On the Complete Elliptic Integrals and Babylonian Identity IV:

The Complete Elliptic Integral of first kind as sum of two Gauss hypergeometric functions

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And they went with haste and found Mary and Joseph, and the baby lying in a manger. And when they saw it, they made known the saying that had been told them concerning this child. And all who heard it wondered at what the shepherds told them. But Mary treasured up all these things, pondering them in her heart. And the shepherds returned, glorifying and praising God for all they had heard and seen, as it had been told them. – Luke 2:16-20.

Abstract. I evaluate the complete elliptic integral of first kind as the sum of two Gauss hypergeometric functions.

1. INTRODUCTION

In this paper, I evaluate the complete elliptic integral of first kind as the sum of two Gauss hypergeometric functions, as follows:

$$K(k) = \frac{\pi}{2} {}_2F_1\left(\frac{1}{2},\frac{1}{2};2;k^2\right) + \frac{\pi k^2}{16} {}_2F_1\left(\frac{3}{2},\frac{3}{2};3;k^2\right).$$

2. THEOREM

Theorem 1. *For* $0 < k < 1$, *then*

$$K(k) = \frac{\pi}{2} {}_2F_1\left(\frac{1}{2},\frac{1}{2};2;k^2\right) + \frac{\pi k^2}{16} {}_2F_1\left(\frac{3}{2},\frac{3}{2};3;k^2\right),$$

where $K(k)$ *is the complete elliptic integral of first kind.*

Proof. In previous paper [1], Theorem 1, I proved that

(1)
$$\frac{K(k)}{\sqrt{\pi}} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^2 \Gamma\left(n+\frac{3}{2}\right)}{\left(\frac{3}{2}\right)_n} \frac{k^{2n}}{n!^2},$$

for $0 < k < 1$.

I know that

(2)
$$\left(\frac{1}{2}\right)_n^2 = \frac{[(2n)!]^2}{2^{4n} n!^2},$$

(3)
$$\left(\frac{3}{2}\right)_n = \frac{(2(n+1))!}{2^{2n+1}(n+1)!}$$

and

(4)
$$\Gamma\left(n+\frac{3}{2}\right) = \sqrt{\pi}\frac{(2n+1)!}{2^{2n+1} n!}.$$

From (1), (2), (3) and (4), it follows that

$$\frac{K(k)}{\sqrt{\pi}} = \sqrt{\pi}\sum_{n=0}^{\infty}\frac{[(2n)!]^2 2^{2n+1}(n+1)!\,(2n+1)!}{2^{4n}\big(2(n+1)\big)!\,n!^2\,2^{2n+1}n!}\frac{k^{2n}}{n!^2}$$

$$= \sqrt{\pi}\sum_{n=0}^{\infty}\frac{[(2n)!]^2(n+1)!\,(2n+1)!}{n!^2\,\big(2(n+1)\big)!\,n!}\frac{k^{2n}}{2^{4n}n!^2}$$

$$= \sqrt{\pi}\sum_{n=0}^{\infty}\left(\frac{n+1}{n+2}\right)\frac{\binom{2n}{n}^2\binom{2n+1}{n}}{\binom{2n+2}{n}}\left(\frac{k}{4}\right)^{2n},$$

ergo,

$$K(k) = \pi\sum_{n=0}^{\infty}\left(\frac{n+1}{n+2}\right)\frac{\binom{2n}{n}^2\binom{2n+1}{n}}{\binom{2n+2}{n}}\left(\frac{k}{4}\right)^{2n}$$

$$= \pi\sum_{n=0}^{\infty}\left(\frac{n}{n+2}\right)\frac{\binom{2n}{n}^2\binom{2n+1}{n}}{\binom{2n+2}{n}}\left(\frac{k}{4}\right)^{2n} + \pi\sum_{n=0}^{\infty}\left(\frac{1}{n+2}\right)\frac{\binom{2n}{n}^2\binom{2n+1}{n}}{\binom{2n+2}{n}}\left(\frac{k}{4}\right)^{2n}$$

$$= \frac{\pi k^2}{16}\,{}_2F_1\left(\frac{3}{2},\frac{3}{2};3;k^2\right) + \frac{\pi}{2}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};2;k^2\right)$$

$$= \frac{\pi}{2}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};2;k^2\right) + \frac{\pi k^2}{16}\,{}_2F_1\left(\frac{3}{2},\frac{3}{2};3;k^2\right). \square$$

Note: For the reader's delight, I construct the Table 1.

