Effective Action Framework for Divergence-Free Quantum Field Theory

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Abstract

We present the basic effective action framework for divergence-free quantum field theory. We describe the loopwise perturbative development of the effective action for a generic field theory, and indicate the manner by which this development is defined in order to evade the conventional divergences of the associated Feynman integrals.

1 Introduction

The divergences of conventional quantum field theory are well known. These occur in the four-dimensional momentum-space integrals that are associated with Feynman graph loops. Whereas it is possible in a restricted class of field theories, termed *renormalizable*, to dispense with the divergences, order by order in perturbation theory, through the process of redefining parameters of the theory (such as masses and coupling constants), this renormalization process is helpless in other theories some of which are physically important such as Einstein's field theory of gravity. Our purpose in this notebook is to present the basic framework of a fully consistent divergence-free approach to quantum field theory.

Our divergence-free approach to quantum field theory is based on the formalism of the *effective quantum action* [1], [2], [3], [4], and its associated effective Feynman graph rules. The key element of the scheme is a subtle analytic definition of the loop-wise perturbative development of the effective action that ensures gauge-invariant and consistent divergence-free results.

To introduce the basic idea in its simplest form, consider the following n-dimensional Euclidean momentum-space integral:

$$\int \mathrm{d}^n p \frac{1}{p^2+m^2} \tag{1}$$

It is clear that this integral is divergent in 4-dimensional Euclidean space. Evaluating for arbitrary dimensionality, we have the result in terms of a gamma function:

$$(\pi^2)^{\frac{n}{2}-1}\Gamma(1-\frac{n}{2})(m^2)^{\frac{n}{2}-1} \tag{2}$$

Notice that for a dimensionality $n = (4 - 2\epsilon)$, close to four, the result exhibits a pole in the limit $\epsilon \to 0$:

$$(\pi^2)^{1-\epsilon}\Gamma(\epsilon - 1)(m^2)^{1-\epsilon} = (\pi^2)^{1-\epsilon}\frac{\Gamma(\epsilon)}{(\epsilon - 1)}(m^2)^{1-\epsilon} \sim (\pi^2)^{1-\epsilon}\frac{\;}{\epsilon(\epsilon - 1)}(m^2)^{1-\epsilon} \tag{3}$$

This evaluation in arbitrary symbolic dimensionality illustrates the method of *dimensional regularization* [5], [6], [7] that is widely used in conventional theory in order to render the perturbative contributions well-defined while applying the renormalization process that dispenses with the divergences [8].

Another method of analytic regularization would utilize the following representation of the integral in four dimensions:

$$\int \mathrm{d}^4p \frac{1}{(p^2 + m^2)^{1+\epsilon}} \tag{4}$$

where ϵ is just a limiting parameter. Evaluated, this gives the result:

$$\pi^2 \frac{\Gamma(-1 + \epsilon)}{\Gamma(1 + \epsilon)}(m^2)^{1-\epsilon} = \pi^2 \frac{(m^2)^{1-\epsilon}}{\epsilon(\epsilon - 1)} \tag{5}$$

with the same pole structure as in dimensional regularization.

Whereas the *parametric* regularization method was noted [9], [10], [11], [12] before the advent of dimensional regularization, the latter was preferred because it respects *gauge invariance*. However, in our approach to the effective quantum action [4], gauge invariance with parametric regularization is always respected since we deal with expressions involving *gauge-covariant* operators.[1] For instance, the above integral would be part of an expression corresponding to the momentum-space matrix trace of the regularized operator:

$$\frac{1}{(\nabla^2 + m^2)^{1+\epsilon}} \tag{6}$$

where ∇^2 is a gauge-covariant Laplacian in four dimensions.

