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The Divergence-Free Effective Action for Quantum Electrodynamics

N.S. Baaklini
nsbqft@aol.com

Abstract

We present a simple introduction to a manifestly gauge-invariant perturbative development for quantum electrodynamics in the effective action formalism, and in the divergence-free framework. The system involves the interaction of the photon field with a Dirac fermionic spinor field. Detailed computations are exhibited, and divergence-free expressions for one and two-loop contributions are provided.

1 Introduction

The interaction of a Dirac fermionic spinor field Ψ, having Dirac conjugate $\bar{\Psi}$, with the bosonic photon field A_μ, a 4-dimensional vector, is described by the Lagrangian density

$$\mathcal{L} = \bar{\Psi}\left(i\gamma\cdot\nabla - m\right)\Psi + \frac{1}{2}A_\mu\Delta_{\mu\nu}A_\nu \tag{1}$$

where we have

$$i\gamma\cdot\nabla = i\gamma\cdot\partial + e\gamma\cdot A \qquad \Delta_{\mu\nu} = \partial^2\eta_{\mu\nu} - \partial_\mu\partial_\nu \tag{2}$$

Here, e is the electromagnetic coupling constant, and γ_μ are the four Dirac matrices in 4-dimensional spacetime that satisfy the anticommutators:

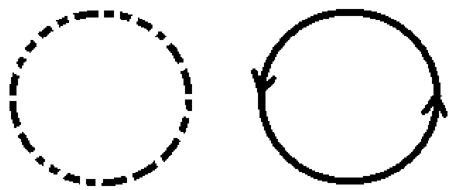
$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu} \qquad \mu,\nu = 0,1,2,3 \tag{3}$$

where $\eta_{\mu\nu}$ is the constant metric of Minkowski spacetime with $\{1,-1,-1,-1\}$ as the respective diagonal elements. In most of this article, we shall suppress the constant e. However, we shall reinstate it when needed.

From our general formalism[1], [2] dealing with similar Fermi-Bose systems, the one-loop corrections are given by

$$\left\{\begin{array}{l} \frac{i}{2}\mathrm{tr}\ln\left(\Delta_{\mu\nu} - \bar{\Psi}\gamma_\mu(i\gamma\cdot\nabla - m)^{-1}\gamma_\nu\Psi - \bar{\Psi}\gamma_\nu(i\gamma\cdot\nabla - m)^{-1}\gamma_\mu\Psi\right) \\ -i\,\mathrm{tr}\ln(i\gamma\cdot\nabla - m) \end{array}\right. \tag{4}$$

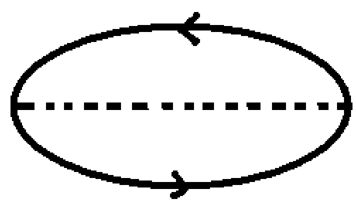
The first term describes an *effective photon loop* with implicit electron insertions, via the terms containing the fermionic fields Ψ and $\bar{\Psi}$. Notice that the latter insertions have other insertions, with respect to the photon field, contained in the covariant derivative. The second term describes an *effective electron loop*, however, with implicit photon insertions. The effective photon loop and the effective electron loop are represented by the following respective diagrams:



The two-loop contribution is given[1], [2] by

$$\left\{ \begin{array}{l} -\frac{1}{2}\left\{\Delta_{\mu\nu} - \bar{\Psi}\gamma_\mu(i\gamma\cdot\nabla - m)^{-1}\gamma_\nu\Psi - \bar{\Psi}\gamma_\nu(i\gamma\cdot\nabla - m)^{-1}\gamma_\mu\Psi\right\}^{-1} \times \\ \mathrm{tr}\left\{(i\gamma\cdot\nabla - m)^{-1}\gamma_\mu(i\gamma\cdot\nabla - m)^{-1}\gamma_\nu\right\} \end{array} \right. \tag{5}$$

This describes an effective electron loop whose *effective propagators* contain photon insertions, being traversed by an *effective photon propagator* which contains electron insertions. The following diagram depicts this contribution:



Notice that we use thick lines to indicate effective propagators with implicit insertions; a *continuous* line with charge arrows for the electron, and a *discontinuous* line for the photon. Later on when we show the expansions with respect to the implicit insertions, thin lines will be used.

Our strategy is to expand the above contributions with respect to the *effective fields* Ψ, $\bar{\Psi}$, and A, and compute the resulting terms of the series, all in a manner preserving effective *gauge invariance*, and in a *divergence-free* framework.

In the following section, we shall deal with the one-loop contributions, going into the details of regularizing, expanding, and computing the various terms. We shall compute the vacuum, bilinear, cubic, and quartic contrributions, and show how the results are gauge invariant, as well as, divergence free.

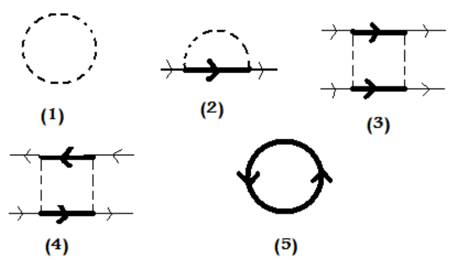
Subsequently, we shall move to present the manner of regularizing the two-loop contribution, and the manner of expanding it with respect to the effective photonic insertions.

