Quantum field theory and non-local geometry:

An intrinsic way of describing quantum behaviour*

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Abstract: The origin of quantum behaviour (or equivalently, wave-particle duality) is an important problem for physics; moreover, Euclidean geometry and Riemannian geometry may be invalid if the small scales of real universe exhibit fractal structure. With this purpose, we attempt to develop a mathematical framework -call it the "non-local geometry"- and meanwhile propose a set of non-local calculus theory for analytically describing fractal (Euclidean geometry and Riemannian geometry are two special cases of fractal whenever the dimension equals an integer). Our study shows that the "Heisenberg Uncertainly Principle" and "non-local entanglement" would naturally emerge in the theoretical framework of non-local geometry. More interestingly, using the non-local geometry we show that if the dimension of time axis is slightly less than 1, then we can directly derive Planck's formula of energy quantum. This means that non-zero Planck's constant itself requires that the dimension of space-time is slightly less than 4; thus, our theory presents a natural explanation for the dimensional regularization of quantum field theory. Our further study shows that all computing results obtained by quantum field theory can be reproduced in the theoretical framework of non-local geometry. To discriminate our theory from current quantum field theory, we suggest a method of measuring the dimension of time axis.

Keywords: Planck's constant; Fractal curve; Non-local space-time; Quantum field theory; Dimensional regularization

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1. Introduction

The quantum field theory is one of the oldest fundamental and most widely used tools in physics. It is spectacularly successful that the value of theoretical calculation is precisely in agreement with experimental data, for example, the anomalous magnet moment of electron. The quantum field theory is established on the basis of the theoretical frameworks including special relativity and quantum mechanics. The starting point of special relativity is the principle of invariant light speed, while the starting point of quantum mechanics is Planck's hypothesis of energy quantum[1-2], i.e.,

$$\varepsilon = h\nu \qquad (1)$$

where, h denotes the Planck's constant and ν the frequency.

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It is well known that the principle of invariant light speed is an intrinsic requirement for the symmetry of Maxwell electromagnetism equations. Nevertheless, the hypothesis of energy quantum (1), which differs from the principle of invariant light speed, is a result of Max Planck's attempts to provide a theoretical explanation for the empirically discovered laws of blackbody radiation. Although the quantum mechanics provides a powerful way to our understanding of microscopic structure of matter, so far there is still no consensus on the origin of the Planck's constant, for instance, whether the Planck's constant can be derived from a deeper principle or not?

More importantly, EPR paradox, presented by Einstein et al [3], indicates that the quantum entanglement would violate the principle of locality of field theory, for example, the locality of field would be violated within the Compton wavelength [4].

In addition to violations of the principle of locality, there further exists ultraviolet divergence in quantum field theory, which need to be removed using the dimensional regularization [5]. Dimensional regularization requires that S matrix should be calculated in a non-integer dimensional space-time [5]. Later, Svozil developed the quantum field theory on fractal space-time (QFTFS), the main advantage of this approach is preservation of gauge invariance and covariance [6]. Svozil's work implies that the dimension of space-time should be $D = 4 - \delta$, where, $0 < \delta << 1$ [6].

Interestingly, recently, the investigation for a consistent theory of quantum gravity strongly indicates that a power-counting renormalizable gravity model can be achieved in a fractional dimensional space-time, for example, the Horava-Lifshitz (HL) gravity model [7-8]. Unfortunately, HL gravity model is not Lorentz invariant. To maintain the Lorentz invariance, Calcagni [9-10] extended the theoretical framework of Svozil's QFTFS so as to contain the case of gravity. Calcagni's work shows that if the Haudorff dimension of space-time $d_H \sim 2$, then the ultraviolet divergence can be removed.

Above developments of quantum field theory and quantum gravity incarnate the popular notion that "the Universe is fractal" at quantum scales [9]. Unfortunately, there is still not a set of rigorous calculus theory for analytically describing fractal. In general, people use the fractional calculus to approximately describe fractal [11]. It is the purpose of this paper to propose a theoretical framework of "non-local geometry" for analytically describing self-similar fractal (e.g., self-correlation between local and global shapes). Our study shows that the non-locality of quantum entanglement can be understood within the framework of the non-local geometry. Interestingly, if we assume that the dimension of time axis is a fraction, then we can derive the formula (1) of energy quantum and the Planck's constant by making use of non-local geometry. Fortunately, fractional dimension is an intrinsic requirement for the dimensional regularization [5-6]; therefore, our theory essentially provides a natural explanation for the dimensional regularization. Later, we shall develop the quantum field theory on the basis of the theoretical framework of non-local geometry, where we would note that Svozil's QFTFS can be reproduced in the theoretical framework of non-local geometry. This means that the non-local geometry will be an intrinsic way of describing quantum field, even for quantum gravity.

The organization of our paper is as follows. In section 2, we introduce the basic ideas of non-local geometry and the definition of non-local derivative. More detailed descriptions of non-local geometry will be introduced in Appendix A and Appendix B. In section 3, we attempt to

derive the Planck's formula of energy quantum using the non-local geometry. In section 4, we present a satisfactorily logical framework for quantum mechanics. In section 5, we develop the quantum field theory on non-local space-time. In section 6, our conclusion follows.

2. Non-local geometry

We have developed the theoretical framework of non-local geometry in Appendix A and Appendix B. In this section, we only present a brief introduction for basic ideas of non-local geometry, and discuss the connection between quantum physics and non-local geometry. More mathematic details see Appendixes A and B.

In classical mathematics, the dimension of a geometric graph is determined by the number of independent variables (i.e., the number of degrees of freedom). For example, every point on a plane can be represented by 2-tuples real number (x_1, x_2), then the dimension of the plane is denoted by 2. Nevertheless, Peano's curve, which is determined by an independent characteristic parameter, would fill up the entire plane [12]. This means that if the classical definition of dimension is taken into account, then the dimension of the plane must be denoted by 1. It is a clear contradiction.

2.1. Shortcoming of Hausdorff measure

The existence of Peano's curve led mathematician to reconsider the definition of dimension, for example, the Hausdorff dimension, which later led to the development of fractal [13]. In general, the Hausdorff dimension is determined by the Hausdorff measure. Unfortunately, Hausdorff measure can determine the dimension of a fractal curve but not describe its analytic properties, for example, the self-similarity between local and global shapes of a fractal curve. To realize this fact, we attempt to check the case of the Cantor set, sees Fig.1. [11, 14].

As shown by Fig.1, the Cantor set is a fractal. Using the Hausdorff measure (A.7) (sees Appendix A) we can compute the dimension of the Cantor set as [14]

$$D = \frac{\ln 2}{\ln 3} = 0.6309...$$

Nevertheless, for the Cantor set, we does not realize any correlation between its local and global segments (i.e., self-similarity) through the Hausdorff measure. For instance, the Hausdorff distance between points $x_2^{(3)}$ and $x_1^{(3)}$ is denoted by

$$H^D\left(x_2^{(3)}, x_1^{(3)}\right) = \left|x_2^{(3)} - x_1^{(3)}\right|^D . \qquad (2)$$

Obviously, Hausdorff distance (2) is independent of the values of points $x_i^{(3)}$, where i runs from 3 to 8. Nevertheless, because of the self-similarity between parts of the Cantor set, any displacement of point $x_i^{(3)}$ $(i = 3,4,...8)$ should influence the distance between $x_2^{(3)}$ and

$x_1^{(3)}$. This is undoubtedly a non-local property. Unfortunately, Hausdorff distance (2) fails to show this property.

In fact, the Hausdorff distance (sees equation (2)), similar to the Euclidean distance, is also a *local* measure using which the distance between two neighboring points on a curve depends only on the positions of these two points. This means, if we want to describe the non-local property of some curves (e.g. fractal curve), we need to construct a *non-local* measure using which the distance between any two neighboring points on a curve may depend on the positions of many points of this curve. Therefore, the geometry, which is on the basis of the non-local measure, will be a more general one compared to Euclidean geometry, Riemannian geometry and Hausdorff geometry. In this paper, we guess that Euclidean geometry and Riemannian geometry may be invalid in the case of describing small scale structure of real universe, which perhaps need to be dealt with using non-local measure.

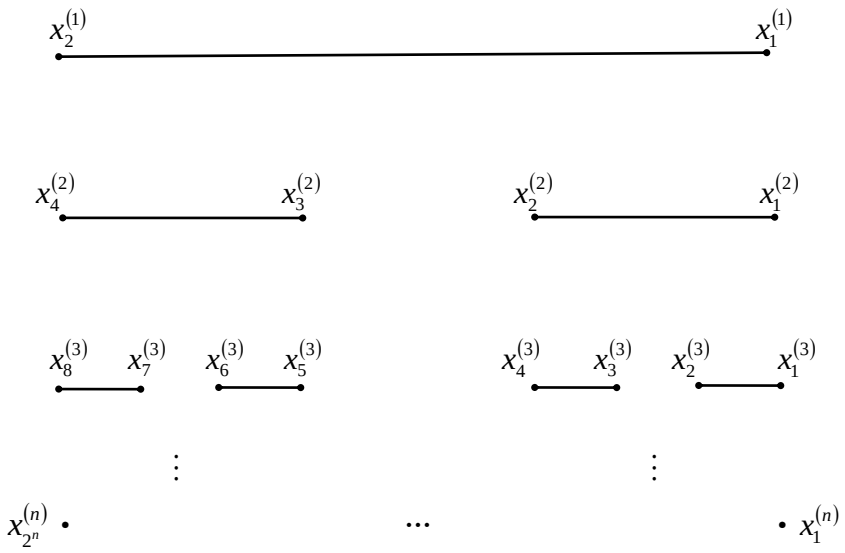

FIG1. The Cantor ternary set is defined by repeatedly removing the middle thirds of line segments [11,14]: (a). One starts by removing the middle third from the interval $\left[x_2^{(1)}, x_1^{(1)}\right]$, leaving $\left[x_4^{(2)}, x_3^{(2)}\right]$ and $\left[x_2^{(2)}, x_1^{(2)}\right]$. (b). Next, the "middle third" of all remaining intervals is removed. (c). This process is continued ad infinitum. Finally, the Cantor ternary set consists of all points in the interval $\left[x_2^{(1)}, x_1^{(1)}\right]$ that are not removed at any step in this infinite process.