Our approach for handling divergences goes *much beyond* mere regularization. It actually dispenses with them altogether in a rather natural and consistent scheme. Doing that *consistently* is based on the fact that the *effective propagator*, in the perturbative development of the effective action, derives from varying the one-loop logarithmic contribution, which takes a form such as

$$\mathrm{tr}\ln(\nabla^2 + m^2) \tag{7}$$

with respect to its argument, and on representing the logarithm by a limiting analytic expression:

$$-\lim_{\epsilon \to 0} \frac{\partial}{\partial \epsilon}\mathrm{tr}(-\nabla^2 + m^2)^{-\epsilon} \tag{8}$$

[1]In our 1987 work[4], we had proposed gauge-invariant cutoff regularization, and suggested that the cutoff scale may perhaps be related consistently to the gravitational scale in a comprehensive theory containing gravitation. We shall return to to the gauge-invariant cutoff theory and its relationship to the present divergence-free framework in a subsequent work.

Consequently, in our perturbative development of the effective action, an *effective propagator* is represented by the limiting expression:

$$\lim_{\epsilon\to 0}\left(\frac{\partial}{\partial\epsilon}\epsilon\right)\frac{1}{(\nabla^2+m^2)^{1+\epsilon}} \tag{9}$$

Notice that the above covariant operator contains the gauge field, and would have to be expanded with respect to the latter before performing Feynman integrations in momentum space. All momentum integrations would have to be performed before applying the limiting procedure, then all results would turn out to be free from divergences. For instance, applying the limiting procedure to the one-loop integral

$$\lim_{\epsilon\to 0}\left(\frac{\partial}{\partial\epsilon}\epsilon\right)[\int \mathrm{d}^4p\frac{1}{(p^2+m^2)^{1+\epsilon}}]=\lim_{\epsilon\to 0}\left(\frac{\partial}{\partial\epsilon}\epsilon\right)[\pi^2\frac{(m^2)^{1-\epsilon}}{\epsilon(\epsilon-1)}] \tag{10}$$

would give the result

$$-\pi^2m^2\left(1-\ln(m^2)\right) \tag{11}$$

Notice that the representation of the logarithmic one-loop contribution of the effective quantum action by a limiting parametric expression is equivalent to the *zeta-function method* [13], [14], [15], [16], [17] of regularization; this is known to yield a finite result in one-loop integrals. In the attempt to generalize the one-loop zeta-function result to higher loops, the socalled method of *operator regularization* [18], [19] had proposed to apply, rather *arbitrarily*, a generalized limiting procedure applying:

$$\lim_{\epsilon\to 0}\left(\frac{\partial}{\partial\epsilon^L}\epsilon^L\right)X(L) \tag{12}$$

to each effective L-loop contribution $X(L)$ (which has a maximal ϵ^{-L} divergence). This prescription, which *does not follow as a natural consequence of the one-loop counterpart*, is rather *ad hoc*, and may turn out to be inconsistent.[2]

However, in our approach, we *associate a limiting procedure with each effective propagator*; this being *a natural consequence of the one-loop effective contribution*, we are led to fully consistent results. Notice that an L-loop contribution to the effective action would contain several effective propagators. In our approach, each effective propagator would require a different limiting parameter such as ϵ and a *pole-removing operator* such as $(\partial/\partial\epsilon)\epsilon$ to be applied to the result after performing integrations over all loop momenta. This would preserve *gauge invariance*, ensure *freedom from divergences*, and guarantee *consistency*.

2 Effective Quantum Action

The effective action of quantum field theory, defined by the functional integral, is an elegant and potentially very powerful framework for computing quantum effects, in a

[2] Later works by the advocates of the scheme proposed in [18] demonstrate the difficulty of securing *unitarity* in the handling of higher-loop contributions.

manner that preserves underlying fundamental symmetries. In the followings, we begin by reviewing the functional-integral definition of the effective action[1], [4]. Then proceed to develop a perturbative formalism in Planck's constant, and derive expressions for mean values on the basis of the Gaussian functional integral. An explicit second-order (two-loop) expression is obtained. Higher-order contributions are prescribed via effective graphical rules.

