On Global Solution of Incompressible Navier-Stokes equations

Algirdas Antano Maknickas
Institute of Mechanics, Vilnius Gediminas Technical University,
Sauletekio al. 11, Vilnius, Lithuania
alm@vgtu.lt

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Abstract

Using equality of full and covariant time derivatives and partial solutions of Helmholtz equation analytical solution of 1D, 2D and 3D Navier-Stokes equations was obtained

1 Introduction

In physics, the NavierStokes equations, named after Claude-Louis Navier and George Gabriel Stokes, describe the motion of fluid substances. The Clay Mathematics Institute has called this one of the seven most important open problems in mathematics and has offered a US$1,000,000 prize for a solution or a counter-example[1].

2 Parabolic equation formulation

Incompressible Navier-Stokes equations are expressed as follow

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \quad (\mathbf{v}\cdot\nabla)\mathbf{v}\right) - \mu\Delta\mathbf{v} + \nabla p = f \tag{1}$$

$$\rho = const \tag{2}$$

First of Navier-Stokes equations could be expressed in full time derivative replacing covariant time derivative by

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \quad (\mathbf{v}\cdot\nabla) \tag{3}$$

So, we obtain

$$\frac{d\mathbf{v}}{dt} - a^2\Delta\mathbf{v} = \frac{1}{\rho}(-\nabla p + f) \tag{4}$$

3 inhomogeneous parabolic like equation for full time derivative, where $a = \sqrt{\mu/\rho}$.

3 One dimensional inhomogeneous solution

Consider the initial-boundary value problem for $v = v(x,t)$

$$\frac{dv}{dt} - a^2\Delta v = \frac{1}{\rho}(-\nabla p + f) \text{ in } \Omega\times(0,\infty) \tag{5}$$

$$v(x,0) = v_0(x)\ x\in\Omega \tag{6}$$

$$\frac{\partial v}{\partial n} = 0 \text{ on } \partial\Omega\times(0,\infty) \tag{7}$$

where $p = p(x,t)$ and $f = f(x,t)$, $\Omega\subset\mathbb{R}^n$, n the exterior unit normal at the smooth parts of $\partial\Omega$, a^2 a positive constant and $v_0(x)$ a given function.

So according to [2] equation (4), when x is normed to $a = 1$, could be rewritten as follow

$$\frac{dv}{dt} = v_{xx} + Q(x,t),\ x\in\Omega,\ t>0 \tag{8}$$

Now, we expand v and Q in the eigenfunctions $\sin(n\pi x)$

$$Q(x,t) = \sum_{n=1}^{\infty} q_n(t) \sin(n\pi x) \tag{9}$$

with

$$q_n(t) = 2\int_{\Omega} Q(x,t)\sin(n\pi x)dx \tag{10}$$

and

$$v(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin(n\pi x) \tag{11}$$

Thus we get the inhomogeneous ODE

$$\dot{u}_n(t) + (n\pi)^2 u_n(t) = q_n(t), \tag{12}$$

whose solution is

$$u_n(t) = u_n(0)e^{-(n\pi)^2 t} + \int_0^t q_n(\tau)e^{-(n\pi)^2(t-\tau)}d\tau \tag{13}$$

where

$$u_n(0) = 2\int_{\Omega} v_0(x)\sin(n\pi x)dx \tag{14}$$

Again, we substitute all obtained equations into (11) and have

$$v(x,t) = \int_{\Omega} v_0(s)(\sum_{n=1}^{\infty} 2\sin(n\pi s)\sin(n\pi x)e^{-(n\pi)^2 t})ds$$
$$+\int_{\Omega} ds \int_0^t Q(s,\tau)(\sum_{n=1}^{\infty} 2\sin(n\pi s)\sin(n\pi x)e^{-(n\pi)^2(t-\tau)}d\tau) \tag{15}$$