2 One-Loop Contributions

Let us begin by expanding the one-loop contributions with respect to the effective fermion field insertions that are implicit in the effective photon propagator. The expansion to fourth order gives for the one-loop contributions:

$$\begin{cases} (1) & \frac{i}{2}\mathrm{tr}\ln(\Delta_{\mu\nu}) \\ (2) & -i\Delta^{-1}_{\mu\nu}\bar{\Psi}\gamma_{\mu}(\gamma\cdot\nabla-m)^{-1}\gamma_{\nu}\Psi \\ (3) & -\frac{i}{2}\Delta^{-1}_{\mu\nu}\bar{\Psi}\gamma_{\nu}(\gamma\cdot\nabla-m)^{-1}\gamma_{\lambda}\Psi\Delta^{-1}_{\lambda\rho}\bar{\Psi}\gamma_{\rho}(\gamma\cdot\nabla-m)^{-1}\gamma_{\mu}\Psi \\ (4) & -\frac{i}{2}\Delta^{-1}_{\mu\nu}\bar{\Psi}\gamma_{\nu}(\gamma\cdot\nabla-m)^{-1}\gamma_{\lambda}\Psi\Delta^{-1}_{\lambda\rho}\bar{\Psi}\gamma_{\mu}(\gamma\cdot\nabla-m)^{-1}\gamma_{\rho}\Psi \\ (5) & -i\,\mathrm{tr}\ln(i\gamma\cdot\nabla-m) \end{cases} \tag{6}$$

The above contributions are shown respectively in the following diagrams:



Notice that the bare photon and fermion propagators and insertions are shown with thin lines, while the effective fermion propagators that include implicit photon insertions are shown with thick lines.

The above terms are formally gauge invariant with respect to the effective photon field, implicit in the covariant operator $i\nabla = i\partial + A$. It is important to preserve this gauge invariance in any attempt to expand the above with respect to the effective photon field, while at the same time regularizing, or rather evading the divergences of the Feynman integrals involved. We shall see that a proper definition of the above expressions will produce the desired results.

Our regularization (or divergence-evasion) technique[1] is based on replacing operators that appear in the form of propagators in such a way that for any such operator Ω,

$$\ln(\Omega) \to \varrho_{\epsilon}\left(-\frac{1}{\epsilon}\Omega^{-\epsilon}\right) \qquad \Omega^{-1} \to \varrho_{\epsilon}\,\Omega^{-(1+\epsilon)} \tag{7}$$

where we have the regularizing (or pole-removing) prescription:

$$\varrho_{\epsilon}\,X(\epsilon) = \lim_{\epsilon\to 0}\left(\frac{\partial}{\partial\epsilon}\epsilon\right)X(\epsilon) \tag{8}$$

and ϵ is a limiting parameter, taken to zero after differentiation, and $X(\epsilon)$ is any regularized expression. It is important that in applying this technique that *each covariant propagator must have its own* limiting parameter ϵ and associated prescription ϱ_ϵ, and that *integration over loop momenta must be done before* applying these prescriptions and taking the limits.

In applying this technique to the operators occurring in the covariant quantum contributions such as displayed above, we shall first *put the vector and spinor propagators in convenient gauge-covariant forms*, then apply the regularizing definitions, and then proceed to expand with respect to the effective photon field.

For the logarithmic bosonic vacuum contribution $\frac{i}{2}\mathrm{tr}\ln(\Delta_{\mu\nu})$, we have the following prescription:

$$\frac{i}{2}\mathrm{tr}\left\{\left(\eta_{\mu\nu}-\frac{\partial_\mu\partial_\nu}{\partial^2}\right)\ln(\partial^2)\right\}\Rightarrow\frac{i}{2}\mathrm{tr}\left\{\left(\eta_{\mu\nu}-\frac{\partial_\mu\partial_\nu}{\partial^2}\right)\varrho_\epsilon\left(-\frac{1}{\epsilon}\frac{1}{(\partial^2+\mu^2)^\epsilon}\right)\right\} \quad (9)$$

Here we have given a small mass μ to the photon which would be taken to zero in the end; in fact, the above contribution will be seen to yield zero.

For the logarithmic fermionic contribution $-i\,\mathrm{tr}\ln(i\gamma\cdot\nabla-m)$, it is clear that it is equivalent to $-i\,\mathrm{tr}\ln(-i\gamma\cdot\nabla-m)$. Hence taking the average of the two, we write and regularize as follows:

$$-\frac{i}{2}\mathrm{tr}\ln\left\{(\gamma\cdot\nabla)^2+m^2\right\}\Rightarrow\frac{i}{2}\varrho_\epsilon\,\mathrm{tr}\left\{\frac{1}{\epsilon}\frac{1}{\{(\gamma\cdot\nabla)^2+m^2\}^\epsilon}\right\} \quad (10)$$

For the photon propagator, we shall write

$$\left.\begin{aligned}\Delta^{-1}_{\mu\nu}&=\tfrac{1}{\partial^2}(\eta_{\mu\nu}-\tfrac{\partial_\mu\partial_\nu}{\partial^2})\\ &\Rightarrow\varrho_\epsilon\tfrac{1}{(\partial^2)^{1+\epsilon}}(\eta_{\mu\nu}-\tfrac{\partial_\mu\partial_\nu}{\partial^2})\end{aligned}\right\} \quad (11)$$

For the fermion propagator, we shall write

$$\left.\begin{aligned}\tfrac{1}{i\gamma\cdot\nabla-m}&=-\tfrac{1}{(\gamma\cdot\nabla)^2+m^2}(i\gamma\cdot\nabla+m)\\ &\Rightarrow-\varrho_\epsilon\tfrac{1}{\{(\gamma\cdot\nabla)^2+m^2\}^{1+\epsilon}}(i\gamma\cdot\nabla+m)\end{aligned}\right\} \quad (12)$$

Notice that the above replacements still exhibit the results in a *manifestly gauge invariant manner*. Now, using methods for expanding the above expressions with regard to the effective photon field, performing loop integration, and then taking the proper limits, we are guaranteed gauge invariant results. We shall also see that our results are *always finite*.