2.1 The Generating Functional for Proper Vertices

The generating functional $Z(J)$ for connected Green functions [8] is defined by the functional integral

$$e^{\frac{i}{\hbar}Z(J)} = \int (d\varphi)\, e^{\frac{i}{\hbar}\{W(\varphi)+\varphi_i J_i\}} \tag{13}$$

where φ_i represents a quantum field component, J_i a corresponding external source, and $W(\varphi)$ is the classical action functional. Here, we use the compact notation where the index i represents all labels and spacetime arguments.

The field φ_i, being functionally integrated over, will be termed the *virtual* field. On the other hand, the *effective* field is defined by

$$\phi_i = \frac{\partial Z(J)}{\partial J_i} \tag{14}$$

The *effective action* $\Gamma(\phi)$, being the generating functional for proper vertices [8], is defined by the Legendre transformation

$$\Gamma(\phi) = Z(J) - \phi_i J_i \tag{15}$$

This gives upon functional differentiation,

$$\frac{\partial \Gamma}{\partial \phi_i} = \frac{\partial Z}{\partial J_k}\frac{\partial J_k}{\partial \phi_i} - J_i - \phi_k \frac{\partial J_k}{\partial \phi_i} = -J_i \tag{16}$$

Hence, we obtain the functional integral expression,

$$e^{\frac{i}{\hbar}\Gamma(\phi)} = \int (d\varphi)\, e^{\frac{i}{\hbar}\{W(\varphi)-(\varphi_i-\phi_i)\Gamma_i\}} \tag{17}$$

where we have denoted by Γ_i the functional derivative of the effective action $\Gamma(\phi)$ with respect to the effective field:

$$\Gamma_i = \frac{\partial \Gamma}{\partial \phi_i} \tag{18}$$

Now making a shift in the integration or virtual field, $\varphi \to \varphi + \phi$, we obtain

$$e^{\frac{i}{\hbar}\Gamma(\phi)} = \int (d\varphi)\, e^{\frac{i}{\hbar}\{W(\phi+\varphi)-\varphi_i\Gamma_i\}} \tag{19}$$

Notice that the effective action Γ on the left side of the above expression is expressed in terms of its functional derivative Γ_i on the right side. This situation calls for an iterative expansion scheme of evaluation.

2.2 Iterative Expansion

The above functional integral expression for the effective action may be computed iteratively in Planck's constant $\hbar$, in the form

$$\Gamma = \Gamma_0 + \hbar\Gamma_1 + \hbar^2\Gamma_2 + \cdots \tag{20}$$

To this end, scale the virtual field $\varphi \to \sqrt{\hbar}\varphi$, and write

$$e^{\frac{i}{\hbar}\Gamma(\phi)} = \int (d\varphi)\, e^{\frac{i}{\hbar}\{W(\phi+\sqrt{\hbar}\varphi)-\sqrt{\hbar}\varphi_i\Gamma_i\}} \tag{21}$$

We write the Taylor expansion

$$W(\phi+\sqrt{\hbar}\varphi) = \begin{cases} W + \sqrt{\hbar}\varphi_i W_i + \hbar\frac{1}{2}\varphi_i\varphi_j W_{ij} \\ +\hbar\sqrt{\hbar}\frac{1}{3!}\varphi_i\varphi_j\varphi_k W_{ijk} + \hbar^2\frac{1}{4!}\varphi_i\varphi_j\varphi_k\varphi_l W_{ijkl} + \cdots \end{cases} \tag{22}$$

where $(W, W_i, W_{ij}, W_{ijk}, \cdots)$ are the classical action functional $W(\phi)$ and its respective totally symmetric derivatives,

$$W_i = \frac{\partial W}{\partial \phi_i} \qquad W_{ij} = \frac{\partial^2 W}{\partial\phi_i\partial\phi_j} \qquad \text{etc.} \tag{23}$$

Immediately realize that $\Gamma_0 = W$, and write in the argument of the functional integral,

$$\Gamma_i = W_i + \hbar(\Gamma_1)_i + \hbar^2(\Gamma_2)_i + \cdots \tag{24}$$

The terms involving W_i cancel, while the term involving $(\Gamma_1)_i$ contributes to Γ_2, and so on.

