Tropical and seasonal year

Wenceslao Segura González
e-mail: wenceslaoseguragonzalez@yahoo.es

Abstract. There is widespread confusion about the concept of tropical year. Although it is correctly defined from the average length of the Sun, at the same time it is identified with the movement of the Sun with respect to the seasonal points. In this study, we distinguish between the tropical year, as it is commonly defined, and the seasonal year, or time between two consecutive passages of the Sun by a particular seasonal point. We found that the mean value of the possible four years seasonal averages (Spring, Summer, Autumn and Winter) coincides with the tropical year. We evaluated the variation of the seasonal years for an interval of time of a few thousands of years. And finally we estimated the durations of the various types of years in universal time units, adapting them for use in the calculation of the calendarists' errors.

1. Introduction

Currently, tropical year is defined as the time that has to elapse so that the geometric mean length of Sun increases in 360 degrees. Formerly, tropical year was understood as the time between two consecutive steps of the Sun by a seasonal point. Both years are conceptually and numerically different. Therefore, it is interesting that today there is widespread confusion when considering both types of years as equivalent.

An example of a typical definition of the tropical year, much repeated in worthy astronomy textbooks, is as follows: «tropical year is defined as the interval between two vernal equinoxes, corresponding to the cycle of the seasons». [1]

In this article we introduce the concept of seasonal year or the time between two consecutive passages of the Sun by a seasonal point, distinguishing between true seasonal year and mean. Using a simplified theory of solar movement, we will calculate the average durations of the four seasonal years, with which we will prove are different from the tropical year.

We found that in numerical evaluations, the tropical year coincides with the arithmetic mean of the four seasonal years. This circumstance is mainly due to the smallness of the variation of the speed of perigee and equally small secular variation of the eccentricity of the Earth orbit.

To conclude, we will establish the relationship between the scale of terrestrial time (TT), within which are measured the durations of the various types of years, and the scale of universal time (UT), which, in its form of coordinated universal time (UTC), is the most used in everyday life. With this information we will express the durations of the years in the scale of UT.

2. Apparent movement of the Sun

We assume that the movement of the Earth around the Sun follows an ellipse whose orbital elements (eccentricity and perihelion longitude) vary secularly as a result of the interaction from the other bodies of the solar system. We consider the apparent motion of an mean Sun whose ecliptic length is given by the geometrical mean length of the true Sun with respect to the mean equinox of date. * In accordance with the theory VSOP82 [2],

$$L_m = 280^\circ.466\,448\,514 + 36\,000^\circ.769\,8231T + 0^\circ.000\,303\,678T^2 + 0^\circ.000\,000\,0212T^3$$

the average length of solar perigee varies according to

$$\omega_m = 282^\circ.937\,348 + 1^\circ.719\,5269T + 0^\circ.000\,459\,62T^2 + 0^\circ.000\,000\,499T^3$$

* Therefore we ignore the precession and nutation. It should be noted that we seeking the average movement of the Sun for large intervals of time, so we can disregard the movements of short period.

and the eccentricity of the orbit have the secular variation

$$e = 0.016\,708\,62 - 0.000\,042\,037T - 0.000\,000\,1236T^2 + .000\,000\,000\,04T^3 \ .$$

In the previous and continuing formulas, *T* is given in Julian centuries from the date J2000.0 on the scale of Terrestrial Time (TT).

The mean anomaly *M*, or angle measured on the ecliptic from perigee to the position of the mean Sun and measured in the direction of the solar motion, is given by

$$M = L_m + 360 - \omega_m = 357°.529\,101 + 35\,999°.050\,296T - 0°.000\,1559T^2 \qquad (1)$$

where we have only retained to the second power of *T*, being more than sufficient for subsequent calculations. Note that in the current era solar perigee is nearing the equinox.

The true anomaly ν or angle measured on the ecliptic from the mean perigee to the true Sun, is

$$\nu = L + 360 - \omega_m$$

L being the length of the true Sun.

The calculation of the true anomaly first requires the determination of the eccentric anomaly *E* by Kepler's law

$$E - e\sin E = M$$

where the angles are currently expressed in radians. Ultimately, the true anomaly is calculated from the eccentric anomaly by the formula

$$\tan\frac{\nu}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}$$

and we are therefore able to calculate the true anomaly of the Sun. [3] Nevertheless, it is easier to solve Kepler's equation through a series expansion of the powers of eccentricity, taking into account that the terrestrial orbit is minimally eccentric. Applying up to the third term, we find

$$L = L_m + \left(2e - e^3/4\right)\sin M + 5/4e^2\sin 2M + 13/12e^3\sin 3M + \ldots \qquad (2)$$

The periodic terms, which in (2) are added to the mean length, are called equation of the centre. [4] This simplified theory is more than sufficient for our further analysis, because, as we have stated, what we seek is the study of the solar movement over a period of some thousands of years.