4 Two dimensional inhomogeneous solution

Consider the initial-boundary value problem for $v = v(x,y,t)$

$$\begin{aligned} \frac{dv^i}{dt} - a^2\Delta v^i &= \frac{1}{\rho}(-\nabla_i p + f_i) \text{ in } \Omega \times (0,\infty) && (16)\\ v^i(x,y,0) &= v_0^i(x,y)\ x,y \in \Omega && (17)\\ \frac{\partial v^i}{\partial n} &= 0 \text{ on } \partial\Omega \times (0,\infty) && (18) \end{aligned}$$

where $p = p(x,y,t)$ and $f = f(x,y,t)$, $\Omega \subset \mathbb{R}^{2n}$, n the exterior unit normal at the smooth parts of $\partial\Omega$, a^2 a positive constant and $v_0^x(x,y), v_0^y(x,y)$ a given function.

So, when x and y are normed to $a = 1$, equation (4) could be rewritten as follow

$$\frac{dv^i}{dt} = v^i_{xx} + v^i_{yy} + Q^i(x,y,t),\ x,y \in \Omega,\ t > 0 \tag{19}$$

Now, we transform to the pole coordinates and expand v and Q in the eigenfunctions $\sin(n\theta)J_n(k_{m,n}r)$ and $\cos(n\theta)J_n(k_{m,n}r)$, where $J_n(k_{m,n}r)$ is Bessel function

$$Q^i(r,\theta,t) = \sum_{m,n=1}^{\infty} (q^i_{1mn}(t)\sin(n\pi x)J_n(k_{m,n}r) + q^i_{2mn}(t)\cos(n\pi x)J_{-n}(k_{m,n}r)) \tag{20}$$

with

$$\begin{aligned} q^i_{1mn}(t) &= \frac{1}{I_{2mn}}\iint_{\Omega} Q^i(r,\theta,t)\sin(n\pi\theta)J_n(k_{m,n}r)rdrd\theta && (21)\\ q^i_{2mn}(t) &= \frac{1}{I_{2m,-n}}\iint_{\Omega} Q^i(r,\theta,t)\cos(n\pi\theta)J_{-n}(k_{m,n}r)rdrd\theta && (22)\\ I_{2mn} &= \iint_{\Omega}(\sin(n\pi\theta)J_n(k_{m,n}r))^2 rdrd\theta && (23) \end{aligned}$$

and

$$v^i(r,\theta,t) = \sum_{m,n=1}^{\infty} (u^i_{1mn}(t)\sin(n\pi x) J_n(k_{m,n}r) + u^i_{2mn}(t)\cos(n\pi x) J_{-n}(k_{m,n}r)) \tag{24}$$

Thus we get the inhomogeneous ODE

$$\dot{u}^i_{jmn}(t) + k^2_{m,n} u^i_{jmn}(t) = q^i_{jmn}(t), \tag{25}$$

whose solution is

$$u^i_{jmn}(t) = u^i_{jmn}(0) e^{-k^2_{m,n}t} + \int_0^t q^i_{jmn}(\tau) e^{-k^2_{m,n}(t-\tau)} d\tau \tag{26}$$

where

$$u^i_{1mn}(0) = \frac{1}{I_{2mn}} \iint_\Omega v^i_0(r,\theta)\sin(n\pi\theta) J_n(k_{m,n}r))rdrd\theta \tag{27}$$

$$u^i_{2mn}(0) = \frac{1}{I_{2m,-n}} \iint_\Omega v^i_0(r,\theta)\cos(n\pi\theta) J_{-n}(k_{m,n}r))rdrd\theta \tag{28}$$