We shall define *mean values* with respect to the Gaussian measure such as

$$\langle\varphi_i\varphi_j\varphi_k\cdots\rangle = \int (d\varphi)\,(\varphi_i\varphi_j\varphi_k\cdots)\, e^{\frac{i}{2}W_{ij}\varphi_i\varphi_j} \tag{25}$$

With this notation we obtain for the effective action,

$$e^{\frac{i}{\hbar}\Gamma} = e^{\frac{i}{\hbar}W}\langle e^{i\{\cdots\}}\rangle \tag{26}$$

$$\{\cdots\} = \sqrt{\hbar}\left\{\frac{1}{3!}\varphi_i\varphi_j\varphi_k W_{ijk} - \varphi_i(\Gamma_1)_i\right\} + \hbar\frac{1}{4!}\varphi_i\varphi_j\varphi_k\varphi_l W_{ijkl} \tag{27}$$

Hence, to order $\hbar^2$,

$$\Gamma = W - i\hbar \ln\left\langle 1 + i\{\cdots\} - \frac{1}{2}\{\cdots\}^2\right\rangle \tag{28}$$

To evaluate this expression, we must compute the mean values for products of φ.

2.3 Gaussian Integral & Mean Values

The Gaussian integral

$$\langle\ \rangle = \int (d\varphi)\, e^{\frac{i}{2}W_{ij}\varphi_i\varphi_j} \tag{29}$$

may be expressed by the functional determinant

$$\langle\ \rangle = (\det W_{ij})^{-\frac{1}{2}} = e^{-\frac{1}{2}\mathrm{tr}\ln(W_{ij})} \tag{30}$$

This expression is valid provided that W_{ij} is a *nonsingular* matrix. We shall deal elsewhere with the singular case which concerns *gauge* theories.

Taking successive derivatives with respect to W_{ij}, obtain

$$\langle\varphi_i\varphi_j\rangle = iW^{-1}_{ij}\langle\ \rangle \tag{31}$$

$$\langle\varphi_i\varphi_j\varphi_k\varphi_l\rangle = i\left\{W^{-1}_{ij}\langle\varphi_k\varphi_l\rangle + W^{-1}_{ik}\langle\varphi_j\varphi_l\rangle + W^{-1}_{il}\langle\varphi_j\varphi_k\rangle\right\} \tag{32}$$

$$\langle\varphi_i\varphi_j\varphi_k\varphi_l\varphi_m\varphi_n\rangle = i\left\{\begin{array}{l} W^{-1}_{ij}\langle\varphi_k\varphi_l\varphi_m\varphi_n\rangle + W^{-1}_{ik}\langle\varphi_j\varphi_l\varphi_m\varphi_n\rangle + W^{-1}_{il}\langle\varphi_j\varphi_k\varphi_m\varphi_n\rangle \\ \\ +W^{-1}_{im}\langle\varphi_j\varphi_k\varphi_l\varphi_n\rangle + W^{-1}_{in}\langle\varphi_j\varphi_k\varphi_l\varphi_m\rangle \end{array}\right\} \tag{33}$$

And so on. Here W^{-1}_{ij} is the symmetric inverse of W_{ij},

$$W^{-1}_{ij}W_{jk} = \delta_{ik} \tag{34}$$

and we have used the relations

$$\delta(\mathrm{tr}\ln W_{ij}) = W^{-1}_{ij}\delta W_{ij} \tag{35}$$

$$\delta(W^{-1}_{ij}) = -W^{-1}_{ik}W^{-1}_{jl}\delta W_{kl} \tag{36}$$

Now, let us take the mean value

$$\left\langle \frac{\partial}{\partial\varphi} \right\rangle = iW_{ij}\langle\varphi_j\rangle \tag{37}$$