3. Tropical year

In the past, tropical year was understood as the time between two consecutive passages of the Sun by the same seasonal point. The ambiguity of the measurements did not permit warning of any variation of this year, no differences were observed according to seasonal point taken as reference. The most accurate definition of the tropical year in antiquity was formulated by Hipparchus, who lived in the 2nd century B.C. [5] He determined the moments in which some equinoxes and solstices occurred in his time, then compared them with other ancient observations, and was thereby able to define fairly accurately the duration of the year.

The procedure used by Hipparchus to make observations is unknown, but it is known that the error made in the determination of the moments in which the seasons began, was the fourth of a day. Even so, averaging between a wide range of years, he was able to ascertain the value of 365^d 5^h 5^m 12^s; the same value Ptolemy found several centuries later. [6]

When celestial mechanics were developed, the average length of the Sun according to time could be determined, henceforth leading to another procedure for the calculation of the tropical year. It consists of determining the time elapsed so that the mean length of the Sun, in reference to the mean Equinox date, increases in 360 degrees. [7]

Assuming that the geometric mean length of the Sun with respect to the mean equinox date will be given by the expression

$$L_m(T) = A + BT + CT^2 + DT^3 \qquad (3)$$

and other terms of higher order which we need not value. The time that has to elapse from the moment *T* so that the length increases by 360° is the tropical year a_T

$$L_m(T)+360=A+B(T+a_T)+C(T+a_T)^2+D(T+a_T)^3$$

where a_T is expressed in Julian centuries of Terrestrial Time. Combining the last two equations results in an algebraic formula that allows us to determine the length of the tropical year which starts the instant *T*.

To avoid the obstacle of solving the above equation of third degree, the problem can be approached in a different way. Given that the acceleration of the mean Sun varies slowly, we can assume without influential error that the mean movement is the same throughout the year, and therefore assume that during that time the mean Sun performs an uniform movement. The average movement of the Sun is deducted from (3)

$$n=\frac{3600}{36525}\frac{dL_m}{dT}=a+bT+cT^2$$

which we have expressed in arcseconds per day, provided the length is expressed in degrees. The tropical year is the time that it takes the Sun to travel a full circumference with the previous movement medium, [8] or

$$a_T=\frac{1\,296\,000}{n}=\frac{1\,296\,000}{a+bT+cT^2}=\frac{1\,296\,000}{a}\left[1-\frac{b}{a}T+\left(\frac{b^2}{a^2}-\frac{c}{a}\right)T^2\right].$$

If we use the length of the VSOP theory, we obtain a mean movement provided by

$$n=3548''.330\,495\,91+5''.986\,2604\cdot10^{-5}\,T+6''.269\cdot10^{-9}\,T^2$$

expressed in arcseconds per day. From here, the length of the tropical year is obtained

$$a_T=365^d.242\,189\,67-0^d.000\,006\,1619T-6^d.4514\cdot10^{-10}\,T^2 \tag{4}$$

which is expressed in days of the TT scale (Terrestrial Time). Using Laskar's NGT theory (New General Theory), the resulting length of tropical year is given by [9]

$$a_T=365^d.242\,189\,67-0^d.000\,006\,1536T-7^d.285\cdot10^{-10}\,T^2 \tag{5}$$

which in all practical purposes coincides entirely with (4).

We must keep in mind that the last two expressions of the tropical year (4) and (5) need not be understood as a continuous function with respect to the variable *T*. Expressions (4) and (5) give us the length of the tropical year which commences on the date given by *T*. Previous expressions are therefore «staggered functions» of the variable *T*, since having chosen a value of this variable, the remaining values of *T* must be spaced between themselves one tropic year. [10] To further clarify, we will call a chronological tropical year one which begins with the calendar year (i.e., midnight on January 1st) and set aside the name of instant tropical year for that which begins any other time of the year.

4. Sidereal and Anomalistic Years

The sidereal year is the time that has to elapse so that the mean length of the Sun increases in 360 degrees with regard to a fixed equinox. The mean annual movement *p* of the equinox through the ecliptic as a result of the precession and which moves in the opposite direction to the annual movement of the Sun, is [11]

$$p=50''.290\,966+0''.022\,2226T$$

henceforth, the sidereal year a_s is

$$a_s=a_T+\frac{p}{n}.$$

Using the VSOP theory, the sidereal year value found is

$$a_s=365^d.256\,362\,805+1^d.006\,935\cdot10^{-7}\,T$$

which is greater than the tropical year (slightly more than twenty minutes) and virtually constant (an increase of some milliseconds each century). As in all our expressions *T* are Julian centuries from the date J2000.0.