2.1 Vacuum Photonic Contribution

Let us begin, for illustration, by computing the rather trivial vacuum contribution of the photon. As we have seen this is given by

$$\frac{i}{2}\mathrm{tr}\left\{\left(\eta_{\mu\nu}-\frac{\partial_\mu\partial_\nu}{\partial^2}\right)\varrho_\epsilon\left(-\frac{1}{\epsilon}\frac{1}{(\partial^2+\mu^2)^\epsilon}\right)\right\} \tag{13}$$

In momentum space we have

$$\frac{3i}{2}\varrho_\epsilon\frac{1}{\epsilon}\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{1}{(-p^2+\mu^2)^\epsilon} \tag{14}$$

where a factor of 3 has resulted from the trace over the vector projection operator.[1] This integral is Minkowskian, which may be evaluated by using the Feynman technique of adding a small imaginary part to μ^2 and choosing an appropriate contour for energy. Equivalently we can choose a contour along the imaginary energy axis and integrate in a corresponding Euclidean space. This is simply done by replacing p_0 by ip_4 and we are left with the Euclidean integral

$$-\frac{3}{2}\varrho_\epsilon\frac{1}{\epsilon}\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{1}{(p^2+\mu^2)^\epsilon} \tag{15}$$

This is easily evaluated and we have the result

$$-\frac{3}{2}\frac{1}{16\pi^2}\varrho_\epsilon\frac{1}{\epsilon}\frac{\Gamma(-2+\epsilon)}{\Gamma(\epsilon)}(\mu^2)^{2-\epsilon} \tag{16}$$

Simplifying, then executing the operation described by ϱ_ϵ, we are left with the finite result:

$$-\frac{3}{4}\frac{\mu^4}{16\pi^2}\left(\frac{3}{2}-\ln(\mu^2)\right) \tag{17}$$

Notice that in the limit $\mu^2\to 0$, the above result is vanishing. However, the above result does illustrate the case for massive bosons as well.

2.2 Fermionic Loop Contribution

Let us consider our regularized fermionic loop contribution

$$\frac{i}{2}\varrho_\epsilon\,\mathrm{tr}\left\{\frac{1}{\epsilon}\frac{1}{\{(\gamma\cdot\nabla)^2+m^2\}^\epsilon}\right\} \tag{18}$$

Using $\gamma\cdot\nabla=\gamma\cdot\partial-i\,\gamma\cdot A$, we shall write $(\gamma\cdot\nabla)^2+m^2=\Delta-Y$ where

$$\begin{cases}\Delta=\partial^2+m^2\\[2ex] Y=i(\gamma\cdot\partial\,\gamma\cdot A+\gamma\cdot A\,\gamma\cdot\partial)+A^2\end{cases} \tag{19}$$

[1] As a matter of fact, we can define our effective action framework for gauge fields (like the photon field in the present case) in a gauge-invariant manner where we can get rid of the non-diagonal part of the projection operator. In our case, this would replace $(\eta_{\mu\nu}-p_\mu p_\nu/p^2)$ by $\eta_{\mu\nu}$. This would give a great computational advantage especially for non-Abelian gauge theories[5].

Notice that Y is *expressed in the operator sense*. Hence our contribution is written in the form

$$\frac{i}{2}\varrho_\epsilon \operatorname{tr}\left\{\frac{1}{\epsilon}\frac{1}{(\Delta - Y)^\epsilon}\right\} \tag{20}$$

and we need to expand with respect to field-dependent part Y. For that purpose, we have the series

$$\operatorname{tr}\left\{\frac{1}{\epsilon}\frac{1}{(\Delta - Y)^\epsilon}\right\} = \operatorname{tr}\left\{\frac{1}{\epsilon}\frac{1}{\Delta^\epsilon} + \frac{1}{\Delta^{1+\epsilon}}Y + \frac{1}{2}\frac{\Gamma(2+\epsilon)}{\Gamma(1+\epsilon)}\underbrace{\frac{1}{\Delta}Y\frac{1}{\Delta}}_{2+\epsilon}Y + \cdots\right\} \tag{21}$$

The underbraced term is the above is understood[1], [2], [3] such that the two propagators, when going to momentum space, are to be combined via a Feynman parameter in the form $1/(\Delta + \cdots)^{2+\epsilon}$.

The *vacuum* contribution, or the term independent of Y, resulting from the above is

$$\frac{i}{2}\varrho_\epsilon \operatorname{tr}\left\{\frac{1}{\epsilon}\frac{1}{(\partial^2 + m^2)^\epsilon}\right\} \tag{22}$$

This gives in Euclidean momentum space,

$$-2\varrho_\epsilon \frac{1}{\epsilon}\int \frac{\mathrm{d}^4 p}{(2\pi)^4}\frac{1}{(p^2+m^2)^\epsilon} \tag{23}$$

where a factor of 4 has resulted from the trace over spinor matrices. Comparing with our computation of the preceding section, the above integral is easily done to obtain

$$-\frac{m^4}{16\pi^2}\left(\frac{3}{2} - \ln(m^2)\right) \tag{24}$$

A contribution that is linear in the photon field is clearly zero. The *bilinear* contribution comes from:

$$\frac{i}{2}\varrho_\epsilon \operatorname{tr}\left\{\frac{1}{\Delta^{1+\epsilon}}Y + \frac{1}{2}\frac{\Gamma(2+\epsilon)}{\Gamma(1+\epsilon)}\underbrace{\frac{1}{\Delta}Y\frac{1}{\Delta}}_{2+\epsilon}Y\right\} \tag{25}$$