The left part of the above equation vanishes due to the convergence of the Gaussian integrand (more precisely, of its Euclidean continuation). Hence if W_{ij} is nonsingular we deduce that $\langle\varphi\rangle = 0$. Taking successive derivatives of the latter with respect to W_{ij}, we conclude that *all mean values with odd number of φ's are vanishing.*

2.4 Higher-Loop Effective Action

Going back to the computation of the effective action, dropping all odd products of φ, obtain to order $\hbar^2$,

$$\Gamma = \left\{\begin{array}{l} W + \hbar\frac{i}{2}\mathrm{tr}\ln(W_{ij}) \\ \\ -\hbar^2\left(\begin{array}{l} \frac{1}{2}W^{-1}_{ij}(\Gamma_1)_i(\Gamma_1)_j + (W^{-1}_{ij}W^{-1}_{kl} + \cdots)(\frac{1}{4!}W_{ijkl} - \frac{i}{3!}(\Gamma_1)_iW_{jkl}) + \\ \\ -\frac{1}{2}(\frac{1}{3!})^2(W^{-1}_{ij}W^{-1}_{kl}W^{-1}_{mn} + \cdots)W_{ijk}W_{lmn} \end{array}\right) \end{array}\right. \tag{38}$$

At this point, use the first-order result

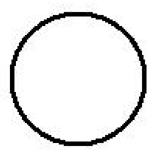
$$\Gamma_1 = \frac{i}{2}\mathrm{tr}\ln(W_{ij}) \qquad (\Gamma_1)_i = \frac{i}{2}W^{-1}_{kl}W_{kli} \tag{39}$$

to obtain

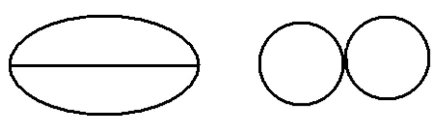
$$\Gamma = \begin{cases} W + \hbar\frac{i}{2}\mathrm{tr}\ln(W_{ij}) \\ +\hbar^2\left(\frac{1}{12}W^{-1}_{il}W^{-1}_{jm}W^{-1}_{kn}W_{ijk}W_{lmn} - \frac{1}{8}W^{-1}_{ij}W^{-1}_{kl}W_{ijkl}\right) \end{cases} \tag{40}$$

The extension of the above result to higher orders is straight-forward. However, the intermediate expressions become very lengthy. As a shortcut, we may turn to graphical rules. The corresponding expressions are obtained by attributing $iW_{ijk\cdots}$ to each vertex, iW^{-1}_{ij} to each connecting line, an overall factor of $-i$, and combinatoric factors. It should be noted that the effective action would only generate one-particle-irreducible vertices; hence we should consider only irreducible graphs.

Whereas the one-loop term $\frac{i}{2}\mathrm{tr}\ln(W_{ij})$ can be represented by the graph

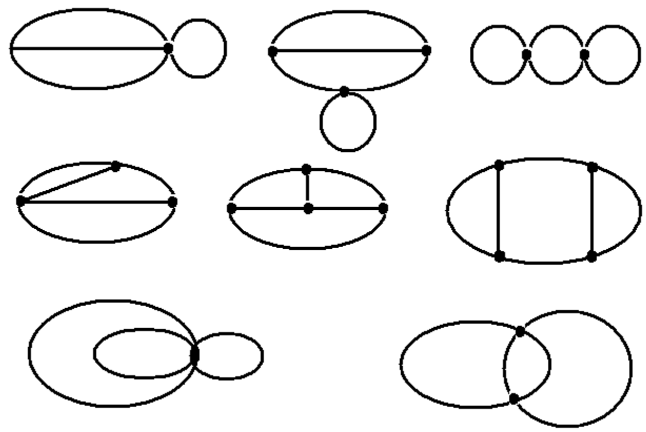


The two-loop terms $\frac{1}{12}W^{-1}_{il}W^{-1}_{jm}W^{-1}_{kn}W_{ijk}W_{lmn}$ and $-\frac{1}{8}W^{-1}_{ij}W^{-1}_{kl}W_{ijkl}$ are represented by the respective graphs