2.2. Non-local measure and non-local derivative

To construct a non-local measure, we attempt to check a m -dimensional volume

$$x(l)=\omega(m)l^m$$

where $\omega(m)$ is a constant factor that depends only on the dimension m, and m may be a fraction.

The fractional derivatives of order m of $x(l)$ gives [11]

$$d^m x(l)=\Gamma(m+1)(dl)^m \sim (dl)^m$$

Obviously, $(dl)^m$, as a m -dimensional volume, is a m -dimensional Hausdorff measure; therefore, the above formula implies that the differences $d^m x(l)$ of order m can be also thought of as a measure for describing the length of a m -dimensional fractal curve. In this case, the order of differences $d^m x(l)$ represents the Hausdorff dimension m.

Using the differences of order m, we define a new distance-call it the "non-local distance"- in the form (sees equation (A.18) in Appendix A):

$$\left|\Delta_m[x(l),x(l-\Delta l)]\right|=\left|\sum_{j=0}^{\infty}\frac{m(m-1)..(m-j+1)(-1)^j}{j!}x(l-j\Delta l)\right|, \qquad (3)$$

where $\left|\Delta_m[x(l),x(l-\Delta l)]\right|$ denotes the non-local distance between points $x(l)$ and $x(l-\Delta l)$.

It is carefully noted that every $x(l-j\Delta l)$ represents a point on a m -dimensional fractal curve, where, $j=0,1,2,\ldots$

Clearly, according to the equation (3), the distance between points $x(l)$ and $x(l-\Delta l)$ would depend on all points $x(l-j\Delta l)$, where, $j=0,1,2,\ldots$

Moreover, it is easy to check that $\left|\Delta_{m=1}[x(l),x(l-\Delta l)]\right|=\left|x(l)-x(l-\Delta l)\right|$. This means that the Euclidean distance is a *special case* of non-local distance whenever the dimension m of the fractal curve equals 1.

If we use the non-local distance (3) to measure the distance between points $x_2^{(3)}$ and $x_1^{(3)}$ (sees Fig. 1), then we shall surprisingly find that the non-local distance

$$\left|\Delta_{m=D}\left[x_2^{(3)}, x_1^{(3)}\right]\right| = \left|\sum_{j=1}^{8} \frac{D(D-1)\ldots(D-j+1)(-1)^j}{j!} x_j^{(3)}\right|,$$

which remarkably differs from the Hausdorff distance (2), would depend on the values of points $x_i^{(3)}$ $(i = 3,4,\ldots 8)$. This means that any displacement of point $x_i^{(3)}$ $(i = 3,4,\ldots 8)$ would change the non-local distance between points $x_2^{(3)}$ and $x_1^{(3)}$. Consequently, non-local distance (3) is indeed an intrinsic way of describing self-similar fractal, since it not only determines the dimension of a fractal curve (e.g., Cantor ternary set) but also reflects the correlation between its parts.

Using the non-local distance we have given a definition for non-local measure in Appendix A (sees equation (A.21)), and we call the corresponding geometry the "non-local geometry". Clearly, fractal is a case of the non-local geometry.

To study the analytic properties of non-local geometry, we define the non-local derivative[1] (sees equation (A. 23) in Appendix A) in the form:

$$\frac{{}_l D_\omega f(x)}{{}_l D_\omega x} = \lim_{\Delta l \to 0} \frac{\Delta_\omega\left[f(l), f(l-\Delta l)\right]}{\Delta_\omega\left[x(l), x(l-\Delta l)\right]}, \qquad (4)$$

where $f(l) = f\left[x(l)\right]$ is a differentiable function with respect to coordinate l, l is a parameter (e.g., the single parameter of Peano's curve) which completely determines the generation of a ω-dimensional fractal curve, and $x(l)$ denotes the length of the corresponding fractal curve.

In particular, non-local derivative (4) is well known the Newton-Leibniz derivative whenever $\omega = 1$.

Using the formula of fractional derivative, the non-local derivative can be rewritten as (see equations (A.24)-(A.26) in Appendix A)

$$\frac{{}_l D_\omega f(x)}{{}_l D_\omega x} = \frac{\dfrac{d^\omega f(l)}{dl^\omega}}{\dfrac{d^\omega x(l)}{dl^\omega}}. \qquad (5)$$

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[1] We introduce a simple way of understanding the non-local derivative. For the case of Newton-Leibniz derivative of $y = f(x)$, x is a 1-dimensional coordinate axis and hence can be measured by a Euclidean scale (ruler). Thus, the differential element of x is a 1-dimensional Euclidean length dx, which gives rise to the Newton-Leibniz derivative $\dfrac{df(x)}{dx}$. Nevertheless, if x is a ω-dimensional fractal curve, then it can not be measured by the Euclidean scale (ruler). In this case, the differential element of x should be a ω-dimensional volume ${}_l D_\omega x$, which gives rive to the non-local derivative $\dfrac{{}_l D_\omega f(x)}{{}_l D_\omega x}$. Details see Appendix A and Fig. 3.

Through the formula (5) we can easily compute the non-local derivative of any differentiable function using the fractional derivative, concrete examples see Appendix B.

2.3. Connection between non-local geometry and quantum behaviour

Now we investigate the connection between quantum mechanics and non-local geometry. For simplicity, we still consider the Cantor set in Fig. 1, where the non-local distance between points $x_2^{(n)}$ and $x_1^{(n)}$ is equal to

$$\Pi\left(x_2^{(n)}, x_1^{(n)}\right) = \left|\Delta_{m=D}\left[x_2^{(n)}, x_1^{(n)}\right]\right| = \left|\sum_{j=1}^{2^n} \frac{D(D-1)\ldots(D-j+1)(-1)^j}{j!} x_j^{(n)}\right|. \tag{6}$$

Clearly, the output value of non-local distance (6) depends on the value of each element in the set $\left\{x_i^{(n)}\right\}$, where i runs from 1 to 2^n. If $n \to \infty$, we will have $\Pi\left(x_2^{(n)}, x_1^{(n)}\right) \to 0$; however, then the number of elements in the set $\left\{x_i^{(n)}\right\}$ would tend to infinity, too. This means that if we want to precisely measure the distance of smaller scale (e.g., $\lim_{n>>1} \Pi\left(x_2^{(n)}, x_1^{(n)}\right)$), correspondingly we shall need to collect the more points $\left\{x_i^{(n)}\right\}$. As a result, if we want to precisely measure the non-local distance between points $\lim_{n\to\infty} x_2^{(n)}$ and $\lim_{n\to\infty} x_1^{(n)}$, we shall need to collect a set of infinite points, i.e., $\left\{x_i^{(\infty)}\right\}_{i=1}^{\infty}$. Unfortunately, we must fail to arrive at this purpose on an actual measurement. In other words, we can not precisely measure the distance of microscopic scale, this fact is consistent with the "Heisenberg Uncertainly Principle"[2].

On the other hand, the non-local distance (6) between points $x_2^{(n)}$ and $x_1^{(n)}$ depends clearly on the position of $x_i^{(n)}\left(i = 1,2,\ldots,2^n\right)$, for example, any displacement of point $x_{2^n}^{(n)}$ would influence the output value of non-local distance (6). It is a clearly non-local correlation (correlations span arbitrarily distances) and similar to quantum entanglement.

The above two facts show that the non-local geometry is indeed an intrinsic way of describing quantum behaviour[3]. In the next section, we attempt to derive the formula (1) of energy quantum and Planck's constant using the non-local derivative (4).

3. Energy quantum and Planck's constant

[2] In fact, the connection between quantum mechanics and fractal has been noticed in some earlier papers [15-18].

[3] In references [19-20], we have shown that the local description is not a way of completely describing physical reality.

Before proceeding to derive the formula (1) of energy quantum, we need to introduce two axioms as follows.

Axiom (i). The total energy of the universe $E(t)$ is independent of time variable t, that is, the total energy of the universe is conserved, i.e., $E(t)=E=const$

Axiom (ii). The dimension of time axis ω is less than 1, i.e., $0<\omega<1$.

For the axiom (i), we need to clarify that the total energy of the universe denotes the sum of all types of energies which exist in the universe, including contributions of dark matters, dark energies and so on.