Now from $Y = i(\gamma\cdot\partial\,\gamma\cdot A + \gamma\cdot A\,\gamma\cdot\partial) + A^2$, we obtain to order A^2,

$$\frac{i}{2}\varrho_\epsilon \operatorname{tr}\left\{\begin{array}{l} 1/(\partial^2+m^2)^{1+\epsilon} + \\ \frac{1}{2}\frac{\Gamma(2+\epsilon)}{\Gamma(1+\epsilon)}\underbrace{\frac{1}{(\partial^2+m^2)}i(\gamma\cdot\partial\,\gamma\cdot A + \gamma\cdot A\,\gamma\cdot\partial)\frac{1}{(\partial^2+m^2)}}_{2+\epsilon}i(\gamma\cdot\partial\,\gamma\cdot A + \gamma\cdot A\,\gamma\cdot\partial)\end{array}\right\} \tag{26}$$

Translating to momentum space, we obtain the bilinear term

$$\int \frac{\mathrm{d}^4 r}{(2\pi)^4}\frac{1}{2}A_\mu(r)A_\nu(-r)\mathcal{X}_{\mu\nu}(r) \tag{27}$$

where r is the external photon momentum, and the one-loop kernel $\mathcal{X}_{\mu\nu}(r)$ is given by

$$i\varrho_\epsilon\,\mathrm{tr}\int\frac{\mathrm{d}^4p}{(2\pi)^4}\begin{pmatrix}1/(-p^2+m^2)^{1+\epsilon}\eta_{\mu\nu}+\\ \frac{1}{2}\frac{\Gamma(2+\epsilon)}{\Gamma(1+\epsilon)}\int_0^1\mathrm{d}x\,\frac{1}{\{(1-x)(-p^2+m^2)+x[-(p+r)^2+m^2]\}^{2+\epsilon}}\times\\ \{\gamma\cdot p\,\gamma_\mu+\gamma_\mu\,\gamma\cdot(p+r)\}\,\{\gamma\cdot(p+r)\,\gamma_\nu+\gamma_\nu\,\gamma\cdot p\}\end{pmatrix} \tag{28}$$

Here we have shown the necessary combination of propagators via a Feynman parameter x. Now the numerator involving the gamma matrices gives

$$(2p_\mu+\gamma_\mu\gamma\cdot r)(2p_\nu+\gamma\cdot r\gamma_\nu) \tag{29}$$

Taking the trace over gamma matrices, we obtain

$$4(4p_\mu p_\nu+2p_\mu r_\nu+2p_\nu r_\mu+r^2\eta_{\mu\nu}) \tag{30}$$

Hence, we obtain

$$4i\,\varrho_\epsilon\int\frac{d^4p}{(2\pi)^4}\begin{pmatrix}1/(-p^2+m^2)^{1+\epsilon}\eta_{\mu\nu}+\\ \frac{1}{2}\frac{\Gamma(2+\epsilon)}{\Gamma(1+\epsilon)}\int_0^1\mathrm{d}x\frac{1}{\{-(p+xr)^2-x(1-x)r^2+m^2\}^{2+\epsilon}}\times\\ (4p_\mu p_\nu+2p_\mu r_\nu+2p_\nu r_\mu+r^2\eta_{\mu\nu})\end{pmatrix} \tag{31}$$

We have to make a shift in the momentum integrand of the second term $p\to(p-xr)$. The numerator becomes under this shift and symmetrization:

$$\left.\begin{aligned}4(p_\mu-xr_\mu)(p_\nu-xr_\nu)+2(p_\mu-xr_\mu)r_\nu+2(p_\nu-xr_\nu)r_\mu+r^2\eta_{\mu\nu}\\ \Rightarrow p^2\eta_{\mu\nu}-4x(1-x)r_\mu r_\nu+r^2\eta_{\mu\nu}\end{aligned}\right\} \tag{32}$$

Hence we obtain

$$4i\,\varrho_\epsilon\int\frac{d^4p}{(2\pi)^4}\begin{pmatrix}1/(-p^2+m^2)^{1+\epsilon}\eta_{\mu\nu}+\\ \frac{1}{2}\frac{\Gamma(2+\epsilon)}{\Gamma(1+\epsilon)}\int_0^1\mathrm{d}x\frac{1}{\{-p^2-x(1-x)r^2+m^2\}^{2+\epsilon}}\times\\ (p^2\eta_{\mu\nu}-4x(1-x)r_\mu r_\nu+r^2\eta_{\mu\nu})\end{pmatrix} \tag{33}$$

Transforming to Euclidean loop momentum, we have

$$-\,4\,\varrho_\epsilon\int\frac{d^4p}{(2\pi)^4}\begin{pmatrix}1/(p^2+m^2)^{1+\epsilon}\eta_{\mu\nu}+\\ \frac{1}{2}\frac{\Gamma(2+\epsilon)}{\Gamma(1+\epsilon)}\int_0^1\mathrm{d}x\frac{1}{\{p^2-x(1-x)r^2+m^2\}^{2+\epsilon}}\times\\ (-p^2\eta_{\mu\nu}-4x(1-x)r_\mu r_\nu+r^2\eta_{\mu\nu})\end{pmatrix} \tag{34}$$