The anomalistic year is the time that it takes the mean Sun to increase 360 degrees from the mean perigee. The mean movement of the Sun with respect to perigee is obtained from the mean anomaly (1)

$$n_a = \frac{dM}{dT} = 3548''.161\,0148 - 3''.073\,183 \cdot 10^{-5}\,T$$

If during the period of one year we assume this mean movement to be constant, the anomalistic year is the time that it takes the Sun to travel a full circumference with the speed n_a

$$a_a = \frac{1\,296\,000}{n_a} = 365^d.259\,635\,793 + 3^d.163\,64 \cdot 10^{-6}\,T$$

as in the other cases, the tropical year is measured in days of 24 hours of the TT scale.

5. Seasonal years

The seasonal year is the time between two consecutive passages of the true Sun by the same seasonal point.* To be more precise, we will call the previous definition true seasonal year, which can be made up of four classes, according to the chosen reference; the Spring equinox, the Summer solstice, the Autumn equinox, or the Winter solstice, which, in order to abbreviate, we will call true Spring, Summer, Autumn, or Winter years.

The duration of these seasonal years is variable, since in addition to the secular terms we must consider the periodic terms of the solar movement. For example, in the period understood between the year 2000 and 3000 the maximum difference between two seasonal years of the same type is 28 minutes, which is almost exclusively due to periodic oscillations. [12]

What interests us now is not the true seasonal year but the average. The average year, which will be simply called seasonal year, is defined as the time that it takes the mean Sun, corrected by the equation of the centre, in two consecutive passages by a seasonal point assuming the orbital parameters have a secular variation, meaning that the length of the Sun is given by equation (2). [13]

If T is the time in which a Spring equinox occurs, the following formula will be fulfilled **

$$L(T) = 0 \tag{6}$$

and the seasonal year of Spring will be given by

$$L(T + a_p) = 360. \tag{7}$$

To establish the duration of the average year of Spring beginning in the given year we adhere to the following procedure. In successive approximations, we find from (2) the time T_p of that year in which the equation (6) is carried out, that is, when the length of the Sun is a multiple of 360°, which represents average of the Spring equinox date. Subsequently, we continue making successive approximations in the equation (2) until we find the elapsed time from T_p, until once again the length of the Sun is a multiple of 360. This time interval is the mean Spring year.§

Applying the above procedure for a period between the years - 3000 and 10000, and then calculating a polynomial curve of adjustment of the values obtained, we find the following time dependency for the mean Spring year §§

---

* We understand that the seasonal points (solstices and equinoxes) are the moments in which the apparent geocentric length of the Sun (affected aberration and nutation) is a multiple of 90 degrees.

** The above is not exact because L given by the formula (2) does not consider all periodic terms of the length of the Sun, but only those coming from the equation of the center. The date obtained from (6) is the average date on which falls the spring. The date of the spring equinox obtained by (6) deviates up to few minutes of the real date, but in our study what interests us are the seasonal mean years and not the date of the seasons.

§ The results obtained by our method match the durations obtained with the formulas that appear in Meeus J.: *Astronomical Algorithms*, Willmann-Bell, 1986, p. 166, the differences between the two results do not exceed, in the period considered, seven tenths of a second. Meeus formulas represent the average values of the dates of the beginning of the four seasons according to the VSOP theory, which essentially coincides with the definition we have given of seasonal year.

§§ The timescale on which are given our results is the Ephemeris Time, which is equivalent to Terrestrial Time (TT). So we can compare these results with the tropical year which also is expressed in the same timescale.

$$a_p = 365^d.242\,374\,08 + 1^d.037 \cdot 10^{-5}\, T - 1^d.344 \cdot 10^{-7}\, T^2 -$$
$$-2^d.373 \cdot 10^{-9}\, T^3 + 1^d.732 \cdot 10^{-11}\, T^4.$$

Equations similar to (6) and (7) apply to the other three seasonal years, with such that in (6) 90° is given (for the year of Summer), 180° (for the year of Autumn), and 270° (for the year of Winter), and in (7) the previous angles will need to be increased by 360°.

The duration of the seasonal year varies with time, and the lengths of the seasonal years also differ amongst themselves. Table 1 demonstrates the differences between each of the seasonal years and the tropical year from the date. [14] Thus, for example, in the year 0 of our era, the Spring year had 14.9 seconds less than the tropical year of that same year 0; while the seasonal year of Autumn lasted for 16.0 seconds more than the tropical year. As shown in table 1, the tropical year sometimes has a longer duration than seasonal years and in other occasions the opposite occurs.