Again, we substitute all obtained equations into (40) and have

$$\begin{aligned} v^i(r,\theta,t) &= \iint_\Omega v^i_0(s',s)(\sum_{m,n=1}^{\infty} \frac{1}{I_{2mn}} \sin(n\pi s) J_n(k_{m,n}s')) \sin(n\pi\theta) J_n(k_{m,n}r)) e^{-k^2_{m,n}t} s'ds'ds \\ &+ \iint_\Omega s'ds'ds \int_0^t Q^i(s',s,\tau)(\sum_{m,n=1}^{\infty} \frac{1}{I_{2mn}} \sin(n\pi s) J_n(k_{m,n}s')) \sin(n\pi\theta) J_n(k_{m,n}r)) e^{-k^2_{m,n}(t-\tau)} d\tau \\ &+ \iint_\Omega v^i_0(s',s)(\sum_{m,n=1}^{\infty} \frac{1}{I_{2m,-n}} \cos(n\pi s) J_{-n}(k_{m,n}s')) \cos(n\pi\theta) J_{-n}(k_{m,n}r)) e^{-k^2_{m,n}t} s'ds'ds \\ &+ \iint_\Omega s'ds'ds \int_0^t Q^i(s',s,\tau)(\sum_{m,n=1}^{\infty} \frac{1}{I_{2m,-n}} \cos(n\pi s) J_{-n}(k_{m,n}s')) \cos(n\pi\theta) J_{-n}(k_{m,n}r)) e^{-k^2_{m,n}(t-\tau)} d\tau \end{aligned} \tag{29}$$

Finally

$$[v^x, v^y]^T = M^2 [v^r, v^\theta]^T \tag{30}$$

where M^2 is transform matrix to x, y coordinates.

5 Three dimensional inhomogeneous solution

Consider the initial-boundary value problem for $v = v(x,y,z,t)$

$$\frac{dv^i}{dt} - a^2 \Delta v^i = \frac{1}{\rho}(-\nabla_i p + f_i) \text{ in } \Omega \times (0,\infty) \tag{31}$$

$$v^i(x,y,z,0) = v^i_0(x,y,z) \; x,y,z \in \Omega \tag{32}$$

$$\frac{\partial v^i}{\partial n} = 0 \text{ on } \partial\Omega \times (0,\infty) \tag{33}$$

where $p = p(x,y,z,t)$ and $f = f(x,y,z,t)$, $\Omega \subset \mathbb{R}^{3n}$, n the exterior unit normal at the smooth parts of $\partial\Omega$, a^2 a positive constant and $v^x_0(x,y,z), v^y_0(x,y,z), v^z_0(x,y,z)$ a given function.

So, when x, y and z are normed to $a = 1$, equation (4) could be rewritten as follow

$$\frac{dv^i}{dt} = v^i_{xx} + v^i_{yy} + v^i_{zz} + Q^i(x,y,z,t), \; x,y,z \in \Omega, \; t > 0 \tag{34}$$

Now, we transform to the sphere coordinates and expand v and Q in the eigenfunctions $Y^m_\ell(\theta,\varphi) j_\ell(kr)$ and $Y^m_\ell(\theta,\varphi) y_\ell(kr)$, where $j_\ell(kr)$ and $y_\ell(kr)$ are the spherical Bessel functions, and $Y^m_\ell(\theta,\varphi)$ are the spherical harmonics [3]

$$Q^i(r,\theta,t) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (q^i_{1\ell m}(t) j_\ell(k_{\ell m}r) + q^i_{2\ell m}(t) y_\ell(k_{\ell m}r)) Y^m_\ell(\theta,\varphi), \tag{35}$$

with

$$
\begin{aligned}
q^i_{1\ell m}(t) &= \frac{1}{I_{3\ell m}} \iiint_\Omega Q^i(r,\theta,t) Y_\ell^m(\theta,\varphi) j_\ell(k_{\ell m} r) r^2 \sin(\theta) dr d\theta d\varphi & (36)\\
q^i_{2\ell m}(t) &= \frac{1}{I'_{3\ell m}} \iiint_\Omega Q^i(r,\theta,t) Y_\ell^m(\theta,\varphi) y_\ell(k_{\ell m} r) r^2 \sin(\theta) dr d\theta d\varphi & (37)\\
I_{3\ell m} &= \iiint_\Omega (Y_\ell^m(\theta,\varphi) j_\ell(k_{\ell m} r))^2 r^2 \sin(\theta) dr d\theta d\varphi & (38)\\
I'_{3\ell m} &= \iiint_\Omega (Y_\ell^m(\theta,\varphi) y_\ell(k_{\ell m} r))^2 r^2 \sin(\theta) dr d\theta d\varphi & (39)
\end{aligned}
$$