The following shows the possible three-loop graphs:[3]

[3]It is possible to write a *Mathematica* package that can generate all possible irreducible graphs for the effective action, at any loop order, and would give the corresponding symbolic terms with the correct combinatorics. My own package (*A Mathematica Package for the Effective Action*) will be introduced, and underlying programming explained, in a separate work.



3 Divergence-Free Effective Action: Fundamental Approach

The starting point of the perturbative development of the effective quantum action is the Gaussian integral which yields the logarithmic one-loop term. This is the term which *defines* the effective propagators that build the higher-loop terms. Replacing the logarithm in the one-loop term by its limiting definition would lead to a new definition of the effective propagators, and is expected to yield a fully divergence-free perturbative development.

Hence, we begin with the divergence-free limiting definition of the determinant:

$$\langle\ \rangle = (\det W_{ij})^{-\frac{1}{2}} = \exp\left(-\frac{1}{2}\mathrm{tr}\ln W_{ij}\right) \Rightarrow \exp\left\{-\frac{1}{2}\varrho_\epsilon \mathrm{tr}\left(-\frac{1}{\epsilon}W_{ij}^{-\epsilon}\right)\right\} \tag{41}$$

Here, we have introduced the symbol ϱ_ϵ for the pole-removing operator with limiting parameter ϵ,

$$\lim_{\epsilon\to 0}\varrho_\epsilon \equiv \left(\frac{\partial}{\partial\epsilon}\epsilon\right) \tag{42}$$

This leads to the limiting definition of the mean value that defines the effective propagator,

$$\langle\varphi_i\varphi_j\rangle = \langle\ \rangle i\varrho_\epsilon W_{ij}^{-(1+\epsilon)} \tag{43}$$

and the higher mean values:

$$\langle\varphi_i\varphi_j\varphi_k\varphi_l\rangle = i\left\{\varrho_\epsilon W_{ij}^{-(1+\epsilon)}\langle\varphi_k\varphi_l\rangle + \varrho_\epsilon W_{ik}^{-(1+\epsilon)}\langle\varphi_j\varphi_l\rangle + \varrho_\epsilon W_{il}^{-(1+\epsilon)}\langle\varphi_j\varphi_k\rangle\right\} \tag{44}$$

$$\langle\varphi_i\varphi_j\varphi_k\varphi_l\varphi_m\varphi_n\rangle = i\left\{\begin{array}{l} \varrho_\epsilon W_{ij}^{-(1+\epsilon)}\langle\varphi_k\varphi_l\varphi_m\varphi_n\rangle + \varrho_\epsilon W_{ik}^{-(1+\epsilon)}\langle\varphi_j\varphi_l\varphi_m\varphi_n\rangle \\ +\varrho_\epsilon W_{il}^{-(1+\epsilon)}\langle\varphi_j\varphi_k\varphi_m\varphi_n\rangle + \varrho_\epsilon W_{im}^{-(1+\epsilon)}\langle\varphi_j\varphi_k\varphi_l\varphi_n\rangle \\ +\varrho_\epsilon W_{in}^{-(1+\epsilon)}\langle\varphi_j\varphi_k\varphi_l\varphi_m\rangle \end{array}\right\} \tag{45}$$

And so on.