Years of our era	Spring year	Summer year	Autum year	Winter year	Sum of differences
-1000	-29.8	-44.0	30.2	43.6	0.0
0	-14.9	-50.5	16.0	49.5	-0.02
1000	0.9	-52.1	0.4	50.8	-0.01
2000	15.9	-48.7	-14.9	47.6	0.01
3000	28.9	-40.6	-28.4	40.1	0.0

Table 1.- **Excesses seconds of the seasonal years with respect to the tropical year. In the last column are the sums of these differences.**

What stands out in table 1 is that the sum of the differences of each of the four seasonal years with the tropical year for a given year is approximately zero (last column of the table 1). This leads to say that with very good approximation the tropical year is the arithmetic mean of the duration of the four seasonal years.

This result is seen in a more general way in figure 1. Represented on the horizontal axis are the years of our era, and marked on the vertical axis is what each of the years exceeds to the tropical year from the year 2000. The descending straight line is the duration of the tropical year, which, as we have already seen, declines over time in a way very similar to the linear form. The other four curves represent the four seasonal years. It is observed that the durations of these years are oscillating with respect to the length of the tropical year. It is further demonstrated that the seasonal year of Summer is in opposition to the Winter year, with relation to the tropical year, and the year of Spring is in opposition to the Autum year.

In figure 1, we see that the periodical variation of each of the four seasonal years occurs around the tropical year, which happens to be very approximately the average of the four seasonal years. Note that maximum differences between each seasonal year and the tropical year decrease with time; the same applies to the period of variation of the seasonal years, for reasons we'll analyze later.

6. Difference between the tropical year and the seasonal years

To explain the distinct durations of the seasonal years and their difference with the tropical year we should consider drawing 1. The ellipse is the path followed by the apparent movement of the Sun. The line of the apside (which is the horizontal line) is rotating with respect to seasonal points, so that currently the perigee is approaching the Spring equinox.

Instead of representing the perigee motion with regard to the equinox, we will do it the other way around: we'll assume the apside line is fixed, being the seasonal points that move with respect to perigee. In such a way, if 1 it represents the position of the vernal equinox with respect to perigee, 1' will be the position of this same equinox after one year. The same is true of the other

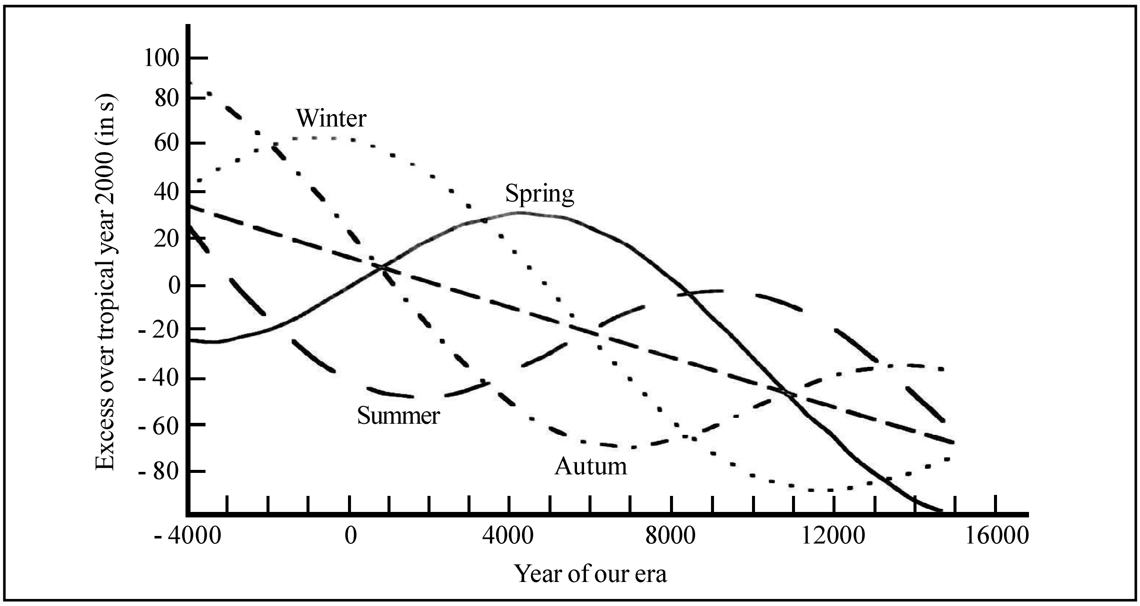


Figure 1.- **Excesses seconds of the seasonal years duration with respect to the tropical year from the year 2000. The straight line represents the tropical year from the date.**

seasonal points, represented by paragraphs 2, 3 and 4. * As shown in drawing 1, the angle rotated by each one of the four seasonal points in one year is the same. ** However, the time that it takes the Sun to travel through each of these angles is not the same, since they are at different points of the elliptical orbit and must meet the second law of Kepler, which requires the constancy of the areolar speed of the apparent motion of the Sun. Therefore, its movement will be slower in the more distant points to Earth than at the nearer points.