Now we show how the axioms (i) and (ii) naturally lead to the formula (1) of energy quantum. To arrive at this purpose, we attempt to compute the non-local derivative of $E(t)$ with respect to coordinate t, that is

$$\frac{{}_{l}D_{\omega}E(t)}{{}_{l}D_{\omega}t}=\frac{{}_{l}D_{\omega}E}{{}_{l}D_{\omega}t}$$

Axioms (ii) implies that the time axis is a fractal curve. For simplicity, we let t be the characteristic parameter (fill parameter) of the fractal time axis, i.e., $l=t$; thus, we only need to compute the following non-local derivative,

$$\frac{{}_{t}D_{\omega}E(t)}{{}_{t}D_{\omega}t}=\frac{{}_{t}D_{\omega}E}{{}_{t}D_{\omega}t}.$$

Using the formula (B.4) in Appendix B, we obtain[4]

$$\frac{{}_{t}D_{\omega}E}{{}_{t}D_{\omega}t}=E(1-\omega)\cdot\frac{1}{t}. \qquad (7)$$

Equation (7) represents the *rate of change of total energy* (RCTE) of a fractal universe at a given time t.

In particular, whenever $\omega=1$, equation (7) gives the Newton-Leibniz derivative

$$\frac{{}_{t}D_{\omega=1}E}{{}_{t}D_{\omega=1}t}=\frac{dE}{dt}=0. \qquad (8)$$

Equation (8) represents the conservation of total energy in an integer dimensional space-time.

Let us order

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[4] To arrive at equation (7), we need to require t to be the fill parameter of a ω-dimensional fractal curve; nevertheless, it is only a special case. In fact, we may require that t satisfies the general formula of ω-dimensional volume, that is, $t\sim l^{\omega}$. Thus, using the formula (B.1) in Appendix B we can arrive at $\frac{{}_{l}D_{\omega}E}{{}_{l}D_{\omega}t}=\frac{E}{\Gamma(1-\omega)\Gamma(1+\omega)}\cdot\frac{1}{t}$, which is equal to $\frac{{}_{t}D_{\omega}E}{{}_{t}D_{\omega}t}$ up to a constant factor $\Gamma(2-\omega)\Gamma(1+\omega)$. However, $\lim_{\omega\to 1}\Gamma(2-\omega)\Gamma(1+\omega)=1$, so we still arrive at equation (7) up to a constant factor which tends to 1.

$$h = E \cdot (1-\omega) \cdot \Delta T , \qquad (9)$$

$$\nu = \frac{1}{t}, \qquad (10)$$

$$\varepsilon = \frac{{}_t D_\omega E}{{}_t D_\omega t} \cdot \Delta T , \qquad (11)$$

where, ΔT denotes the least time interval (i.e., characteristic size), which depends on the limit of natural measure. Henceforth, we call ΔT the critical time scale.

The physical meaning of ΔT is clear: We can only observe the physics above critical time scale ΔT. Notably, it does not mean that there is no physics below ΔT. Conversely, there exists physics below ΔT, and it is still described by non-local geometry. However, we can not observe it because of the restriction of measuring instrument[5]. This is consistent with the spirit of Copenhagen interpretation.

Interestingly, from the viewpoint of *effective field theory*, the existence of such a critical time scale ΔT is very necessary, for example, then we will have a natural momentum cutoff parameter $\Lambda = \frac{h}{c\Delta T}$. As a result, in the framework of non-local geometry, two important properties of renormalization theory will be satisfied naturally; they are respectively: (a). The cutoff parameter Λ exists; (b). Dimension of spacetime $D < 4$ (since $D = 3 + \omega$ and $0 < \omega < 1$).

The equation (7) shows a curious fractal effect, and however it does not occur in an integer dimensional space-time. Intuitively, RCTE (i.e., $\frac{{}_t D_\omega E}{{}_t D_\omega t}$) at time t looks like "fluctuation of total energy" at time t. In particular, the equation (7) can be rewritten as a non-local integral equation (Sees definition (B.10) in Appendix B):

$$E = \int {}_t D_\omega E = \int \frac{{}_t D_\omega E}{{}_t D_\omega t} {}_t D_\omega t = \int E(1-\omega) \frac{1}{t} \frac{d^\omega t}{dt^\omega} (dt)^\omega , \qquad (12)$$

where, we have used the formula of fractional integral [11],

$$\int t^{-\omega} (dt)^\omega = \frac{1}{\Gamma(\omega)} \int_0^t (t-z)^{\omega-1} z^{-\omega} dz = \Gamma(1-\omega), \quad (0 < \omega < 1)$$

and the formula of fractional derivative [11]

$$\frac{d^\omega t}{dt^\omega} = \frac{t^{1-\omega}}{\Gamma(2-\omega)} .$$

Equation (12) undoubtedly implies that the total energy of a fractal universe is invariant. Not only that, it even indicate that the total energy of a fractal universe is composed of all RCTEs at different time $t \in [0, T_a]$, where T_a denotes the age of universe. This means that the RCTE at

[5] This is because we are a part of universe, and so we can only use the measuring instrument of corresponding scale to observe a limited scale part of universe, and ΔT is a limit. In other words, the scale of ours (or measuring instrument) itself restricts our ability of observing nature.

every time $t \in [0, T_a]$ will exist for ever. Also, consider that $\frac{{}_t D_\omega E}{{}_t D_\omega t}$ is not a function of space variable (x, y, z), the RCTE will randomly appear at any position of universe. Thus, the RCTE looks like a "physical effect", and thereby should be observed in the space-time. Because we can only observe the physical effect above critical time scale ΔT, so we can (at most) observe the RCTE on time interval ΔT which is denoted by $\varepsilon = \frac{{}_t D_\omega E}{{}_t D_\omega t} \cdot \Delta T$. We can call ε the Planck's energy quantum, and $E(1-\omega)\Delta T$ the Planck constant.

As such, substituting equations (9)-(11) into equation (7) we get Planck's formula (1) of energy quantum, i.e., $\varepsilon = h\nu$.

Moreover, let us insert $0 < \omega < 1$ into equation (9), then it yields

$$E \cdot \Delta T > h, \qquad (13)$$

which is consistent with the "Heisenberg Uncertainly Principle".

Inequality (13) implies that non-commutativity may be an intrinsic property of the non-local geometry.

Undoubtedly, in order to guarantee the value of Planck's constant enough small, we need to require $1-\omega << 1$ (Sees equation (9)). When ω is very close to 1, we can regard the dimension of time axis as 1 in the macroscopic scale. However, $\omega \neq 1$ would emerge as quantum effect in the microscopic scale. That is to say, the quantum effect arises because the dimension of time axis is less than 1. On the other hand, from the Fig. 1 we can note that the Cantor set is discrete[6]. In fact, for any geometric graph, if its dimension is less than 1, then it would be discrete. The closer its dimension tends to 1, the closer it approaches continuum, sees Appendix D. From this meaning, the time axis should be discrete (since $\omega < 1$). This fact can be regarded as a natural explanation for quantization.

In particular, if we assume that the dimension of space is 3, then the dimension of space-time would be $D = 3 + \omega = 4 - \delta$, where $0 < \delta << 1$. This is consistent with the requirement of QFTFS [6]. However, we does not require the dimension of space must be 3 (e.g., the case of quantum gravity [10]).

4. Klein-Gordon equation

In section 3, we have derived the formula (1) of energy quantum using the non-local derivative. Our derivation shows that the quantum behaviour (or equivalently, wave-particle duality) arises because the dimension of time axis is less than 1.

To derive the centre equation of quantum mechanics (i.e., dynamical equation of matter wave), we need to introduce the third axiom as follow.

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[6] It is carefully noted that the Cantor set, as a subset of the interval $[0,1]$, is discrete, but is still uncountable. Sees Appendix C.

Axiom (iii). The energy and momentum satisfy the relativistic relation,

$$E^2 = P^2 c^2, \qquad (14)$$

where, E denotes the energy, P denotes the momentum and c the light speed.

In the equation (14), we have assumed that the rest mass is equal to zero[7]. In other words, the rest mass, which differs from the energy and momentum, is not the most fundamental physical entity. The existence of rest mass depends on the Higgs mechanism [21-22], which is a foundation of the standard model of particle physics [23].

Substitution of equation (14) into equation (1) leads to the de Broglie relation

$$P = \frac{h}{\lambda}, \qquad (15)$$

where, λ is the wave length of matter wave.

Using equation (1) and (15), we can obtain the function of matter wave

$$\phi(x) = C \exp\left[\frac{i}{\hbar} P_\mu x^\mu\right], \qquad (16)$$

where, P_μ denotes 4-dimensional momentum, x^μ denotes 4-dimensional coordinates and $\hbar = \frac{h}{2\pi}$.

The non-local derivatives of order 2 of equation (16) with respect to coordinate x^μ gives[8]

$$-\hbar^2 \partial^\mu \partial_\mu \phi = P^\mu P_\mu \phi. \qquad (17)$$

Equation (14) shows $P^\mu P_\mu = 0$, so the equation (17) can be rewritten as

$$-\hbar^2 \partial^\mu \partial_\mu \phi = 0, \qquad (18)$$

which is the Klein-Gordon equation with zero mass.

If we take into account the principle of gauge invariance [24], then using equation (18) we can arrive at the wave equation coupled with gauge fields,

$$-\hbar^2 D^\mu D_\mu \phi = 0, \qquad (19)$$

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[7] The complete relativistic relation is $P^\mu P_\mu = m^2 c^2$, where P_μ denotes 4-dimensional momentum. We can arrive at equation (14) by ordering $m = 0$.

