Another Proof that the Catalan's Constant is Irrational

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Put all your hope in God, not looking to your reason for support.

Proverbs 3:5

ABSTRACT. We use the contradiction method for prove, again, that the Catalan's constant is irrational.

1. INTRODUCTION

In Mathematics, the Catalan's constant [1] is defined by

$$G := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}. \tag{1.1}$$

The Catalan's constant was named after Eugène Charles Catalan (30 May 1814 – 14 February 1894), a French and Belgian mathematician.

In previous paper [2], we prove that the constant G is irrational. In this paper, we damos outra prova de que the constant G is irrational.

2. THE PROOF

LEMMA. *The Catalan's constant have the following representation in series*

$$G = 8\sum_{n=0}^{\infty} \frac{2n+1}{(16n^2+16n+3)^2}.$$

Proof. We developed the power series formula from the definition of Catalan's constant as follows

$$G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}$$

$$= \frac{(-1)^0}{(2\cdot 0+1)^2} + \frac{(-1)^1}{(2\cdot 1+1)^2} + \frac{(-1)^2}{(2\cdot 2+1)^2} + \frac{(-1)^3}{(2\cdot 3+1)^2} + \frac{(-1)^4}{(2\cdot 4+1)^2} + \frac{(-1)^5}{(2\cdot 5+1)^2} + \cdots$$

$$= \frac{1}{(2\cdot 0+1)^2} + \frac{1}{(2\cdot 2+1)^2} + \frac{1}{(2\cdot 4+1)^2} + \cdots - \frac{1}{(2\cdot 1+1)^2} - \frac{1}{(2\cdot 3+1)^2} - \frac{1}{(2\cdot 5+1)^2} \cdots$$

$$= \frac{1}{(2\cdot 0+1)^2} + \frac{1}{(2\cdot 2+1)^2} + \frac{1}{(2\cdot 4+1)^2} + \cdots - \left[\frac{1}{(2\cdot 1+1)^2} + \frac{1}{(2\cdot 3+1)^2} + \frac{1}{(2\cdot 5+1)^2} + \cdots\right]$$

$$= \frac{1}{[2\cdot(2\cdot 0)+1]^2} + \frac{1}{[2\cdot(2\cdot 1)+1]^2} + \frac{1}{[2\cdot(2\cdot 2)+1]^2} + \cdots$$
$$- \left\{\frac{1}{[2\cdot(2\cdot 0+1)+1]^2} + \frac{1}{[2\cdot(2\cdot 2+1)+1]^2} + \frac{1}{[2\cdot(2\cdot 2+1)+1]^2} + \cdots\right\}$$

$$= \frac{1}{(4\cdot 0+1)^2}+\frac{1}{(4\cdot 1+1)^2}+\frac{1}{(4\cdot 2+1)^2}+\cdots-\left[\frac{1}{(4\cdot 0+3)^2}+\frac{1}{(4\cdot 1+3)^2}+\frac{1}{(4\cdot 2+3)^2}+\cdots\right]$$

$$= \sum_{n=0}^{\infty}\frac{1}{(4n+1)^2}-\sum_{n=0}^{\infty}\frac{1}{(4n+3)^2}$$

$$= \sum_{n=0}^{\infty}\frac{(4n+3)^2-(4n+1)^2}{(4n+1)^2(4n+3)^2}$$

$$= \sum_{n=0}^{\infty}\frac{16n+8}{(4n+1)^2(4n+3)^2}$$

$$= 8\sum_{n=0}^{\infty}\frac{2n+1}{[(4n+1)(4n+3)]^2}$$

$$= 8\sum_{n=0}^{\infty}\frac{2n+1}{(16n^2+16n+3)^2}. \square$$

THEOREM. *The Catalan's constant is irrational.*

Proof. We will use the *reductio ad absurdum*.