Looking at this issue in more detail; if we calculate the angle that describes the length of the Sun when, starting with the vernal equinox, one tropical year passes

$$\Delta L = L\left(T_p + a_T\right) - L\left(T_p\right) \approx \left(\frac{dL}{dT}\right)_p a_T$$

being T_p the moment when the Sun is in average in the Spring equinox. Using the formula (2) and taking up to the second order in series expansion we find

$$\Delta L \approx -2e\alpha\cos M_p - \frac{5}{2}e^2\alpha\cos 2M_p + 2\beta\sin M_p + \frac{5}{2}e\beta\sin 2M_p$$

The increase in the mean length of the Sun is null since at the end of a tropical year it takes the same value (by definition of the tropical year). M_p is the mean anomaly of the Spring equinox and the α y β parameters are

$$\alpha = \frac{d\omega_m}{dT}a_T\,; \qquad \beta = \frac{de}{dT}a_T$$

a_T expressed in Julian centuries, α in degrees, and β dimensionless. In the previous formulas the subscript p means that the values are taken at the time of the vernal equinox.

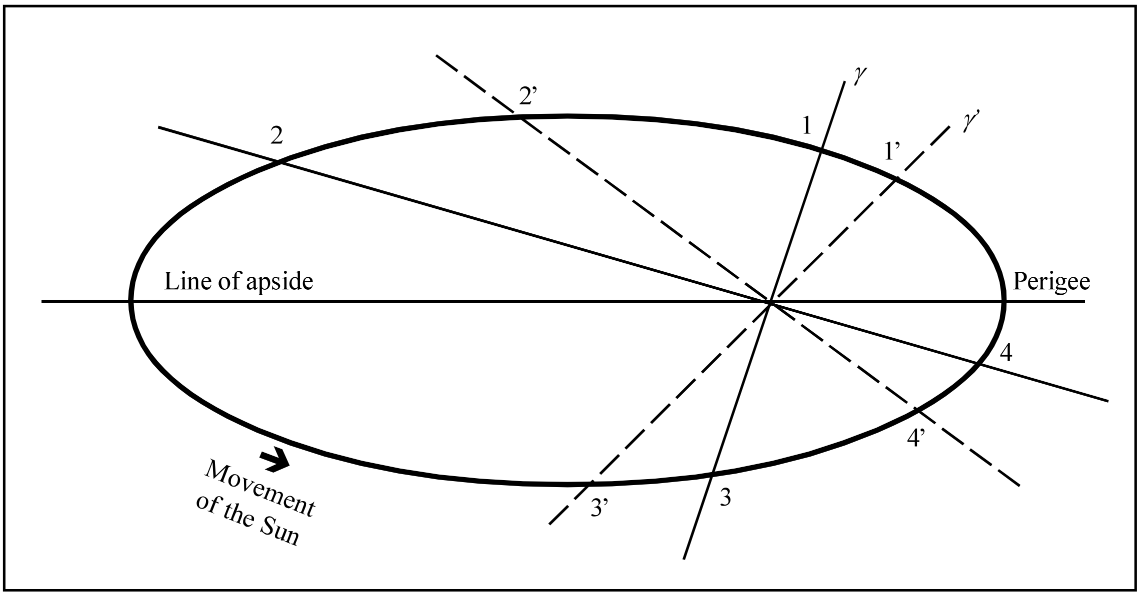
Given that the angle α is small, we do not commit an appreciable error if we assume that the Sun describes it with an angular velocity n that we assume constant and approximate value

$$\frac{1}{n} = \frac{365.25\cdot 86\,400}{360} = 87\,660$$

in seconds of time per degree. Therefore, the time that the Sun would take in travelling the angle L is

* No matter what type of year is used in the calculation. The differences between the different types of averages years considered is at most a few tens of seconds, and during this small time interval is negligible movement of the equinox relative to perigee.