Now integrating over p, we obtain

$$-\frac{1}{4\pi^2}\varrho_\epsilon \left\{ \begin{array}{l} \frac{\Gamma(-1+\epsilon)}{\Gamma(1+\epsilon)}(m^2)^{1-\epsilon}\eta_{\mu\nu} \\ -\frac{1}{2}\int_0^1 \mathrm{d}x \left(\begin{array}{l} 2\frac{\Gamma(-1+\epsilon)}{\Gamma(1+\epsilon)}\left\{-x(1-x)r^2+m^2\right\}^{1-\epsilon}\eta_{\mu\nu}+ \\ \frac{\Gamma(\epsilon)}{\Gamma(1+\epsilon)}\left\{-x(1-x)r^2+m^2\right\}^{-\epsilon}\left(4x(1-x)r_\mu r_\nu - r^2\eta_{\mu\nu}\right) \end{array} \right) \end{array} \right\} \tag{35}$$

Simplifying and executing the ϱ_ϵ prescription, we obtain

$$\frac{1}{4\pi^2}\left\{ \begin{array}{l} m^2\left(1-\ln(m^2)\right)\eta_{\mu\nu} \\ -\frac{1}{2}\int_0^1 \mathrm{d}x \left(\begin{array}{l} 2\left\{-x(1-x)r^2+m^2\right\}\left\{1-\ln\left[-x(1-x)r^2+m^2\right]\right\}\eta_{\mu\nu} \\ +\left(4x(1-x)r_\mu r_\nu - r^2\eta_{\mu\nu}\right)\ln\left[-x(1-x)r^2+m^2\right] \end{array} \right) \end{array} \right\} \tag{36}$$

Expanding to 4th order in the external photon momentum r, integrating over the Feynman parameter x, and simplifying, we obtain the gauge invariant result:

$$\frac{1}{12\pi^2}\left(r^2 - r_\mu r_\nu\right)\left\{\ln(m^2) - \frac{1}{5}\frac{r^2}{m^2} + \cdots\right\} \tag{37}$$

2.3 Fermion Self-Energy

The one-loop contribution to the fermion self-energy is represented by the term

$$-i\,\Delta^{-1}_{\mu\nu}\bar{\Psi}\gamma_\mu(i\gamma\cdot\nabla - m)^{-1}\gamma_\nu\Psi \tag{38}$$

According to our gauge-invariant regularization prescription, this should take the form

$$i\,\varrho_a\,\varrho_b\,\mathrm{tr}\left\{\frac{1}{(\partial^2)^{1+a}}\left(\eta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\partial^2}\right)\bar{\Psi}\gamma_\mu\frac{1}{\left\{(\gamma\cdot\nabla)^2+m^2\right\}^{1+b}}(i\gamma\cdot\nabla + m)\gamma_\nu\Psi\right\} \tag{39}$$

where a, b are two independent limiting parameters. As a matter of fact, the number of limiting parameters needed to guarantee the finiteness of an L-loop contribution does not have to be greater that L. In our one-loop calculation, we do not need both a, b, and we shall dispense with one of them, b, and call the remaining one ϵ (otherwise, we could proceed including both of them till the very end, when we shall discover that one of them is not necessary). Hence we shall work with the expression:

$$-i\,\varrho_\epsilon\,\mathrm{tr}\left\{\frac{1}{(\partial^2)^{1+\epsilon}}\left(\eta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\partial^2}\right)\bar{\Psi}\gamma_\mu\frac{1}{(i\gamma\cdot\nabla - m)}\gamma_\nu\Psi\right\} \tag{40}$$

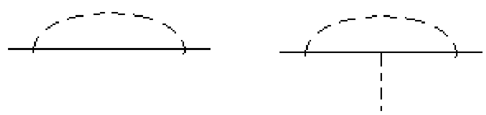
We shall expand the effective fermion propagator to first order in the photon field, hence, together with the fermion self-energy, we shall obtain the correction to the minimal coupling and the magnetic moment. For that purpose we have

$$\frac{1}{(i\gamma\cdot\nabla - m)} = \frac{1}{(i\gamma\cdot\partial - m)} - \frac{1}{(i\gamma\cdot\partial - m)}\gamma\cdot A\frac{1}{(i\gamma\cdot\partial - m)} + \cdots \tag{41}$$

Hence we have the two terms:

$$\left\{\begin{array}{l} i\,\varrho_\epsilon\,\mathrm{tr}\left\{\frac{1}{(\partial^2)^{2+\epsilon}}\left(\eta_{\mu\nu}\partial^2-\partial_\mu\partial_\nu\right)\bar{\Psi}\gamma_\mu\frac{1}{\partial^2+m^2}(i\gamma\cdot\partial+m)\gamma_\nu\Psi\right\}+ \\ i\,\varrho_\epsilon\,\mathrm{tr}\left\{\frac{1}{(\partial^2)^{2+\epsilon}}\left(\eta_{\mu\nu}\partial^2-\partial_\mu\partial_\nu\right)\bar{\Psi}\gamma_\mu\frac{1}{\partial^2+m^2}(i\gamma\cdot\partial+m)\gamma\cdot A\frac{1}{\partial^2+m^2}(i\gamma\cdot\partial+m)\gamma_\nu\Psi\right\}+\cdots \end{array}\right. \tag{42}$$

The first term describes the one-loop correction to the fermion self-energy. The second term describes the correction to the vertex with a photon insertion. These two terms correspond to the familiar two following graphs, respectively,



We shall convert these expressions to momentum space and compute them.
The *self-energy* term takes the following form in momentum space:

$$-\,i\,\varrho_\epsilon\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{1}{(-p^2)^{2+\epsilon}}(\eta_{\mu\nu}p^2-p_\mu p_\nu)\times\bar{\Psi}(r)\gamma_\mu\left\{\frac{1}{-(p+r)^2+m^2}\right\}\{\gamma\cdot(p+r)+m\}\gamma_\nu\Psi(r) \tag{43}$$