3.1 Divergence-Free Loop Contributions

The pertrurbative expressions of the effective quantum action that result from the above modifications are easily constructed. The divergence-free effective action expression to two-loop order is given by:

$$\Gamma = \begin{cases} W - \hbar \frac{i}{2} \varrho_\epsilon \mathrm{tr}\left(\frac{1}{\epsilon} W_{ij}^{-\epsilon}\right) \\ +\hbar^2 \begin{pmatrix} \frac{1}{12} \varrho_a \varrho_b \varrho_c W_{il}^{-(1+a)} W_{jm}^{-(1+b)} W_{kn}^{-(1+c)} W_{ijk} W_{lmn} \\ -\frac{1}{8} \varrho_a \varrho_b W_{ij}^{-(1+a)} W_{kl}^{-(1+b)} W_{ijkl} \end{pmatrix} \end{cases} \tag{46}$$

Notice that *each effective propagator* is associated with its own limiting parameter $(a, b, c, \cdots)$ and pole-removing operator. We should note here that in translating the compact notation of the above expressions to the counterparts involving momentum-space integrations, all dummy momentum integrations must be executed before applying the divergence-removing operators, as we have illustrated in many other articles.

The extension of the above result to higher orders is straight-forward. Turning to graphical rules, the corresponding expressions are obtained by attributing $iW_{ijk\cdots}$ to each vertex, $i\varrho_\epsilon W_{ij}^{-(1+\epsilon)}$ to each connecting line (not forgetting that each effective propagator will have its own limiting parameter), an overall factor of $-i$, and combinatoric factors.

3.2 Computing with Effective Propagators

To compute the loop contributions to the effective action that are represented by the expressions of the preceding section, we must know how to expand the effective propagators in terms of bare propagators and effective field terms. To that end, consider splitting the effective bilinear kernel W_{ij} into a bare part Δ_{ij} and a field-dependent part Y_{ij}. In matrix form, we have $\mathbf{W} = \Delta + Y$.

The divergence-free one-loop contribution then takes the form:

$$-\frac{i}{2} \varrho_\epsilon \mathrm{tr}\left(\frac{1}{\epsilon} \frac{1}{(\Delta + Y)^\epsilon}\right) \tag{47}$$

Next, we use the following integral representation for the power of an operator:

$$\frac{1}{(\Delta + Y)^\epsilon} = \frac{1}{\Gamma(\epsilon)} \int_0^\infty \mathrm{d}\lambda\, \lambda^{\epsilon - 1} e^{-\lambda(\Delta + Y)} \tag{48}$$

We now utilize the following series expansion of the trace of the exponential of $A + B$

with two operators A and B:

$$\mathrm{tr}(e^{A+B}) = \mathrm{tr}\left(\begin{array}{l} e^A + e^A B + \frac{1}{2}\int_0^1 \mathrm{d}x\, e^{(1-x)A} B e^{xA} B + \\ \\ +\frac{1}{3}\int_0^1 \mathrm{d}x \int_0^x \mathrm{d}y\, e^{(1-x)A} B e^{(x-y)A} B e^{yA} B + \cdots \end{array}\right) \tag{49}$$

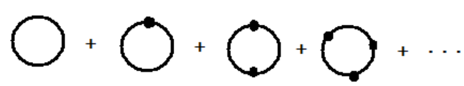
The resulting one-loop series takes the form:

$$-\frac{i}{2}\varrho_\epsilon\, \mathrm{tr}\left(\begin{array}{l} \frac{1}{\Gamma(1+\epsilon)}\int_0^\infty \mathrm{d}\lambda\, \lambda^{\epsilon-1} \times \\ \\ \left\{ e^{-\lambda\Delta} - \lambda e^{-\lambda\Delta} Y + \frac{\lambda^2}{2}\int_0^1 \mathrm{d}x\, e^{-\lambda(1-x)\Delta} Y e^{-\lambda x\Delta} Y + \cdots \right\} \end{array}\right) \tag{50}$$