** The angles are measured from the position of the Earth.



Drawing 1.- **Mobility of seasonal points with respect to perigee.**

$$\Delta\tau \approx 87\,660\,\Delta L$$

expressed in seconds. This quantity of time is that which the tropical year exceeds the year of Spring. The duration of the year of Spring will be approximately

$$a_p \approx 365^d.242\,189\,67 - 0^d.000\,006\,1619T + \\ +1^d.014\,583\left(2e\alpha\cos M_p + \frac{5}{2}e^2\alpha\cos 2M_p - 2\beta\sin M_p - \frac{5}{2}e\beta\sin 2M_p\right) \tag{8}$$

and similar formulas for the other three seasonal years. The sum of the durations of the four seasonal years, since it is obtained by the formula (8), is very approximately four times the tropical year.* In (8) we observe that there are two reasons that the four seasonal years are different from each other and different from the tropical year. One of them, and most important, is the rotation of the apside line, and the other is the variation of the eccentricity. Note that if the orbit were circular, that is, with zero eccentricity, the seasonal years would be equal to the tropical year.

The circumstance that α (speed with which rotates the perigee) has a small variation is responsible for the fact that the arithmetic mean of the four seasonal years is roughly equal to the tropical year. Indeed, in (8), α is the variation of the perigee at the time T_p. In formulas equivalent to (8) that give the other three types of seasonal years, α will be the variation of the perigee at each of the seasonal points. If α had a quick variation, there would be a noticeable difference from its value in each seasonal point, so that, in adding the length of the seasonal four years, we would find expressions like

$$2e\alpha_p\cos M_p + 2e\alpha_v\cos M_v + 2e\alpha_o\cos M_o + 2e\alpha_i\cos M_i$$

which would not be zero, as it would be if we take the same value of α in the four seasonal points that we have represented by subscripts *p, v, o, i.*

The maximum difference that can exist between a seasonal year and the tropical year is approximately

$$1^d.014\,583\cdot 2e\alpha$$

as deduced from (8). We have only taken the first term of the correction. This situation would occur when the seasonal point is in the perigee of the orbit, that is, when the mean anomaly is null and therefore $\cos M = 1$. Applying the values of the current era, we find the difference between

* For this deduction we must indicate that the difference between mean anomalies of seasonal points consecutives is 90 degrees, so that the sum of trigonometric functions corresponding to each of the four stations shown in (8) will be zero.

the tropical year and a seasonal year to be around 50 seconds. * Continuing with the same reasoning, we can say that the maximum difference, at the present time, which could exist between two seasonal years would be double the previous value, i.e. about 100 seconds: this situation would arise when a seasonal point is at perigee and the other is at the apogee (thereby $\cos M = -1$). The formula (8) also provides us an explanation as to why the Spring year is in opposition to the Autumn year, with respect to the tropical year, as we have previously pointed out. In effect, if M_p is the mean anomaly of the Spring equinox, it follows that the anomaly of the autumnal equinox will be $M_o = M_p + 180$ and therefore

$$\cos M_o = \cos\left(M_p + 180\right) = -\cos M_p$$

which means that while the Spring year exceeds (or instead is inferior to) the tropical year, the Autumn year will be likewise the opposite. Note that in accordance with this reasoning we have, of course, assumed that eccentricity squared is minimal, as is its variation, leaving us therefore with only the first of the periodical terms of (8).

7. Difference between the Terrestrial Time and Universal Time scales

The scale used to measure the astronomical years previously discussed is Terrestrial Time. However, the basic unit of time used in calendars is the mean solar day, composed of 24 hours from Universal Time. In fact, the civil time scale is the UTC (Universal Time Coordinated), but as this scale deviates only 0.9 seconds maximum from Universal Time, we can consider without noticeable error that the calendar uses UT as the basic unit.

We call lengthening of the day A_D the time which the duration of the mean solar day exceeds the 24 hours Terrestrial Time. The mean solar day always has 24 hours of Universal Time but has a duration that varies when measured in terrestrial time.

An mean solar day has 24 hours of Universal Time

$$UT_2 - UT_1 = 24,$$

where 1 is the initial moment of the beginning of the mean solar day and 2 is its completion. During this period, terrestrial time increases in 24 hours plus a certain amount

$$TT_2 - TT_1 = 24 + A_D$$

where TT means the Terrestrial Time scale. Combining the last two expressions, we find

$$A_D = \left(TT_2 - UT_2\right) - \left(TT_1 - UT_1\right) = \Delta T_2 - \Delta T_1$$

where we have defined ΔT with

$$\Delta T = TT - UT$$

which we express in seconds. Since the moments 1 and 2 differ in one day, the result is very approximately

$$A_D = \Delta T\left(T + \frac{1}{36\,525}\right) - \Delta T\left(T\right) = \frac{1}{36\,525}\frac{d\Delta T}{dT} \qquad (9)$$

a quantity which is a function of time T (in Julian centuries measured from a certain time), and which is expressed in seconds per day. Whether the Julian centuries are measured in Terrestrial Time or Universal Time does not matter.