We shall simplify the factor involving the projection operator and the Dirac matrices:

$$\left.\begin{array}{r} (\eta_{\mu\nu}p^2-p_\mu p_\nu)\gamma_\mu\{\gamma\cdot(p+r)+m\}\gamma_\nu \\ \Rightarrow -p^2(3\gamma\cdot p+\gamma\cdot r)-2(p\cdot r)(\gamma\cdot p)+3p^2m \end{array}\right\} \tag{44}$$

Hence we have $\bar{\Psi}(r)\{\cdots\}\Psi(r)$, where the kernel is given by

$$-\,i\,\varrho_\epsilon\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{1}{(-p^2)^{2+\epsilon}}\frac{1}{-(p+r)^2+m^2}\left\{-p^2(3\gamma\cdot p+\gamma\cdot r)-2(p\cdot r)(\gamma\cdot p)+3p^2m\right\} \tag{45}$$

The propagators will be combined using a Feynman parameter x to give

$$-\,i\,\varrho_\epsilon\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\Gamma(3+\epsilon)}{\Gamma(2+\epsilon)}\int_0^1\mathrm{d}x\,(1-x)^{1+\epsilon}\frac{\{-p^2(3\gamma\cdot p+\gamma\cdot r)-2(p\cdot r)(\gamma\cdot p)+3p^2m\}}{\{-(p+xr)^2-x(1-x)r^2+xm^2\}^{3+\epsilon}} \tag{46}$$

We now need to make a shift in the loop momentum $p\to(p-xr)$. Under this shift and under loop-momentum symmetrization, the numerator becomes

$$3\left\{p^2\left(\frac{1}{2}(-1+3x)(\gamma\cdot r)+m\right)+x^2mr^2+(-1+x)x^2r^2(\gamma\cdot r)\right\} \tag{47}$$

Hence, after transforming to Euclidean loop momentum, we have

$$3\,\varrho_\epsilon\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\Gamma(3+\epsilon)}{\Gamma(2+\epsilon)}\int_0^1\mathrm{d}x\,(1-x)^{1+\epsilon}\frac{\{p^2\left(\frac{1}{2}(1-3x)(\gamma\cdot r)-m\right)+x^2mr^2+(-1+x)x^2r^2(\gamma\cdot r)\}}{\{p^2-x(1-x)r^2+xm^2\}^{3+\epsilon}} \tag{48}$$

Now integrating over momentum p, executing the prescription ϱ_ϵ, and integrating over the Feynman parameter x, we obtain the finite result:

$$\frac{3}{32\pi^2}\left\{(\gamma\cdot r)-2m(1-\ln m^2)+2m\left(1-\frac{m^2}{r^2}\right)\ln\left(1-\frac{r^2}{m^2}\right)\right\} \tag{49}$$

Notice that to second order in r, we have

$$\frac{3}{32\pi^2}\left\{(\gamma\cdot r)+3m\ln(m^2)-\frac{r^2}{m}+\ldots\right\} \tag{50}$$

The corresponding correction to the effective quantum action is

$$\frac{3e^2}{32\pi^2}\bar{\Psi}\left\{i(\gamma\cdot\partial)+\frac{1}{m}\partial^2+\cdots+2m\ln(m^2)\right\}\Psi \tag{51}$$

Here we have included the electromagnetic coupling constant e^2 that was suppressed throughout the computation. Notice that the first term contributes to the renormalization of the fermion field, the second term is a correction to the propagator, and the last term contributes to mass renormalization.

2.4 Vertex Correction

From the preceding section, the one-loop vertex correction takes the form

$$i\,\varrho_\epsilon\,\mathrm{tr}\left\{\frac{1}{(\partial^2)^{2+\epsilon}}\left(\eta_{\mu\nu}\partial^2-\partial_\mu\partial_\nu\right)\bar{\Psi}\gamma_\mu\frac{1}{\partial^2+m^2}(i\gamma\cdot\partial+m)\gamma\cdot A\frac{1}{\partial^2+m^2}(i\gamma\cdot\partial+m)\gamma_\nu\Psi\right\} \tag{52}$$

Translating this to momentum space we obtain

$$\left.\begin{array}{c}-i\varrho_\epsilon\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{1}{(-p^2)^{2+\epsilon}}(\eta_{\mu\nu}p^2-p_\mu p_\nu)\frac{1}{-(p+r+s)^2+m^2}\frac{1}{-(p+r)^2+m^2}\times\{\cdots\}\\ \{\cdots\}=\bar{\Psi}(r+s)\gamma_\mu\{\gamma\cdot(p+r+s)+m\}\gamma\cdot A(s)\{\gamma\cdot(p+r)+m\}\gamma_\nu\Psi(r)\end{array}\right\} \tag{53}$$

In the above, we suppress integrations over external momenta carried by the effective fields. The propagators will be combined using two Feynman parameters x and y. First we have

$$\left.\begin{array}{r}(1-x)\{-(p+r)^2+m^2\}+x\{-(p+r+s)^2+m^2\}\\ \Rightarrow -p^2-2(p\cdot r)-r^2-2x(p\cdot s)-xs^2-2x(r\cdot s)+m^2\end{array}\right\} \tag{54}$$