[8] Consider that $\lim_{\omega \to 1} \frac{{}_t D_\omega}{{}_t D_\omega t} = \frac{d}{dt}$ and $1-\omega$ is enough small, we can use the Newton-Leibniz derivative $\frac{d}{dt}$ to replace the non-local derivative $\frac{{}_t D_\omega}{{}_t D_\omega t}$.

where D_μ denotes the covariant derivative $D_\mu = \partial_\mu - i\frac{q}{\hbar}A_\mu$ and A_μ the gauge fields.

Clearly, using equation (18) or (19) we can derive the Weyle equation; however, to derive the Dirac equation, we need to obtain the rest mass using Higgs mechanism [21-22].

So far, we have reproduced the quantum mechanics using axioms (i)-(iii). Here, it is emphasized that the properties (1) and (15) of matter are no longer two hypotheses presented by de Broglie [25], but are the conclusions of axioms (i)-(iii).

In the next section, we shall develop the quantum field theory on non-local space-time.

5. Quantum field theory on non-local space-time

In previous sections, we focus only on the non-local derivative and integral referring to a single variable. Now we turn to investigate the multi-variable integral. For convenience, we take $\hbar = c = 1$.

The theorem 2 in Appendix B shows as follow:

For a D-dimensional ball Ω and a spherically symmetric function $f(r) = f(r_1, r_2, r_3)$, the non-local multi-variable integration of $f(r)$ over Ω equals[9]

$$\int_{\beta_{D_1}(r_1)}\int_{\beta_{D_2}(r_2)}\int_{\beta_{D_3}(r_3)} f(r_1, r_2, r_3)\prod_{i=1}^{3}{}_{r_i}D_{D_i}x_i = \frac{2\pi^{\frac{D}{2}}}{\Gamma\left(\frac{D}{2}\right)}\int_0^R f(r)r^{D-1}dr\,, \tag{20}$$

where, $D = D_1 + D_2 + D_3$, $x_i = \frac{\pi^{\frac{D_i}{2}}}{\Gamma\left(\frac{D_i}{2}+1\right)}r_i^{D_i}$ and $\Omega = \beta_{D_1}(r_1)\otimes\beta_{D_2}(r_2)\otimes\beta_{D_3}(r_3)$, details see Appendix B.

In the other direction, Svozil has proved that [6] the Lebesgue-Stieltjes integral of $f(x) = f(r)$ over Ω equals

$$\int_\Omega f(x)d\mu_H(x) = \frac{2\pi^{\frac{D}{2}}}{\Gamma\left(\frac{D}{2}\right)}\int_0^R f(r)r^{D-1}dr\,, \tag{21}$$

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[9] Through equation (20) and theorem 1 in Appendix B, now we can understand the profound meaning of Peano's curve. For example, equation (20) shows that a 3-dimensional ball Ω may be a Cartesian product of three 1-dimensional curves $\beta_{D_i=1}(r_i)$, where, $i = 1,2,3$. Not only that, the theorem 1 in Appendix B even implies that this 3-dimensional ball Ω can be filled up by a 3-dimensional Peano's curve $\beta_{D=3}(r)$, the latter is completely determined by an independent parameter r.

where, $\int_\Omega d\mu_H(x)$ denotes the Lebesgue-Stieltjes integral and $\mu_H(x)$ denotes the Hausdorff measure.

Comparing equation (20) and equation (21), we have

$$\int_\Omega f(x)_r D_D x = \int_\Omega f(x) d\mu_H(x), \qquad (22)$$

where, ${}_r D_D x = \prod_i {}_{r_i} D_{D_i} x_i$.

Equation (22) shows that, for the case of spherically symmetric function, D-dimensional Lebesgue-Stieltjes integral is equivalent to non-local integral. Notably, the equation (21) is the starting point of QFTFS [6]. This means that the computing results obtained by QFTFS shall be all reproduced in the theoretical framework of non-local geometry.

For example, to compute the Feynman diagrams in momentum space, we can still use the Fourier-Stieltjes transformation since there have:

$$g(x) = C(D)\int_{\tilde{X}} \tilde{g}(p) \exp[i p x]_r D_D p = \frac{1}{(2\pi)^D}\int_{\tilde{X}} \tilde{g}(p) \exp[i p x] d\mu_H(p),$$

$$\tilde{g}(p) = \int_X g(x) \exp[-i p x]_r D_D x = \int_X g(x) \exp[-i p x] d\mu_H(x),$$

where, X denotes the coordinate space and $\tilde{X}$ the momentum space.

Using equations (20) and (22), the vacuum-vacuum amplitude (partition function) in non-local space-time can be written in the form:

$$Z[J] = \int [d\phi] \exp\left\{i\int d^D x \left[L(\phi, \partial_\mu \phi) + \phi J\right]\right\}, \qquad (23)$$

where, J is a source and we have already integrated out momenta.

Because $0 < \omega < 1$, and also because the dimension of space-time $D = 3 + \omega$, we shall have $D < 4$, which is consistent with the requirement of dimensional regularization [5-6]. This means that there is no divergence in the theoretical framework of quantum field theory on non-local space-time, and that the dimensional regularization is an intrinsic requirement of non-zero Planck's constant. Interestingly, Svozil [6] suggested a way of measuring the dimension of space-time D using the difference between experimental and theoretical values of a_e, where a_e is the form factor of the electromagnetic current proportional to magnetic moment. The suggested difference equals [6]

$$a_e(D=4) - a_e(D) \approx \frac{\alpha}{8\pi}(\wp + \log(\tau))(4 - D).$$

Go a step further, consider that, in the framework of non-local geometry, Planck's constant is completely determined by the critical time scale ΔT, the dimension of time axis ω and total energy of universe E, we also attempt to propose some method of measuring independently ΔT, ω and E, details see Appendix E. Thus, our formula (9) can be tested in principle. Not only that, the formula (9) is essentially a new result which can not be predicted by quantum mechanics or quantum field theory.

In addition, the starting point of quantum gravity [9-10] suggested by Calcagni is the theoretical framework of QFTFS [6]; therefore, Calcagni's quantum gravity should be also reproduced in the theoretical framework of non-local geometry.

So far, we have reproduced the quantum field theory in the theoretical framework of non-local geometry. Nevertheless, a natural question is whether there were an intrinsic and deeper physical theory where the quantum field theory is only a special case? We shall discuss this question in future work.

6. Conclusion

Quantum behaviour (equivalently, wave-particle duality) arises because the time axis is a fractal curve. To describe the analytic properties of fractal, we develop a theoretical framework of "non-local geometry" and meanwhile propose the definitions of non-local derivative and integral. Using the non-local derivative we can directly derive Planck's formula of energy quantum, and thereby naturally present a theoretical foundation for quantum mechanics. Thereafter, we attempt to develop the quantum field theory using the way of non-local geometry. Our study shows that all computing results obtained by quantum field theory can be reproduced in the theoretical framework of non-local geometry. Therefore we conclude that the non-local geometry is an intrinsic way of describing quantum behaviour. Moreover, to discriminate our theory from current quantum field theory, we suggest to test our formula $h = E \cdot (1-\omega) \cdot \Delta T$ by measuring three important constant, that is, the critical time scale ΔT, the dimension of time axis ω and total energy of universe E.

Appendix A

In Euclidean geometry, the dimension of a geometric graph is determined by the number of independent variables (i.e., the number of degrees of freedom). For example, every point on a plane can be represented by 2-tuples real number (x_1, x_2), then the dimension of the plane is denoted by 2. Nevertheless, the existence of Peano's curve powerfully refutes this viewpoint. Peano's curve, which is determined by an independent characteristic parameter (i.e., fill parameter), would fill up the entire plane [12]. Therefore, mathematicians have to reconsider the definition of dimension. The most famous one of all definitions of dimension is the Hausdorff dimension, which is defined through the Hausdorff measure [11].

A.a. Hausdorff measure and Hausdorff dimension

In order to bring the definition of Hausdorff dimension, we firstly introduce the Hausdorff measure. [11]

Let W be a non-empty subset of n-dimensional Euclidean space R^n, the diameter of W is defined as

$$diam(W) = \sup\{d(x, y),\ x, y \in W\}, \qquad \text{(A.1)}$$

where, $d(x, y)$, which is the distance between points x and y, is a real-valued function on $W \otimes W$, such that the following four conditions are satisfied.

$d(x, y) \geq 0$ for all $x, y \in W$; (A.2)

$d(x, y) = 0$ if and only if $x = y$; (A.3)

$d(x, y) = d(y, x)$ for all $x, y \in W$; (A.4)

$d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in W$. (A.5)

For example, the distance of n-dimensional Euclidean space R^n can be defined as

$$d_E(x, y) = |x - y| = \left(\sum_{i=1}^{n} |x_i - y_i|^2 \right)^{\frac{1}{2}} \qquad \text{(A.6)}$$

Then, it is easy to check that equation (A.6) satisfies conditions (A.2)-(A.5).

Now, let us consider a countable set $\{E_i\}$ of subsets of diameter at most ε that covers W, i.e.,

$W \subset \bigcup_{i=1}^{\infty} E_i$, $diam(E_i) \leq \varepsilon$ for all i.