The lengthening of the day A_D can be expressed in days per days

$$A_D = \frac{1}{86\,400}\frac{1}{36\,525}\frac{d\Delta T}{dT}$$

We shall refer to the complete analysis made by Stephenson who investigated more than

* This amount would be that would be obtained if an equinox (or a solstice) were found in the perigee or apogee. This is not what happens at present, hence the differences between seasonal and the tropical year current dates do not reach the 50 seconds. Note also that the parameters *e*, *a* y *n* vary over time. So for the year - 4000 the maximum value of this difference was 59 seconds and to the year 4000 will drop to about 48 seconds.

three hundred ancient observations. [15] From their findings he concluded that there is a constant secular braking of the rotation of the Earth, at least in a period understood between the year - 500 and today, which is expressed by the equation

$$\Delta T = 31t^2 - 20 \tag{10}$$

given in seconds. t are the elapsed centuries since the year 1820, which correspond to the era of Newcomb's observations, or, in other words, to the moment in which the Ephemeris Time scale progressed at the same rate as the Universal Time scale. The 20 seconds are placed to make the results of ancient records match with the modern observations. [16]

Equation (10) can be expressed as

$$\Delta T = 80.44 + 111.6T + 31T^2$$

in seconds. T are Julian centuries counted from J2000.0. From (9) and (11) the lengthening of the day is obtained as

$$A_D = u + wT = 3.5364 \cdot 10^{-8} + 1.9647 \cdot 10^{-8} T \tag{11}$$

in days per days.

8. The duration of the years in the Universal Time scale

As we have said, astronomical years are expressed in the uniform timescale, while the civil year, i.e. that used in calendars, has Universal Time as its unit. Said unit is not uniform, but does have the advantage of conforming to the apparent movement of the Sun, which ultimately is that which governs human activity. For this reason it is important to express the duration of astronomical years in Universal Time.

Let's assume that the difference between Terrestrial Time and Universal Time fulfills the relationship of Stephenson (10). If we use d to represent the 24 hour day of UT and D to represent an interval of 24 hours of TT, we have for the lengthening of the day

$$d = D + A_D$$

The length of the tropical year is given in unit D by

$$a_T = a + bT + cT^2$$

since the unit D is related to d through

$$D = d - u - wT$$

where we have used (11). Hence, the tropical year in unit d, or the same in mean solar days, will be

$$a_T = (a - au) + (b - bu - aw)T + (c - cu - bw)T^2$$

where we have ignored the term of third order due to its extreme smallness. For the tropical year that results from the VSOP theory, we find its duration in Universal Time to be

$$a_T = 365^d.242\,176\,754 - 0^d.000\,013\,338T - 6^d.4502 \cdot 10^{-10} T^2$$

which means a decrease of 1.15 seconds each century instead of the 0.53 second reduction experienced by the tropical year when expressed in TT. When using this expression, we must be remember that it is approximate, as it is the formula of Stephenson. Making a similar calculation for the expression that gives us the year of Spring, the following is obtained

$$a_p = 365^d.242\,3611 + 3^d.19 \cdot 10^{-6} T - 1^d.344 \cdot 10^{-7} T^2$$

valid for the period understood between the years - 3000 and 10000. [17]

9. Conclusions

We have defined the seasonal years as the average time between two consecutive passages of the Sun by each of the seasonal points, distinguishing between the true seasonal year and the average year. We prove that the seasonal years are different from the tropical year, defined from the mean length of the Sun.

We conclude that the tropical year is the arithmetic mean of the average of the four seasonal years. Moreover, we calculate the variation of the mean seasonal years, verifying that they oscillate respect of the tropical year.

We explain this behavior as being due to 1) the smallness of the variation of the speed with which the Earth perihelion rotates and 2) the extreme smallness of the variation of the eccentricity of the orbit.

Lastly, we express the astronomical years in function, not according to the uniform time scale, but rather in Universal Time.

10. References

[1] DOGGETT L. E.: «Calendars» en *Explanatory supplement to the astronomical almanac*, University Science Books, 1992, p. 576. Among the limited literature that clearly states the difference between the two years noted: MEEUS J.; SAVOIE D.: «The history of the tropical year», *Journal of the British Astronomical Association* **102** (1992) 40-42; HEYDARI-MALAERI M.: «A concise review of the iranian calendar», arXiv:astro-ph/0409620 v2, 2004 y MEEUS J.: *More Mathematical Astronomy Morsels*, Willmann-Bell, 2002, pp. 357-366.