Then we have

$$\left.\begin{array}{r}(1-y)\{-p^2\}+y\{-p^2-2(p\cdot r)-r^2-2x(p\cdot s)-xs^2-2x(r\cdot s)+m^2\}\\ \Rightarrow -p^2-2y(p\cdot r)-yr^2-2yx(p\cdot s)-yxs^2-2yx(r\cdot s)+ym^2\end{array}\right\} \tag{55}$$

This may be written in the form

$$-(p+yr+yxs)^2-y(1-y)r^2-yx(1-yx)s^2-2xy(1-y)r\cdot s+ym^2 \tag{56}$$

Hence our integral becomes

$$\left\{\begin{array}{l} -i\varrho_\epsilon \frac{\Gamma(4+\epsilon)}{\Gamma(2+\epsilon)} \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y (1-y)^{1+\epsilon} y \times \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \times \\ \frac{1}{\{-(p+yr+yxs)^2 - y(1-y)r^2 - yx(1-yx)s^2 - 2xy(1-y)r\cdot s + ym^2\}^{4+\epsilon}} \end{array}\right\} \bar{\Psi}(r+s)\{\cdots\}\Psi(r) \tag{57}$$

where

$$\{\cdots\} = \left\{\begin{array}{l} p^2\gamma_\mu[\gamma\cdot(p+r+s)+m]\gamma\cdot A(s)[\gamma\cdot(p+r)+m]\gamma_\mu \\ -\gamma\cdot p[\gamma\cdot(p+r+s)+m]\gamma\cdot A(s)[\gamma\cdot(p+r)+m]\gamma\cdot p \end{array}\right. \tag{58}$$

We must make a shift in the loop momentum $p \to (p - yr - yxs)$. Under this shift, and upon symmetrizing with respect to loop momentum, the fermion kernel becomes an expression of the form $(p^2 M + N)$. Our integral, after going to Euclidean loop momentum, takes the form

$$\left\{\begin{array}{l} \varrho_\epsilon \frac{\Gamma(4+\epsilon)}{\Gamma(2+\epsilon)} \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y (1-y)^{1+\epsilon} y \times \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \times \\ \frac{1}{\{-p^2 - y(1-y)r^2 - yx(1-yx)s^2 - 2xy(1-y)r\cdot s + ym^2\}^{4+\epsilon}} \end{array}\right\} \bar{\Psi}(r+s)\{-p^2 M + N\}\Psi(r) \tag{59}$$

For the correction to the minimal coupling, a term of the integral independent of r, s, we find

$$\frac{3e^2}{32\pi^2}\bar{\Psi}\gamma\cdot A\Psi \tag{60}$$

Comparing to the result for the bare self-energy, of the preceding section, this shows that the fermion self energy is gauge invariant with a correction of the form

$$\frac{3e^2}{32\pi^2}\bar{\Psi}i\gamma\cdot\nabla\Psi \tag{61}$$

Investigating terms of the integral that are linear in the external momenta r, s, we are led to the action term

$$-\frac{3e^2}{32\pi^2}\frac{1}{m}i\left\{\bar{\Psi}\partial_\mu\Psi - (\partial_\mu\bar{\Psi})\Psi\right\}A_\mu \tag{62}$$

Comparing with the correction to the bare self-energy, of the preceding section, which is second-order in the momentum,

$$-\frac{3e^2}{32\pi^2}\frac{1}{m}\partial_\mu\bar{\Psi}\partial_\mu\Psi \tag{63}$$

we realize that the above term comes from the gauge invariant counterpart

$$-\frac{3e^2}{32\pi^2}\frac{1}{m}\nabla_\mu\bar{\Psi}\nabla_\mu\Psi \tag{64}$$

2.5 Four-Fermion Contribution

The general form of the four-fermion one-loop contribution is given by

$$\left\{ \begin{array}{l} -\frac{i}{2}\Delta^{-1}_{\mu\nu}\bar{\Psi}\gamma_\nu(\gamma\cdot\nabla-m)^{-1}\gamma_\lambda\Psi\Delta^{-1}_{\lambda\rho}\bar{\Psi}\gamma_\rho(\gamma\cdot\nabla-m)^{-1}\gamma_\mu\Psi \\ \\ -\frac{i}{2}\Delta^{-1}_{\mu\nu}\bar{\Psi}\gamma_\nu(\gamma\cdot\nabla-m)^{-1}\gamma_\lambda\Psi\Delta^{-1}_{\lambda\rho}\bar{\Psi}\gamma_\mu(\gamma\cdot\nabla-m)^{-1}\gamma_\rho\Psi \end{array} \right. \tag{65}$$

In fact, the contributions coming from this are finite even without a regularizing prescription. We shall be interested here in the first-order term without photon insertions, and we shall compute the contributions that do not depend on the momenta of the external fermions. Hence, the corresponding expression in momentum space is

$$\begin{array}{c} -\frac{i}{2}\int\frac{d^4p}{(2\pi)^4}\frac{1}{(-p^2)^4}\frac{1}{(-p^2+m^2)^2}(p^2\eta_{\mu\nu}-p_\mu p_\nu)(p^2\eta_{\lambda\rho}-p_\lambda p_\rho) \\ \times\left\{ \begin{array}{l} \bar{\Psi}\gamma_\mu(\gamma\cdot p+m)\gamma_\lambda\Psi\bar{\Psi}\gamma_\rho(\gamma\cdot p+m)\gamma_\nu\Psi+ \\ \\ \bar{\Psi}\gamma_\mu(\gamma\cdot p+m)\gamma_\lambda\Psi\bar{\Psi}\gamma_\nu(\gamma\cdot p+m)\gamma_\rho\Psi \end{array} \right\} \end{array} \tag{66}$$