For a positive D and each $\varepsilon > 0$, we consider covers of W by countable families $\{E_i\}$ of (arbitrary) sets E_i with diameter less than ε, and takes the infimum of the sum of $[diam(E_i)]^D$. Then we have

$$H_\varepsilon^D(W) = \inf\left\{\sum_{i=1}^{\infty}[diam(E_i)]^D : W \subset \bigcup_{i=1}^{\infty} E_i,\ diam(E_i) \le \varepsilon\right\}. \qquad \text{(A.7)}$$

If the following limit exists

$$H^D(W) = \lim_{\varepsilon \to 0} H_\varepsilon^D(W) = finite,$$

then the value $H^D(W)$ is called the D-dimensional Hausdorff measure.

A.b. Shortcoming of Hausdorff measure

In general, D may be a fraction. In 1967, Mandelbrot realized that [13] the length of coastline can be measured using Hausdorff measure (A.7) rather than Euclidean measure (A.6), and then the dimension of coastline is a fraction. Mandelbrot calls such geometric graph the "fractal".

The fractal is self-similar between its local and global shapes. Unfortunately, Hausdorff measure can determine the dimension of fractal but not reflect the connection (e.g., self-similarity) among the parts of the corresponding fractal. To see this, we consider the Koch's curve in Fig. 2.

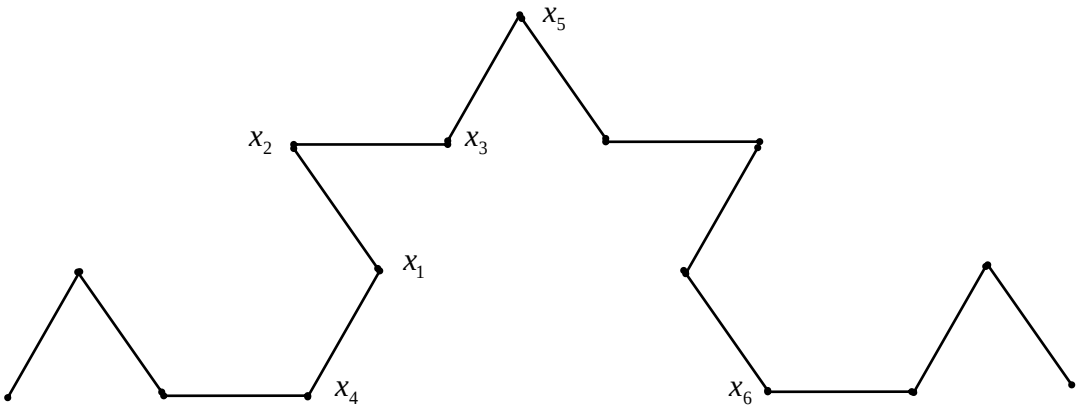

FIG.2. The Koch's curve, which is similar to the generation of Cantor ternary set (sees Fig.1), is defined by repeatedly adding the middle thirds of line segments [13].

Clearly, the congruent triangle $\Delta x_1 x_2 x_3$ is similar to $\Delta x_4 x_5 x_6$. If we use the Hausdorff measure (A.7) to measure the local distance of Koch's curve (e.g., the distance between points x_1 and x_3), then we have

$$H^D(x_1, x_3) = |x_1 - x_2|^D + |x_2 - x_3|^D, \qquad \text{(A.8)}$$

where, D is the dimension of Koch curve.

Equation (A.8) shows that the Hausdorff distance between points x_1 and x_3 depends only on the positions of points x_i $(i = 1,2,3)$, and is thereby independent of the positions of points x_j $(j = 4,5,6)$. Nevertheless, because of the self-similarity of Koch's curve, any displacements of points x_j $(j = 4,5,6)$ would influence the positions of x_i $(i = 1,2,3)$ and hence change the distance between points x_1 and x_3. That is to say, the local shape $(e.g., \Delta x_1 x_2 x_3)$ is closely related to the global shape $(e.g., \Delta x_4 x_5 x_6)$. Unfortunately, the Hausdorff distance (A.8) undoubtedly fails to reflect this fact. Therefore, we need to find a new measure of describing the analytic properties of fractal.

A.c. Definition of non-local measure

Hausdorff measure (A.7) does not reflect the self-similarity of fractal, so we can not establish the calculus theory of fractal using the Hausdorff measure. In general, people often use the fractional calculus to approximately describe the analytic properties of fractal [11, 26].

The fractional calculus is a theory of integrals and derivatives of any arbitrary real order. For example, the fractional derivatives of order m of the function $y(l) = cl^n$ equals [11]

$$\frac{d^m y(l)}{dl^m} = c \frac{\Gamma(n+1)}{\Gamma(n-m+1)} l^{n-m}, \qquad \text{(A.9)}$$

where, $\Gamma(x)$ denotes the Gamma function and m is an arbitrary real number.

Now, let us consider a m-dimensional volume

$$x(l) = \omega(m) l^m, \qquad \text{(A.10)}$$

where, $\omega(m)$ is a constant which depends only on the dimension m.

Using formula (A.9), the fractional derivatives of order m of equation (A.10) equals

$$\frac{d^m x(l)}{dl^m} = \Gamma(m+1)\omega(m). \qquad \text{(A.11)}$$

Equation (A.11) can be written as

$$\Delta^m x(l) = \Gamma(m+1)\omega(m)(\Delta l)^m + o\left[(\Delta l)^m\right], \qquad \text{(A.12)}$$

where, $\Delta^m x(l)$ denotes the differences of order m, and $o\left[(\Delta l)^m\right]$ denotes the infinitesimal terms of higher order compared to $(\Delta l)^m$.

Equation (A.12) implies that

$$\Delta^m x(l) \sim (\Delta l)^m, \qquad \text{(A.13)}$$

and thereby

$$\sum_{i=1}^{N} \Delta^m x(l_i) \sim \sum_{i=1}^{N} (\Delta l)^m. \qquad \text{(A.14)}$$

Obviously, $(\Delta l)^m$ is a m-dimensional Hausdorff measure, which can describe the length of a m-dimensional fractal curve. Consequently, equations (A.13) and (A.14) together imply that $\Delta^m x(l)$ can be also thought of as a m-dimensional measure. In this case, the order of differences $\Delta^m x(l)$ represents the Hausdorff dimension m. Because of this fact, we next attempt to use the differences of order m to define a new measure.

Let us consider the left-shift operator with step Δl and the identity operator as follows:

$$L_{\Delta l} x(l) = x(l - \Delta l), \qquad \text{(A.15)}$$

$$L_0 x(l) = x(l). \qquad \text{(A.16)}$$

Using the Left-shift operator $L_{\Delta l}$ and the identity operator L_0, we can define the difference operator of order m in the form:

$$(L_0 - L_{\Delta l})^m = \sum_{j=0}^{\infty} \frac{m(m-1)\ldots(m-j+1)(-1)^j}{j!} L_{j\Delta l}. \qquad \text{(A.17)}$$

Using equation (A.17), we define a new distance between points $x(l)$ and $x(l-\Delta l)$ in the form:

$$\left|\Delta_m\left[x(l), x(l-\Delta l)\right]\right| = \left|\sum_{j=0}^{\infty} \frac{m(m-1)\ldots(m-j+1)(-1)^j}{j!} x(l - j\Delta l)\right|. \qquad \text{(A.18)}$$

We call the equation (A.18) the "non-local distance", which describes the length of a m-dimensional fractal curve.

Whenever $m=1$, the non-local distance (A.18) returns to the Euclidean distance, i.e.,

$$\left|\Delta_{m=1}\left[x(l),x(l-\Delta l)\right]\right|=\left|x(l)-x(l-\Delta l)\right|. \qquad \text{(A.19)}$$

In general, the non-local distance (A.18) does not satisfies the general properties[10] (A.4)-(A.5) of distance but reflects the connection between local and global segments of fractal. To understand the latter, we need to realize that the output value of equation (A.18) would depend on the values of all points $x(l-j\Delta l)$ $(j=0,1,2...)$ rather than only on points $x(l)$ and $x(l-\Delta l)$.

For instance, in Fig.2, the non-local distance $\left|\Delta_D\left[x_5,x_3\right]\right|$ between points x_3 and x_5 would depend on the positions of points x_i $(i=1,2,3,4,5)$ rather than only on points x_3 and x_5. Therefore, the non-local distance (A.18) is indeed an intrinsic way of describing fractal, since it not only determines the dimension but also reflects the connection between local and global segments of fractal.

Using the non-local distance (A.18), we can propose a definition for non-local measure.

Let W be a non-empty subset of n-dimensional Euclidean space R^n. We consider a countable set $\{F_i\}$ of subsets of diameter at most ε that covers W, i.e.,

$$W\subset\bigcup_{i=1}^{\infty}F_i,\ \ diam^D(F_i)\le\varepsilon \ \text{ for all } \ i.$$

where, $diam^D(F_i)$ defined by using the non-local distance (A.18) denotes the diameter of F_i, i.e.,

$$diam^D(F_i)=\sup\left\{\left|\Delta_D\left[x,y\right]\right|,\ x,y\in F_i\right\}. \qquad \text{(A.20)}$$

Non-local measure: For a positive D and each $\varepsilon>0$, we consider covers of W by countable families $\{F_i\}$ of (arbitrary) sets F_i with diameter less than ε, and takes the infimum of the sum of $diam^D(F_i)$. Then we have

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[10] For example, the non-local distance between points $x(l)$ and $x(l+\Delta l)$, i.e., $\left|\Delta_m\left[x(l),x(l+\Delta l)\right]\right|$, need to be defined using the Right-shift operator $R_{\Delta l}$ which leads to $R_{\Delta l}x(l)=x(l+\Delta l)$. The corresponding difference operator reads $(R_{\Delta l}-R_0)^m$. Then, the non-local distance would not satisfies condition (A.4).

$$\Pi_{\varepsilon}^{D}(W)=\inf\left\{\sum_{i=1}^{\infty} diam^{D}(F_i):\ W\subset\bigcup_{i=1}^{\infty}F_i,\ diam^{D}(F_i)\le\varepsilon\right\}. \qquad \text{(A.21)}$$

If the following limit exists

$$\Pi^{D}(W)=\lim_{\varepsilon\to 0}\Pi_{\varepsilon}^{D}(W)=fini,$$

then the value $\Pi^{D}(W)$ is called the D-dimensional non-local measure; meanwhile, the dimension of W equals D.