REFERENCES

[1] Guedes, Edigles, *On the complete elliptic integrals and babylonian Identity 1: the* $\frac{1}{\pi}$ *formulaes involving gamma functions and summations*, 2013.

Table 1

In this table, I have: first column: m; second column: $k = 1/m$; third column: $\frac{\pi}{16m^2}\,{}_2F_1\left(\frac{3}{2},\frac{3}{2};3;\frac{1}{m^2}\right)$; fourth column: $\frac{1}{2}\pi\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};2;\frac{1}{m^2}\right)$, fifth column: $\frac{\pi}{16m^2}\,{}_2F_1\left(\frac{3}{2},\frac{3}{2};3;\frac{1}{m^2}\right)+\frac{1}{2}\pi\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};2;\frac{1}{m^2}\right)$; sixth column: $K(k)$.

2	$\frac{1}{2}$	0.06055480897	1.625195545	1.685750354	1.685750354
3	$\frac{1}{3}$	0.0238082898	1.593578445	1.617386735	1.617386735
4	$\frac{1}{4}$	0.01287669071	1.583365531	1.596242222	1.596242222
5	$\frac{1}{5}$	0.008097213128	1.578770634	1.586867847	1.586867847
6	$\frac{1}{6}$	0.005570305291	1.576308131	1.581878436	1.581878436
7	$\frac{1}{7}$	0.004069461643	1.574834457	1.578903918	1.578903918
8	$\frac{1}{8}$	0.003104358801	1.573882412	1.576986771	1.576986771
9	$\frac{1}{9}$	0.002446732173	1.57323169	1.575678422	1.575678422
10	$\frac{1}{10}$	0.00197833762	1.572767224	1.574745561	1.574745561
11	$\frac{1}{11}$	0.001632847064	1.572424101	1.574056948	1.574056948
12	$\frac{1}{12}$	0.00137067899	1.572163429	1.573534108	1.573534108
13	$\frac{1}{13}$	0.001167011611	1.571960744	1.573127756	1.573127756
14	$\frac{1}{14}$	0.00100563207	1.571800032	1.572805664	1.572805664
15	$\frac{1}{15}$	0.0008755836452	1.571670449	1.572546032	1.572546032
16	$\frac{1}{16}$	0.0007692443155	1.571564443	1.572333687	1.572333687
17	$\frac{1}{17}$	0.000681178131	1.57147662	1.572157798	1.572157798
18	$\frac{1}{18}$	0.0006074233099	1.571403046	1.572010469	1.572010469
19	$\frac{1}{19}$	0.0005450369919	1.571340797	1.571885834	1.571885834
20	$\frac{1}{20}$	0.0004917960419	1.571287661	1.571779457	1.571779457
21	$\frac{1}{21}$	0.0004459956038	1.571241943	1.571687938	1.571687938
22	$\frac{1}{22}$	0.0004063105236	1.571202322	1.571608632	1.571608632
23	$\frac{1}{23}$	0.0003716981682	1.571167761	1.571539459	1.571539459
24	$\frac{1}{24}$	0.0003413290826	1.571137433	1.571478762	1.571478762
25	$\frac{1}{25}$	0.0003145367283	1.571110674	1.571425211	1.571425211
26	$\frac{1}{26}$	0.0002907805271	1.571086946	1.571377726	1.571377726
27	$\frac{1}{27}$	0.0002696183309	1.571065806	1.571335424	1.571335424
28	$\frac{1}{28}$	0.0002506856666	1.571046892	1.571297578	1.571297578
29	$\frac{1}{29}$	0.0002336799135	1.571029902	1.571263582	1.571263582
30	$\frac{1}{30}$	0.0002183481196	1.571014584	1.571232932	1.571232932
31	$\frac{1}{31}$	0.0002044775275	1.571000724	1.571205202	1.571205202
32	$\frac{1}{32}$	0.0001918881458	1.570988144	1.571180032	1.571180032