By hypothesis, we suppose that G is a rational number. Of course, there exist two positive integers a and b, such that $G = a/b$, where, clearly, $b > 1$. Firstly, we define the number

$$(2.1) \qquad x := \frac{(16b^2+16b+3)!^2}{4^{8b^2+8b+1}(8b^2+8b+1)!\,(8b^2+8b)!}\cdot\left(G-8\sum_{n=0}^{b}\frac{2n+1}{(16n^2+16n+3)^2}\right).$$

If G is rational, then x is an integer. We substitute $G = a/b$ into this definition to find

$$(2.2)\ x = \frac{(16b^2+16b+3)!^2}{4^{8b^2+8b+1}(8b^2+8b+1)!\,(8b^2+8b)!}\cdot\left(\frac{a}{b}-8\sum_{n=0}^{b}\frac{2n+1}{(16n^2+16n+3)^2}\right)$$

$$= \frac{(16b^2+16b+3)!^2\,a}{4^{8b^2+8b+1}b(8b^2+8b+1)!\,(8b^2+8b)!}$$

$$-8\sum_{n=0}^{b}\frac{(16b^2+16b+3)!^2\,(2n+1)}{4^{8b^2+8b+1}(8b^2+8b+1)!\,(8b^2+8b)!\,(16n^2+16n+3)^2}.$$

It is obvious that the first term is an integer; because, for $b > 1$, then $4^b(b!)^2 < (2b+1)!^2$. The second term is an integer; because, for $b > 1$, then $(2n+1)^2 4^b b((b-1)!)^2 < (2b+1)!^2$. Hence x is an integer.

We, now, demonstrate that $0 < x < 1$.

First, we demonstrate that x is strictly positive, we insert the series representation of G into the definition of x and we find

$$(2.3)\, x = \frac{(2b+1)!^2}{4^b b((b-1)!)^2}\left|\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2} - \sum_{n=0}^{b}\frac{(-1)^n}{(2n+1)^2}\right|$$
$$= \frac{(2b+1)!^2}{4^b b((b-1)!)^2}\left|\sum_{n=b+1}^{\infty}\frac{(-1)^n}{(2n+1)^2}\right| = \frac{(2b+1)!^2}{4^b b((b-1)!)^2}\left|\sum_{n=b+1}^{\infty}\frac{\cos(\pi n)}{(2n+1)^2}\right|$$
$$> \frac{(2b+1)!^2}{4^b b((b-1)!)^2}\left|\int_{b+1}^{\infty}\frac{\cos(\pi x)}{(2x+1)^2}dx\right|$$
$$= \frac{(2b+1)!^2}{4^b b((b-1)!)^2}\left|-\frac{1}{4}\pi\,\mathrm{Ci}\left(\left(b+\frac{3}{2}\right)\pi\right) - \frac{\cos(\pi b)}{4b+6}\right| > 0.$$

On the other hand, for all terms with $2n+1 \geq 2b+2$, i.e., $2n \geq 2b+1$, we have the upper estimate

$$(2.4) \qquad \frac{(2b+1)!}{(2n+1)!} \leq \frac{1}{(2b+2)^{2n-2b}}.$$

This inequality is strict for every $2n+1 \geq 2b+3$, i.e., $n \geq b+1$. Thereof, we substitute (1.1) and (2.4) in (2.1)

$$(2.5) \qquad x = \frac{(2b+1)!^2}{4^b b((b-1)!)^2}\left|\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2} - \sum_{n=0}^{b}\frac{(-1)^n}{(2n+1)^2}\right|$$
$$= \frac{(2b+1)!^2}{4^b b((b-1)!)^2}\left|\sum_{n=b+1}^{\infty}\frac{(-1)^n}{(2n+1)^2}\right| < \frac{(2b+1)!^2}{4^b b((b-1)!)^2}\left|\sum_{n=b+1}^{\infty}\frac{(-1)^n}{(2n+1)!^2}\right|$$
$$= \frac{1}{4^b b((b-1)!)^2}\left|\sum_{n=b+1}^{\infty}\frac{(-1)^n(2b+1)!^2}{(2n+1)!^2}\right| < \frac{1}{4^b b((b-1)!)^2}\left|\sum_{n=b+1}^{\infty}\frac{(-1)^n}{(2b+2)^{2n-2b}}\right|$$
$$= \frac{1}{4^b b((b-1)!)^2}\left|-\frac{(-1)^b}{4b^2+8b+5}\right| < 1.$$

Since there is no integer strictly between 0 and 1, we have get in a contradiction, and so G must be irrational. ◻

REFERENCES

[1] http://en.wikipedia.org/wiki/Catalan's_constant, available in July 12, 2013.

[2] Guedes, Edigles, *An Elegant Proof that the Catalan's Constant is Irrational,* July 12, 2013, vixra.