Henceforth we call these geometric graphs, which are described by non-local measure $\Pi^{D}(W)$, the "non-local geometry".

A.d. Definition of non-local derivative

Fractal is a case of non-local geometry (e.g., there exists a non-local self-similarity between local and global segments of fractal) and can be therefore described by the non-local measure $\Pi^{D}(W)$. To describe the analytic properties of non-local geometry (e.g., fractal), we need to introduce non-local derivative.

Before proceeding to introduce the definition of non-local derivative, let us consider a ω-dimensional fractal curve $\beta_{\omega}(l)$ (sees Fig. 3), which is determined by an independent characteristic parameter l (e.g., the fill parameter of Peano's curve), filling up a ω-dimensional region. Assume that the length of the fractal curve is determined by a ω-dimensional volume $x(l)$, then the (non-local) length between points $a=\beta_{\omega}(l_0)$ and $b=\beta_{\omega}(l_n)$ in Fig.3 should be denoted by

$$\Pi^{D}([a,b])=\lim_{\Delta l\to 0}\inf\left\{\sum_{i=1}^{n}|\Delta_{\omega}[x(l_{i-1}),x(l_i)]|\right\}, \qquad \text{(A.22)}$$

where, we have used the non-local measure (A.21).

It is carefully noted that the length between points $a=\beta_{\omega}(l_0)$ and $b=\beta_{\omega}(l_n)$ can not be measured by Euclidean scale, sees Fig. 3.

As such, we can present a definition for non-local derivative as follow.

Non-local derivative: For any differentiable function $y=f(x)$, if $x=x(l)$, which is a ω-dimensional volume, describes the length of a ω-dimensional fractal curve $\gamma_{\omega}(l)$, then the non-local derivative of $y=f(x)$ with respect to the fractal curve $\gamma_{\omega}(l)$ is defined as

$$\frac{{}_{l}D_{\omega}f(x)}{{}_{l}D_{\omega}x}=\lim_{\Delta l\to 0}\frac{\Delta_{\omega}[f(l),f(l-\Delta l)]}{\Delta_{\omega}[x(l),x(l-\Delta l)]}, \qquad \text{(A.23)}$$

where, $f(l)=f[x(l)]$.

Clearly, if $\omega=1$, then (A.23) will return to the Newton-Leibniz derivative, and meanwhile $\gamma_{\omega=1}(l)$ is restored to a 1-dimensional coordinate axis.

In general, the fractional derivatives of order ω of any differentiable function $f(l)$ is defined in the form [27]:

$$\frac{d^{\omega}f(l)}{dl^{\omega}}=\lim_{\Delta l\to 0}\frac{\sum_{j=0}^{\infty}\frac{\omega(\omega-1)..(\omega-j+1)(-1)^{j}}{j!}f(l-j\Delta l)}{(\Delta l)^{\omega}}. \qquad \text{(A.24)}$$

Comparing (A.18) and (A.24), we have

$$\frac{d^{\omega}f(l)}{dl^{\omega}}=\lim_{\Delta l\to 0}\frac{\Delta_{\omega}[f(l),f(l-\Delta l)]}{(\Delta l)^{\omega}}. \qquad \text{(A.25)}$$

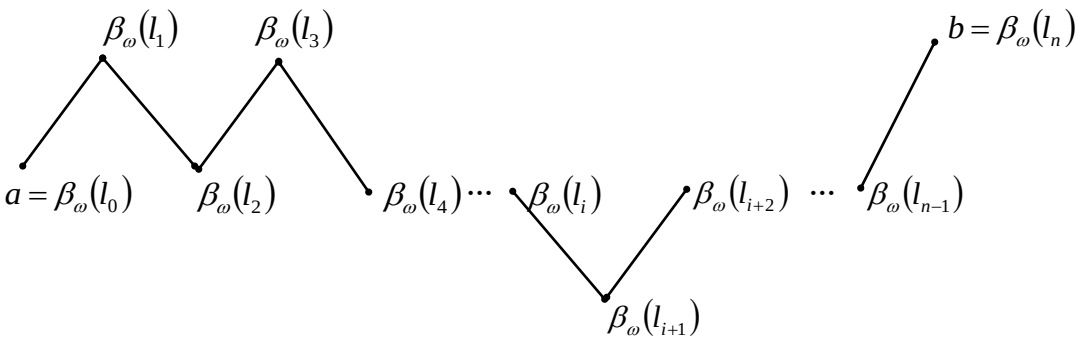
Using formula (A.25), the formula (A.23) can be rewritten as

$$\frac{{}_{l}D_{\omega}f(x)}{{}_{l}D_{\omega}x}=\lim_{\Delta l\to 0}\frac{\Delta_{\omega}[f(l),f(l-\Delta l)]}{\Delta_{\omega}[x(l),x(l-\Delta l)]}\cdot\lim_{\Delta l\to 0}\frac{\frac{1}{(\Delta l)^{\omega}}}{\frac{1}{(\Delta l)^{\omega}}}=\frac{\lim_{\Delta l\to 0}\frac{\Delta_{\omega}[f(l),f(l-\Delta l)]}{(\Delta l)^{\omega}}}{\lim_{\Delta l\to 0}\frac{\Delta_{\omega}[x(l),x(x-\Delta l)]}{(\Delta l)^{\omega}}}.$$

$$=\frac{\frac{d^{\omega}f(l)}{dl^{\omega}}}{\frac{d^{\omega}x(l)}{dl^{\omega}}}$$

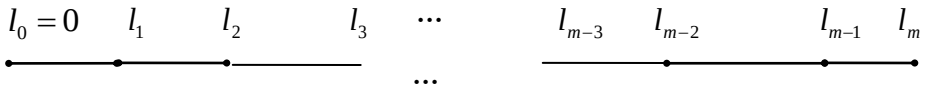
(A.26)

Formula (A.26) indicates that we can compute the non-local derivative using the fractional derivative.



Fractal curve $\beta_\omega(l)$

Non-local scale $x(l)$



Euclidean scale l

FIG.3. The fractal curve $\beta_\omega(l)$ consists of the union $\bigcup_{i=1}^{n}[\beta_\omega(l_{i-1}),\beta_\omega(l_i)]$, where $n=n(m)$. The dimension of fractal curve $\beta_\omega(l)$ equals $\omega=\lim_{m\to\infty}\frac{\ln n(m)}{\ln m}$. Since $\beta_\omega(l)$ is a fractal curve, the distance between points $\beta_\omega(l_0)$ and $\beta_\omega(l_n)$ can not be measured using the Euclidean scale (ruler) l; otherwise, we shall have $d[\beta_\omega(l_0),\beta_\omega(l_n)]=\lim_{m\to\infty}n(m)\frac{l_m-l_0}{m}=\infty$ or $=0$. However, the fractal curve $\beta_\omega(l)$ can be measured using the non-local scale (ruler) $x(l)$, the way of measure sees the formula (A.22).

Appendix B

In Appendix A, we have noted that the non-local derivative can be computed using the formula (A.26). In this appendix, we compute two useful non-local derivatives and introduce the definition of non-local integral.

B.a. Computing examples of non-local derivative

Example 1. If $x(l)=\omega(m)l^m$ describes the length of a m-dimensional fractal curve, then the non-local derivative of the constant function $f(x)=C$ with respect to x equals

$$\frac{{}_l D_m C}{{}_l D_m x}=\frac{C}{\Gamma(1-m)\Gamma(1+m)}\cdot\frac{1}{x}, \qquad \text{(B.1)}$$

where $\omega(m)$ is a constant that depends only on m.