The integration over λ may be done, once the operators are expressed in terms of matrix elements in momentum space. The resulting series takes the form:

$$-\frac{i}{2}\varrho_\epsilon\, \mathrm{tr}\left(\frac{1}{\epsilon}\frac{1}{\Delta^\epsilon} - \frac{1}{\Delta^{1+\epsilon}} Y + \frac{1}{2}\frac{\Gamma(2+\epsilon)}{\Gamma(1+\epsilon)} \underbrace{\frac{1}{\Delta} Y \frac{1}{\Delta}}_{2+\epsilon} Y - \cdots \right) \tag{51}$$

Pictorially, each term of the series may be represented by a closed loop having a number of insertions equal to the degree of Y.



In the above series terms, it should be understood that two or several propagators separated by field insertions (the underbraced factor) are actually combined using Feynman parameters, with a power equal to the argument of the associated upper gamma function. For instance, the second-degree term may be represented in momentum space such as:

$$\int_0^1 \mathrm{d}x \int_p \frac{Y(r)Y(-r)}{\{(1-x)\Delta(p) + x\Delta(p+r)\}^{2+\epsilon}} \tag{52}$$

where p is the loop momentum, and r is an external momentum carried by Y.

For the expansion of the effective propagators of the form

$$\frac{1}{(\Delta+Y)^{1+\epsilon}} = \frac{1}{\Gamma(1+\epsilon)}\int_0^\infty \mathrm{d}\lambda\, \lambda^\epsilon e^{-\lambda(\Delta+Y)} \tag{53}$$

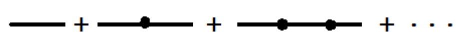
that appear in the higher loop contributions, we utilize the following series for the exponential function:

$$e^{A+B} = e^A + \int_0^1 \mathrm{d}x\, e^{(1-x)A} B e^{xA} + \int_0^1 \mathrm{d}x \int_0^x \mathrm{d}y\, e^{(1-x)A} B e^{(x-y)A} B e^{yA} + \cdots \tag{54}$$

In momentum space, and after integrating over λ, the series associated with each effective propagator takes the form

$$\frac{1}{\Delta^{1+\epsilon}} - \frac{\Gamma(2+\epsilon)}{\Gamma(1+\epsilon)} \left(\frac{1}{\Delta} Y \frac{1}{\Delta} \right)_{2+\epsilon} + \frac{1}{2} \frac{\Gamma(3+\epsilon)}{\Gamma(1+\epsilon)} \underbrace{\frac{1}{\Delta} Y \frac{1}{\Delta} Y \frac{1}{\Delta}}_{3+\epsilon} - \cdots \tag{55}$$

Each term of this series may be depicted by a line having a corresponding number of Y insertions.



The momentum-space propagators (underbraced) are understood to be combined using Feynman parameters with a total power equal to their number plus ϵ (argument of the associated upper gamma function).

4 Discussion

The effective action scheme for the divergence-free regularization of quantum field theory, introduced in this article, was conceived by the author long time ago, in the early 1990s. However, becoming aware of the existence of the zeta-function scheme[13]-[17] and the scheme of operator regularization[18]-[19] made us feel no urgency for proposing our framework, thinking that all such schemes might be equivalent.

However, we continued to develop our own scheme, and continued to apply it to computations in various quantum field theories, from simple scalar field systems, to quantum electrodynamics, to gauge theories, and to quantum gravity. We have even developed symbolic manipulation software (under *Mathematica* and otherwise) which can handle the tedious underlying computations rather automatically.[20]

Eventually we had realized that our scheme is much simpler than others, while the other schemes turn out to be too complicated to handle higher-loop contributions, *if at all applicable*. Hence we have decided to present our scheme which may actually be described as *the correct and the natural effective action method of regularization* for gauge-invariant and divergence-free quantum field theory. Whereas this article has laid down the foundations of our scheme, other articles will deal with specific applications.[21], [22], [23], [24]

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For Those Who Seek True Comprehension of Fundamental Theoretical Physics