Symmetrizing with respect to loop momentum, transforming to Euclidean loop momentum, simplifying and integrating, we obtain the result:

$$-\frac{11e^2}{32\pi^2}\frac{1}{m^2}\left\{(\bar{\Psi}\Psi)^2+\frac{3}{22}\bar{\Psi}\gamma_\mu\Psi\bar{\Psi}\gamma_\mu\Psi\right\} \tag{67}$$

3 Two-Loop Contributions

The two-loop contributions in quantum electrodynamics are described by the effective term:

$$\left\{ \begin{array}{l} -\frac{1}{2}\left(\Delta_{\mu\nu}-\bar{\Psi}\gamma_\mu(i\gamma\cdot\nabla-m)^{-1}\gamma_\nu\Psi-\bar{\Psi}\gamma_\nu(i\gamma\cdot\nabla-m)^{-1}\gamma_\mu\Psi\right)^{-1}\times \\ \\ \mathrm{tr}\left[(i\gamma\cdot\nabla-m)^{-1}\gamma_\mu(i\gamma\cdot\nabla-m)^{-1}\gamma_\nu\right] \end{array} \right. \tag{68}$$

This includes fermion and photon insertions that are obtained by expanding the above with respect to the implicit fermion and photon fields. The above may be written as

$$-\frac{1}{2}\Pi_{\mu\nu}\times\mathrm{tr}\left[(i\gamma\cdot\nabla-m)^{-1}\gamma_\mu(i\gamma\cdot\nabla-m)^{-1}\gamma_\nu\right] \tag{69}$$

This form shows that we can compute the effective fermion loop $\mathrm{tr}[\cdots]$, to any desired order in the associated photon insertions, without worrying about the structure of $\Pi_{\mu\nu}$. The latter and the associated bosonic loop computation may be done afterwards. To take care of divergences in the first loop, we replace each effective fermion propagator by its regular counterpart as follows:

$$\frac{1}{(i\gamma\cdot\nabla-m)}\Rightarrow\varrho_a\left\{\frac{1}{\left[(\gamma\cdot\nabla)^2+m^2\right]^{1+a}}(i\gamma\cdot\nabla+m)\right\} \tag{70}$$

Here a is a limiting parameter, and ϱ_a is the associated divergence-removal prescription. Hence we have for the preceding expression, with two parameters a and b,

$$-\frac{1}{2}\Pi_{\mu\nu}\varrho_a\,\varrho_b\,\mathrm{tr}\left\{\frac{1}{\left[(\gamma\cdot\nabla)^2+m^2\right]^{1+a}}(i\gamma\cdot\nabla+m)\gamma_\mu\frac{1}{\left[(\gamma\cdot\nabla)^2+m^2\right]^{1+b}}(i\gamma\cdot\nabla+m)\gamma_\nu\right\} \tag{71}$$

We can expand the above to any desired order in the photon field. For example, if we need to compute the effective action to second order in the photon field, we begin by using

$$\begin{cases} i\gamma\cdot\nabla = i\gamma\cdot\partial+\gamma\cdot A \\ (\gamma\cdot\nabla)^2=\partial^2-(i\gamma\cdot\partial)(\gamma\cdot A)-(\gamma\cdot A)(i\gamma\cdot\partial)-A^2 \end{cases} \tag{72}$$

For the factor

$$\frac{1}{\left[(\gamma\cdot\nabla)^2+m^2\right]^{1+a}} \tag{73}$$

We write, to second order,

$$\begin{cases} \frac{1}{(\partial^2+m^2)^{1+a}}+ \\ \frac{\Gamma(2+a)}{\Gamma(1+a)}\underbrace{\frac{1}{\partial^2+m^2}\{(i\gamma\cdot\partial)(\gamma\cdot A)+(\gamma\cdot A)(i\gamma\cdot\partial)\}\frac{1}{\partial^2+m^2}}_{2+a}+ \\ \frac{\Gamma(2+a)}{\Gamma(1+a)}\underbrace{\frac{1}{\partial^2+m^2}A^2\frac{1}{\partial^2+m^2}}_{2+a}+ \\ \frac{1}{2}\frac{\Gamma(3+a)}{\Gamma(1+a)}\underbrace{\frac{1}{\partial^2+m^2}\{(i\gamma\cdot\partial)(\gamma\cdot A)+(\gamma\cdot A)(i\gamma\cdot\partial)\}\frac{1}{\partial^2+m^2}\{(i\gamma\cdot\partial)(\gamma\cdot A)+(\gamma\cdot A)(i\gamma\cdot\partial)\}\frac{1}{\partial^2+m^2}}_{3+a}+\cdots \end{cases} \tag{74}$$

The meaning of the above underbrace notation should be clear from preceding work. It is clear that making the above substitutions, and converting to momentum space, any desired term may be obtained, and would be gauge invariant and free from divergences, provided the limiting procedure is applied correctly.

4 Discussion

In this article, we have given basic formalism, and detailed computations, more than enough, in order to show that our basic framework for the effective action is easily applicable to the theory of quantum electrodynamics. The renormalization procedure, which amounts to redefining the basic fields and parameters of the Lagrangian, via *finite corrections* in our case, as well as *setting the basic vacuum contributions to zero*, so that the flat spacetime computations remain consistent, can all be done without problems comparing with the example of the simple scalar field system[3]. More detailed and complete results for quantum electrodynamics, with a computing package[4], together with rules, and algorithms, to obtain results to any desired perturbative order, will be presented in other, more specialized, articles.

Extensions to non-Abelian gauge field[5], and to quantum gravitational[6], systems will be treated in other articles, as well.

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