Proof. The fractional derivatives of order m of the constant C and the power function $y(l)=al^n$ are respectively as follows [11]:

$$\frac{d^m C}{dl^m}=\frac{C}{\Gamma(1-m)}l^{-m}, \qquad \text{(B.2)}$$

$$\frac{d^m y(l)}{dl^m}=a\frac{\Gamma(n+1)}{\Gamma(n-m+1)}l^{n-m}. \qquad \text{(B.3)}$$

Using formulas (A.26), (B.2) and (B.3), the non-local derivative $\frac{{}_l D_m f(x)}{{}_l D_m x}$ can be computed as follow:

$$\frac{{}_l D_m C}{{}_l D_m x}=\frac{\frac{d^m C}{dl^m}}{\frac{d^m\left[\omega(m)l^m\right]}{dl^m}}=\frac{\frac{C}{\Gamma(1-m)}l^{-m}}{\omega(m)\Gamma(1+m)}=\frac{C}{\Gamma(1-m)\Gamma(1+m)}\cdot\frac{1}{x}.$$

The proof is complete. □

Example 2. If x is the characteristic parameter of a m-dimensional fractal curve, and meanwhile it also describes the length of this fractal curve, then the non-local derivative of the constant function $f(x)=C$ with respect to x equals

$$\frac{{}_{x}D_{m}C}{{}_{x}D_{m}x}=C(1-m)\cdot\frac{1}{x}. \qquad \text{(B.4)}$$

Proof. Using formulas (A.26), (B.2) and (B.3) we have

$$\frac{{}_{x}D_{m}f(x)}{{}_{x}D_{m}x}=\frac{\frac{d^{m}C}{dx^{m}}}{\frac{d^{m}x}{dx^{m}}}=\frac{\frac{C}{\Gamma(1-m)}x^{-m}}{\frac{\Gamma(2)}{\Gamma(2-m)}x^{1-m}}=C\frac{\Gamma(2-m)}{\Gamma(1-m)}\cdot\frac{1}{x}. \qquad \text{(B.5)}$$

Consider the property of Gamma function

$$\Gamma(x+1)=x\Gamma(x), \qquad \text{(B.6)}$$

we have

$$1-m=\frac{\Gamma(2-m)}{\Gamma(1-m)}. \qquad \text{(B.7)}$$

Substituting equation (B.7) into equation (B.5) we arrive at

$$\frac{{}_{x}D_{m}C}{{}_{x}D_{m}x}=C(1-m)\cdot\frac{1}{x}.$$

The proof is complete. □

Comparing equations (B.1) and (B.5) we note that $\frac{{}_{l}D_{m}C}{{}_{l}D_{m}x}$ up to a constant factor is equal to $\frac{{}_{x}D_{m}C}{{}_{x}D_{m}x}$, that is,

$$\frac{{}_{x}D_{m}C}{{}_{x}D_{m}x}=\Gamma(2-m)\Gamma(1+m)\frac{{}_{l}D_{m}C}{{}_{l}D_{m}x}. \qquad \text{(B.8)}$$

However, $\Gamma(2-m)\Gamma(1+m)$ tends to 1 whenever $m\to 1$; that is,

$$\lim_{m\to 1}\frac{{}_{x}D_{m}C}{{}_{x}D_{m}x}=\lim_{m\to 1}\frac{{}_{l}D_{m}C}{{}_{l}D_{m}x}. \qquad \text{(B.9)}$$

B.b. Definition of non-local integral

Now we propose a definition for non-local integral as follow.

Non-local integral: If $\frac{{}_{l}D_{m}f(x)}{{}_{l}D_{m}x}=g(x)$, then the non-local integral of $g(x)$ on a m-dimensional fractal curve $\beta_m(l)$ is defined in the form:

$$\int {}_{l}D_{m}f(x)=\int_{\beta_m(l)}g(x){}_{l}D_{m}x=\int_{W}g(x)\frac{d^{m}x}{dl^{m}}(dl)^{m}, \qquad \text{(B.10)}$$

where W denotes the definitional domain of the characteristic parameter l; also, the parameter l completely determines the generation of the m-dimensional fractal curve $\beta_m(l)$.

Using the definition of non-local integral we present two useful theorems as follows.

***Theorem 1*:** Let $x(r)=\omega(m)r^{m}$, which describes the length of a m-dimensional fractal curve $\beta_m(r)$, denote the volume of a m-dimensional sphere Ω, and let $f(x)=f(r)$ be a spherically symmetric function, then the non-local integration of $f(x)$ on the m-dimensional fractal curve $\beta_m(r)$ equals

$$\int_{\beta_m(r)}f(x){}_{r}D_{m}x=\frac{2\pi^{\frac{m}{2}}}{\Gamma\left(\frac{m}{2}\right)}\int_{0}^{R}f(r)r^{m-1}dr, \qquad \text{(B.11)}$$

where, $\omega(m)=\dfrac{\pi^{\frac{m}{2}}}{\Gamma\left(\frac{m}{2}+1\right)}$.

Proof. The Riemann-Livouville fractional integrals of order m of $f(l)$ is defined in the form [11]:

$$\int f(l)(dl)^{m}=\frac{1}{\Gamma(m)}\int_{y}^{R}(l-y)^{m-1}f(l)dl. \qquad \text{(B.12)}$$

Using equations (A.25) and (B.3)we have

$${}_{r}D_{m}x=\frac{d^{m}x}{dr^{m}}(dr)^{m}=\omega(m)\Gamma(1+m)(dr)^{m}. \qquad \text{(B.13)}$$

Using equations (B.10) and (B.13) we arrive at

$$\int_{\beta_m(r)}f(x){}_{r}D_{m}x=\int_{\Omega}f(r)\omega(m)\Gamma(1+m)(dr)^{m}. \qquad \text{(B.14)}$$

Inserting equation (B.12) into equation (B.14) leads to

$$\int_{\beta_m(r)} f(x)_r D_m x = \omega(m)\frac{\Gamma(1+m)}{\Gamma(m)}\int_y^R (r-y)^{m-1} f(r)dr . \qquad \text{(B.15)}$$

Using $\omega(m)=\dfrac{\pi^{\frac{m}{2}}}{\Gamma\left(\dfrac{m}{2}+1\right)}$ and the formula (B.6), the equation (B.15) can be rewritten as

$$\int_{\beta_m(r)} f(x)_r D_m x = \frac{\pi^{\frac{m}{2}} m}{\Gamma\left(\dfrac{m}{2}+1\right)}\int_y^R (r-y)^{m-1} f(r)dr . \qquad \text{(B.16)}$$

For $y=0$, equation (B.16) yields

$$\int_{\beta_m(r)} f(x)_r D_m x = \frac{\pi^{\frac{m}{2}} m}{\Gamma\left(\dfrac{m}{2}+1\right)}\int_0^R r^{m-1} f(r)dr . \qquad \text{(B.17)}$$

Using the formula (B.6) we have

$$\frac{m}{2}\Gamma\left(\frac{m}{2}\right)=\Gamma\left(\frac{m}{2}+1\right). \qquad \text{(B.18)}$$

Substitution of equation (B.18) into equation (B.17) leads to

$$\int_{\beta_m(r)} f(x)_r D_m x = \frac{2\pi^{\frac{m}{2}}}{\Gamma\left(\dfrac{m}{2}\right)}\int_0^R f(r)r^{m-1}dr .$$

The proof is complete. □

Clearly, for $f(x)=1$, the formula (B.11) gives the volume of a m-dimensional sphere, $\dfrac{\pi^{\frac{m}{2}}}{\Gamma\left(\dfrac{m}{2}+1\right)}R^m$.

Finally, we present a theorem of non-local multi-variable integration as follow.

Theorem 2: Let $x_i(r_i)=\omega(m_i)r_i^{m_i}\ (i=1,2,3)$, which describes the length of a m_i-dimensional fractal curve $\beta_{m_i}(r_i)$, denote the volume of a m_i-dimensional sphere Ω_i, and let $f(r)=f(r_1,r_2,r_3)$ be a spherically symmetric function, then the non-local integration of $f(r)$ on the fractal graph $\beta_{m_1}(r_1)\otimes\beta_{m_2}(r_2)\otimes\beta_{m_3}(r_3)$ equals

$$\int_{\beta_{m_1}(r_1)}\int_{\beta_{m_2}(r_2)}\int_{\beta_{m_3}(r_3)} f(r_1,r_2,r_3)\prod_{i=1}^{3}{}_{r_i} D_{m_i} x_i = \frac{2\pi^{\frac{m}{2}}}{\Gamma\left(\frac{m}{2}\right)}\int_0^R f(r) r^{m-1} dr\,, \qquad \text{(B.19)}$$

where, $\omega(m_i) = \dfrac{\pi^{\frac{m_i}{2}}}{\Gamma\left(\frac{m_i}{2}+1\right)}$, $r^2 = r_1^2 + r_2^2 + r_3^2$ and $m = m_1 + m_2 + m_3$.

Proof. Reference [11] has proved that (sees the formula (1.85) in [11])

$$\int_{\Omega_1}\int_{\Omega_2}\int_{\Omega_3} f(r_1,r_2,r_3)\prod_{i=1}^{3}\frac{2\pi^{\frac{m_i}{2}}}{\Gamma\left(\frac{m_i}{2}\right)} r_i^{m_i-1} dr_i = \frac{2\pi^{\frac{m}{2}}}{\Gamma\left(\frac{m}{2}\right)}\int_0^R f(r) r^{m-1} dr\,. \qquad \text{(B.20)}$$

Using equations (B.11) and (B.20) we easily arrive at equation (B.19)

The proof is complete. □

From the theorem 2, we note that the constant (geometric) factor $\omega(m_i)$ is important. It would guarantee that the dimension m of integral domain can be smoothly linked by the dimension m_i of each variable.

Appendix C

Theorem 3: The Cantor set is uncountable.

Proof. Suppose the Cantor set W is countable, and list its elements as

$$W = \{x_1, x_2, x_3, .\}.$$

Now look at the base 3 expansion of each of those numbers.

First, let us consider

$$\begin{aligned} x_1 &= 0.a_{11}a_{12}a_{13}\ldots \\ x_2 &= 0.a_{21}a_{22}a_{23}\ldots \\ &\vdots \\ x_k &= 0.a_{k1}a_{k2}a_{k3}\ldots \end{aligned}, \qquad \text{(C.2)}$$

where, $a_{ij} = 0$ or 2 for all i, j.