and

$$
v^i(r,\theta,t) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (u^i_{1\ell m}(t) j_\ell(k_{\ell m} r) + u^i_{2\ell m}(t) y_\ell(k_{\ell m} r)) Y_\ell^m(\theta,\varphi) \tag{40}
$$

Thus we get the inhomogeneous ODE

$$
\dot{u}^i_{j\ell m}(t) + k^2_{\ell m} u^i_{j\ell m}(t) = q^i_{j\ell m}(t), \tag{41}
$$

whose solution is

$$
u^i_{j\ell m}(t) = u^i_{jmn}(0) e^{-k^2_{\ell m} t} + \int_0^t q^i_{jmn}(\tau) e^{-k^2_{\ell m}(t-\tau)} d\tau \tag{42}
$$

where

$$
\begin{aligned}
u^i_{1\ell m}(0) &= \frac{1}{I_{3\ell m}} \iiint_\Omega v^i_0(r,\theta,\varphi) Y_\ell^m(\theta,\varphi) j_\ell(k_{\ell m} r) r^2 \sin(\theta) dr d\theta d\varphi & (43)\\
u^i_{2\ell m}(0) &= \frac{1}{I'_{3\ell m}} \iiint_\Omega v^i_0(r,\theta,\varphi) Y_\ell^m(\theta,\varphi) y_\ell(k_{\ell m} r) r^2 \sin(\theta) dr d\theta d\varphi & (44)
\end{aligned}
$$

Again, we substitute all obtained equations into (40) and have

$$
\begin{aligned}
v^i(r,\theta,t) &= \iiint_\Omega v^i_0(s'',s',s) (\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{I_{3\ell m}} Y_\ell^m(\theta,\varphi) j_\ell(k_{\ell m} r) Y_\ell^m(s',s) j_\ell(k_{\ell m} s'')) e^{-k^2_{\ell m} t} d\Omega \\
&+ \iiint_\Omega d\Omega \int_0^t Q^i(s'',s',s,\tau) (\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{I_{3\ell m}} Y_\ell^m(\theta,\varphi) j_\ell(k_{\ell m} r) Y_\ell^m(s',s) j_\ell(k_{\ell m} s'')) e^{-k^2_{\ell m}(t-\tau)} d\tau \\
&+ \iiint_\Omega v^i_0(s'',s',s) (\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{I'_{3\ell m}} Y_\ell^m(\theta,\varphi) y_\ell(k_{\ell m} r) Y_\ell^m(s',s) j_\ell(k_{\ell m} s'')) e^{-k^2_{\ell m} t} d\Omega \\
&+ \iiint_\Omega d\Omega \int_0^t Q^i(s'',s',s,\tau) (\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{I'_{3\ell m}} Y_\ell^m(\theta,\varphi) j_\ell(k_{\ell m} r) Y_\ell^m(s',s) y_\ell(k_{\ell m} s'')) e^{-k^2_{\ell m}(t-\tau)} d\tau
\end{aligned} \tag{45}
$$

where $d\Omega = s''^2 \sin(s') ds'' ds' ds$. Finally

$$
[v^x, v^y, v^z]^T = M^3 [v^r, v^\theta, v^\varphi]^T \tag{46}
$$

where M^3 is transform matrix to x, y coordinates.

6 Conclusions

Using equality of full and covariant time derivatives and partial solutions of Helmholtz equation analytical solution of 1D, 2D and 3D Navier-Stokes equations was obtained

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