[2] BRETAGNON P.: «Théorie du mouvement de emsemble des planètes. Solution VSOP82», *Astronomy and Astrophysics* **114** (1982) 278-288. The VSOP (acronym for Secular Variations of the Planetary Orbits) theory has been redesigned in version VSOP87 that evaluates solar and planetary ephemeris by rectangular and spherical coordinates, BRETAGNON P., FRANCON G.: «Planetary theories in rectangular and spherical variables.VSOP87 solutions», *Astronomy and Astrophysics* **202** (1988) 309-315. A reduced version of the VSOP theory is in BRETAGNON P., SIMON J.L.: *Planetary programs tables from -4000 to +2800*, Willmann-Bell, 1986. A more detailed version of the above quote is in MEEUS J.: *Astronomical Algorithms*, Willmann-Bell, 1991, pp. 197-204 y pp. 381-422.

[3] GOLDSTEIN H.: *Mecánica Clásica*, Reverté, 1988, pp. 122-128.

[4] SMART W. M.: *Textbook on spherical astronomy*, Cambridge University Press, 1990, pp. 118-120.

[5] To improve lunisolar calendars ancient Greek astronomers measured the length of the year, getting different values, all very acceptable, SAMUEL A. E.: *Greek and roman chronology. Calendars and years in classical antiquity*, Verlag C. H. Beck, 1972, pp. 35-55.

[6] Another method for determining the tropical year (within the meaning given in the text) was by precession. It was based on the sidereal year, which was well known as you could easily measure and then subtracted as long as the sun took to walk the angle described by the precession in a year, the result was the tropical, PROVERBIO E.: «Copernico and the determination of the length of the tropical year», en Gregorian refo*rm of the calendar. Proceedings of the Vatican conference to commemorate its 400th anniversary 1582-1982*, edited by Coyne, Hoskin y Pederson, Specola Vaticana, 1983, pp. 129-133.

[7] VIVES T. J.: *Astronomía de posición*, Alhambra, 1971, pp. 303-310.

[8] BORKOWSKI K. M.: «The tropical year and solar calendar», *Journal of the Royal Society of Canada* **85** (1991) 121-130.

[9] LASKAR J.: «Secular terms of classical planetary theories using the results of general theory», *Astronomy and Astrophysics* **157** (1986) 59-70. The results of this theory are valid for the years ranging from -5000 to 5000.

[10] This means that to estimate errors of a calendar as a result of using a different tropical year, we can not integrate the tropical year as a function of time, as more than one author has done wrong.

[11] LIESKE J. H.; LEDERLE T.; FRICKE W.; MORANDO B.: «Expressions for the Precession Quantities Based upon the IAU (1976) System of Astronomical Constants», *Astronomy and Astrophysics* **58** (1977) 1-16.

[12] To calculate this number we used the value based on the simplified procedure of the VSOP theory (J. Meeus, Astronomical Algorithms, ob. cit., pp. 165-170), which gives an error of only a few seconds. A list of dates starting from seasons from year 0 to 3000 using the VSOP87 theory in MEEUS J.: *Astronomical tables of the Sun, Moon and the planets*, Willmann-Bell, 1995, pp.. 101-175.

[13] This definition of the seasonal year has been suggested to us by KORT S. J.: «Astronomical appreciation of the gregorian calendar», *Ricerche astronomiche* **6** (1949) 109-116.

[14] Table 1 results coincide with durations that are obtained using the formulas that appear in J. Meeus, *Astronomical Algorithms*, ob. cit., p. 166, the differences between both results do not exceed, in the period considered, the 7 tenths of a second. Meeus's formulas represent averages of the dates for the start of the four seasons according to the VSOP theory, which essentially matches the definition we have given seasonal year.

[15] This result has been partially revised by Stephenson in collaboration with Morrison doing the analysis of 449 records obtained in the period - 700 to 1600. Finding that the coefficient of has a value of 33 seconds per century square, which means a variation of the lengthening of the day of 1.8 milliseconds per century square, MORRISON L. V., STEPHENSON F. R.: "Ancient eclipse and the earth's rotation", *Highlights of Astronomy* **12** (2002) 338-341.

[16] Have devised various empirical formulas of to modern observations, SCHMADEL L. D., D. ZECH: «Empirical transformations from U.T. to E.T. for the period 1800-1980 ", *Astronomische Nachrichten* **309** (1998) 219-221.
[17] An analysis of the relationship between astronomical and calendar appears in SEGURA GONZÁLEZ, Wenceslao: *Hemerología. La ciencia de los calendarios*, Acento 2000, 2006.