Second, let

$$y_1 = 0.b_1b_2b_3.., \qquad \text{(C.3)}$$

where, $b_i = \begin{cases} 0 & if \ \ a_{ii} = 2 \\ 2 & if \ \ a_{ii} = 0 \end{cases}$.

Comparing (C.2) and (C.3), we note that $y \neq x_1$ since $b_1 \neq a_{11}$, $y \neq x_2$ since $b_2 \neq a_{22}$, and so on. This implies that $y \notin W$, but this is a contradiction since $b_i \in \{0,2\}$ for each i and then $y \in W$. Therefore, the cantor set is uncountable.

The proof is complete. □

Notably, this proof implies that the fractal set with dimension $D(m) = \dfrac{\ln 2}{\ln 2 + \ln \dfrac{m}{m-1}}$ $(m > 1)$ is uncountable, where $m = 3$ is the case of theorem 3. When $m = 1$, $D(m = 1) = 0$ is the dimension of a countable set; when $m = \infty$, $D(m = \infty) = 1$ is the dimension of an interval, e.g., $[0,1]$. Go a step further, if we admit an axiom that the cardinal number of a set is always monotonically non-decreasing with respect to the dimension of this set, then we can conclude that there are only two cardinal numbers in the interval $[0,1]$, that is, countable cardinal

number a and uncountable cardinal number c. There would be no any number lying between a and c.

The construction of fractal set with dimension $D(m)=\dfrac{\ln 2}{\ln 2+\ln\dfrac{m}{m-1}}$ $(m>1)$ sees Fig. 4 in Appendix D.

Appendix D

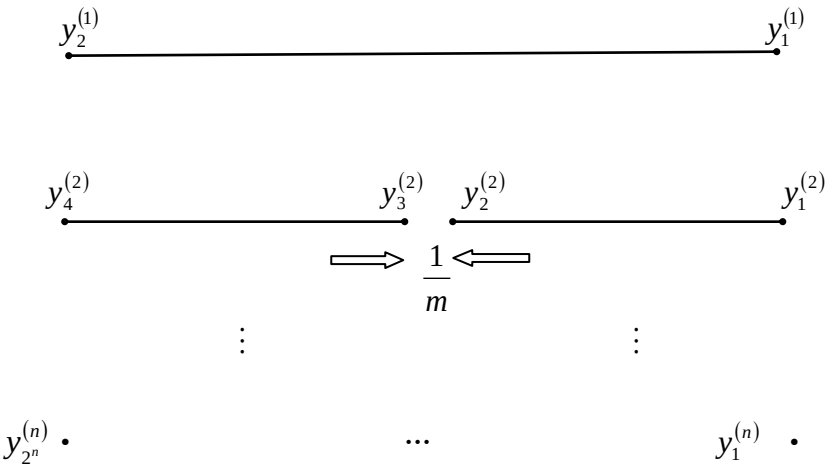


FIG.4. The fractal set with dimensions $D(m)=\dfrac{\ln 2}{\ln 2+\ln\dfrac{m}{m-1}}$ $(m>1)$ is defined by repeatedly removing the middle $\dfrac{1}{m}$ of line segments: (a). One starts by removing the middle $\dfrac{1}{m}$ from the interval $\left[y_2^{(1)},y_1^{(1)}\right]$, leaving $\left[y_4^{(2)},y_3^{(2)}\right]$ and $\left[y_2^{(2)},y_1^{(2)}\right]$. (b). Next, the "middle $\dfrac{1}{m}$" of all remaining intervals is removed. (c). This process is continued ad infinitum. Finally, the $D(m)$-dimensional fractal consists of all points in the interval $\left[y_2^{(1)},y_1^{(1)}\right]$ that are not removed at any step in this infinite process.

For simplicity, let the length of interval $\left[y_2^{(1)}, y_1^{(1)}\right]$ equal 1 (sees Fig. 4). It is easy to see that the Euclidean length of the interval $\left[y_2^{(n)}, y_1^{(n)}\right]$ equals $\left(\frac{1}{2}-\frac{1}{2m}\right)^{n-1}$, and that the total number of remaining intervals equals 2^{n-1}. Therefore, the Hausdorff dimension of this fractal equals

$$D(m)=-\frac{\ln 2^{n-1}}{\ln\left(\frac{1}{2}-\frac{1}{2m}\right)^{n-1}}=\frac{\ln 2}{\ln 2+\ln\frac{m}{m-1}}.$$

Clearly, $\lim_{m\to\infty} D(m)=1$ and $D(m)$ is monotonically increasing with respect to m. This means that the closer $D(m)$ tends to 1, the closer the fractal curve shown by Fig.4 approaches continuum. Conversely, the more $D(m)$ deviates 1, the closer the fractal curve approaches a set of discrete points.

Appendix E

First, we consider how to measure the total energy of universe E.

As pointed out in section 3, E denotes the sum of all types of energies which exist in the universe, including contributions of dark matters, dark energies and so on. In general, according to the Modern cosmology, E is composed of three parts which are visible matter, dark matter and dark energy respectively. If we refer to ρ_a as the mean density of the universe, and V as the volume of the universe, then $E=\rho_a Vc^2$, where c denotes the light speed.

On the one hand, if we can measure the age of universe, then we can find the scale of universe using the Hubble formula, and thereby compute the volume V.

On the other hand, we can measure the mean density of some representational galaxy ρ, and thereafter estimate the contributions ρ' of dark matter and dark energy. Using ρ and ρ', we can find $\rho_a=\rho+\rho'$.

In fact, the measures of ρ_a and V are two main objectives of Modern cosmology. Once ρ_a and V are measured, then we can compute $E=\rho_a Vc^2$.

Second, we restrict our attention to how to measure ω and ΔT.

Consider that the time axis is a fractal curve, we shall propose the following three steps to measure the length of a time interval L.

(a). We use a standard clock to specify a time interval whose length is L according to this standard clock.

(b). We find another clock i whose step length is $\frac{L}{N_i}$ (and is less than the step length of standard clock), where $N_i > 1$. For example, the step length of a wrist watch is generally denoted by a step of second hand.

(c). Let the standard clock and the clock i go simultaneously. When the standard clock arrive at L from 0, then we count the number $M_i = M_i(N_i)$ of steps that the clock i has gone.

If the dimension of time axis equals ω, then according to the fractal theory we have:

$$\lim_{N_i \to \infty} \left(\frac{L}{N_i} \right)^{\omega} M_i = L^{\omega}. \qquad \text{(E.1)}$$

Equation (E.1) shows that when we use the clock i to measure "L", then the value of "L" equals L^{ω}, where $\omega = \lim_{N_i \to \infty} \frac{\ln M_i}{\ln N_i}$.

In an actual measurement, we can use the clock with different step length to approach the limit $N_i \to \infty$.

$$N_1 < N_2 < ... < N_i < ... < N_n. \qquad \text{(E.2)}$$

Of course, consider that there exists a critical time scale ΔT, the step length of clock can not approach zero, that is to say, the limit step length is denoted by:

$$\lim_{N_i \to \infty} \left(\frac{L}{N_i} + \Delta T \right) = \Delta T. \qquad \text{(E.3)}$$

Thus, thinking of the existence of ΔT, the equation (E.1) must be rewritten as

$$\left(\frac{L}{N_i} + \Delta T \right)^{\omega} M_i = L^{\omega}. \qquad \text{(E.4)}$$

$i = 1,2,...n$

Using the equation (E.4) we arrive at

$$\omega \ln \left(\frac{1}{N_i} + \delta \right) = \ln \frac{1}{M_i}. \qquad \text{(E.5)}$$

where, $\delta = \frac{\Delta T}{L}$.

Consider that $\delta << 1$, using the Taylor expansion

$$\ln\left(\frac{1}{N_i}+\delta\right)\approx \ln\frac{1}{N_i}+N_i\delta,$$

the equation (E.5) can be written as

$$\frac{\ln M_i}{N_i}=-\omega\delta+\omega\frac{\ln N_i}{N_i}. \qquad \text{(E.6)}$$

If we order $y_i=\frac{\ln M_i}{N_i}$, $x_i=\frac{\ln N_i}{N_i}$, and $\varepsilon=-\omega\delta$, then the equation (E.6) can be written in the form:

$$y_i=\varepsilon+\omega x_i. \qquad \text{(E.7)}$$

Obviously, using the measured value (M_i,N_i), we can compute (x_i,y_i), where $i=1,2,\ldots,n$.

So we have the following estimated values:

$$\omega=\frac{\sum_{i=1}^{n}x_iy_i-\frac{1}{n}\left(\sum_{i=1}^{n}x_i\right)\left(\sum_{i=1}^{n}y_i\right)}{\sum_{i=1}^{n}x_i^2-\frac{1}{n}\left(\sum_{i=1}^{n}x_i\right)^2},$$

$$\varepsilon=\frac{1}{n}\sum_{i=1}^{n}y_i-\frac{\omega}{n}\sum_{i=1}^{n}x_i.$$

Now that ω and ε have been found, using the formulas $\delta=\frac{\Delta T}{L}$ and $\varepsilon=-\omega\delta$, we can obtain the estimated value of ΔT. Here we suggest to make this measurement using the atomic clock.

Finally, using the estimated values of E, ω, and ΔT, we can compute the theoretical value of Planck's constant.

Therefore, our formula $h=E\cdot(1-\omega)\cdot\Delta T$ can be tested in principle.

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