditions.

For instance, assuming invariance of the form of the coordinate transformations under a change of the stationary coordinate system may or may not be correct. In the above example of measuring the dimensions of a spherical light pulse there are two physically different cases, that of the emitter moving relative to the detectors and vice versa, and in each of these cases there are the values of the coordinates as measured in the two different coordinate systems.

If the emitting coordinate system is moving at +v relative to the detecting system, then the x-coordinates of the photons along the x-axis in the emitting system on arrival at the detectors are $\xi = (c-v)\tau$ and $-(c+v)\tau$. In the detecting coordinate system, they are $x = \pm ct$. If the clocks are synchronized and the moment of arrival of the photon in the detecting system can be communicated to the emitting system immediately, then $t = \tau$.

If the detecting coordinate system is moving at -v relative to the emitter, however, the x-coordinates in the emitting system are $\xi = \pm c\tau$, and the positions in the detecting coordinate system are $x = (c+v)t$, and $-(c-v)t$. Again, if the clocks are synchronized and the moment of arrival of the photon in the detecting system can be communicated to the emitting system immediately, then $t = \tau$. The time values for the arrival at the detectors in this case, of course, are different from those in the previous case.

Similarly, in the photon-emitting clock example, if the clock is moving relative to the detectors, then the time coordinate will be the same in both coordinate systems if measured by the front detector, but differ if measured by the rear detector: $t = \tau\,(1 - (v/(c+v)))$.

If the clock is stationary and the detectors move relative to it, then the motion of the detectors during the time of travel of the photons must be accounted for. This does not affect the time measurement of the front detector, but the transformation for the rear detector is changed to $t = \tau (1 - v/c)$.

Another assumption is that the relationship between the coordinates in the moving system, ξ, τ, and the stationary system must be expressed as a function of only the coordinates in the stationary system, x, t, and the relative velocity, v, or the relative velocity plus c. That is, $\xi,\tau = f_{\xi,\tau}(x,t,v)$, or $f_{\xi,\tau}(x,t,v,c)$. This is not generally true, as we can see in the example of the photon-emitting clock. If the rear detector, for example, is located off-axis, then the measured time of travel of the clock will depend upon the distance of the detector from the clock during the time of motion, $d = r/\sin \theta$, where $r = \sqrt{y^2+z^2}$, and θ is the angle of the detector relative to the x-axis at the location of the clock during the time of travel, or equivalently, $d = \sqrt{x^2+y^2+z^2}$.

Similarly, it seems to me that formal properties such as the invariance of the transformed form of the unbounded Maxwell and quantum wave equations in free space do not necessarily imply the correctness of the transformations when applied to equations with arbitrary boundary conditions and including potential functions. That is, the effects from the changes in the spatial and temporal boundaries and potentials on observable quantities due to the transformations would have to be verified empirically. It should also be noted that some of the issues treated here have been raised elsewhere. [3]

[1] A. Einstein, *Annalen der Physik*, 17:891, 1905. 1923 English translation

[2] V. Yakovenko, Lecture Notes for Course Phys171H, U. of Maryland, 15 Nov. 2004

[3] *Unsolved Problems in Special and General Relativity*, F. Smarandache, F. Yuhua, Z. Fengjuan, ed., Education Publishing, 2013.