A NEW CALCULUS ON THE RING OF SYMMETRIC FUNCTIONS AND ITS APPLICATIONS

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Abstract

This paper develops a new calculus on the ring of symmetric functions $\Lambda_{\mathbb{Q}}$ and introduces its application. In the last of this paper, the author describes a new general method to expand any symmetric function in terms of a basis in $\Lambda_{\mathbb{Q}}$. For application of it, the author also mentions a general way to evaluate the transition matrix between any two bases in $\Lambda_{\mathbb{Q}}$.

Keywords: the ring of symmetric functions; combinatorics; representation theory; transition matrices

1 Notations and Definitions

Throughout this paper, $\mathbb{N} = \{0, 1, 2, 3, \cdots\}$, $\lambda \vdash n$ denotes that λ is a partition of n. ϕ denotes a partition of 0. $\Lambda_{\mathbb{Q}}$ is the ring of symmetric functions in infinitely many variables $x_1, x_2, x_3, ...$ with $\mathbb{Q}$ coefficients.

Basically, we use the notation and the definition of [2]. $p_n, h_n, e_n, m_\lambda, s_{\lambda/\mu}$ denote the power sum symmetric function, the complete homogeneous symmetric function, the elementary symmetric function, the monomial symmetric function and the skew schur symmetric function associated with the integer n or the partition λ, μ respectively. For any $f_n \in \Lambda_{\mathbb{Q}}$, $f_\lambda = f_{\lambda_1} f_{\lambda_2} f_{\lambda_3} \cdots$ is automatically defined.

Definition 1. $\bigcirc_n$ is an linear transformation on $\Lambda_{\mathbb{Q}}$, that has the following rules. $A, B \in$

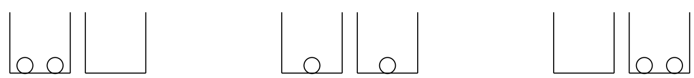
$\Lambda_{\mathbb{Q}}; \alpha, \beta \in \mathbb{Q}; n, m \in \mathbb{N}$.

$$\bigcirc_0 A = A$$
$$\bigcirc_n (\bigcirc_m A) = \bigcirc_m (\bigcirc_n A)$$
$$\bigcirc_n (\alpha A + \beta B) = \alpha (\bigcirc_n A) + \beta (\bigcirc_n B)$$
$$\bigcirc_n (AB) = \sum_{m=0}^{n} (\bigcirc_m A)(\bigcirc_{n-m} B)$$

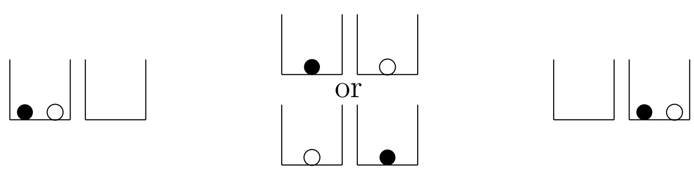
$\bigcirc_n^m$ denotes m times action of $\bigcirc_n$. Let us call $\bigcirc_n$ 'ball putting operator' or 'n indistinguishable balls', $\bigcirc_1^n$ 'n distinguishable balls', and $A \in \Lambda_{\mathbb{Q}}$ 'box' or 'boxes'. The following figures depict the reason for these names.

$$\bigcirc_1 (AB) \quad = \quad (\bigcirc_1 A) B \quad + \quad A (\bigcirc_1 B)$$

$$\bigcirc_2 (AB) \quad = \quad (\bigcirc_2 A) B \quad + \quad (\bigcirc_1 A)(\bigcirc_1 B) \quad + \quad A (\bigcirc_2 B)$$



$$\bigcirc_1^2 (AB) \quad = \quad \left(\bigcirc_1^2 A\right) B \quad + \quad 2 (\bigcirc_1 A)(\bigcirc_1 B) \quad + \quad A \left(\bigcirc_1^2 B\right)$$



Definition 2. $\bigcirc_n$ acts on the power sum symmetric function p_m $(m \geqslant 1)$ as following rules.

$$\bigcirc_n p_m = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases} \quad \text{, if } n \neq 0$$

Theorem 3. *One can easily see that*

$$\bigcirc_n p_m^l = \begin{cases} \binom{l}{n/m} p_m^{l-n/m} & m \mid n \\ 0 & m \nmid n \end{cases} \quad \textit{, if } n \neq 0$$

Example 4.

$$\bigcirc_n p_1^m = \binom{m}{n} p_1^{m-n}$$

Definition 5. $\bigcirc_n^{-1}$ denotes inverse operation of $\bigcirc_n$. For instance, $\bigcirc_n^{-1}\bigcirc_n A = A + B$, where B satisfies the condition $\bigcirc_n B = 0$. For example, $\bigcirc_1^{-1} p_1 = \frac{1}{2}p_1^2 + A$, where $\bigcirc_1 A = 0$. Let us call this 'ball picking operator'.

Because the power sum symmetric functions form a basis of $\Lambda_{\mathbb{Q}}$, any symmetric function can be expressed in terms of the power sum symmetric functions. Throughout the paper, we consider that symmetric function is a 'function' of the power sum symmetric function. For $A, q_n, r_\lambda \in \Lambda_{\mathbb{Q}}$, $A(q)$ denotes a symmetric function obtained by replacing p_n with q_n, and $A(r)$ a symmetric function obtained by replacing p_λ with r_λ.

Note that this notation is different from the conventional notation in the area of symmetric functions, such as $e(x)$. In this paper, we never care about the variables $x_1, x_2, x_3, \cdots$, but we always care about how a symmetric function is expressed in terms of the power sum symmetric function.

For convenience, we introduce other notations of substitution,

$$A|_{p\to q}$$
$$A|_{p\to r}$$

For instance, if $A = \frac{1}{2}p_2 + \frac{1}{2}p_1^2$, then

$$\begin{aligned} A(h) &= \frac{1}{2}h_2 + \frac{1}{2}h_1^2 = \frac{1}{4}p_2 + \frac{3}{4}p_1^2 \\ A(s) &= \frac{1}{2}s_{(2)} + \frac{1}{2}s_{(1,1)} = \frac{1}{2}p_1^2 \end{aligned}$$

2 Equation Of Symmetric Function

The complete homogeneous symmetric functions are expressed by the power sum symmetric functions as follows.

$$\begin{aligned} h_0 &= 1 \\ h_1 &= p_1 \\ h_2 &= \tfrac{1}{2}p_2 + \tfrac{1}{2}p_1^2 \\ h_3 &= \tfrac{1}{3}p_3 + \tfrac{1}{2}p_2p_1 + \tfrac{1}{6}p_1^3 \end{aligned}$$

One may notice that $\bigcirc_1 h_3 = h_2$ and $\bigcirc_2 h_3 = h_1$. Actually, h_n satisfies the following equation.

Lemma 6. *The complete homogeneous symmetric function has equations below*

$$\begin{aligned} h_0 &= 1 \\ \bigcirc_n h_m &= h_{m-n} && (1) \\ h_m(0) &= 0 \quad \textit{if } m \neq 0 && (2) \end{aligned}$$

If $n > m$, $\bigcirc_n h_m = 0$. (2) states that h_n is homogeneous. One can see it as a definition of h_n.

Proof. It suffices to prove that the known relation $nh_n = \sum_{m=1}^{n} p_m h_{n-m}$ satisfies (2). $\square$

Let us make sure that (2) yields p_n expression of h_n. First, let us evaluate h_1. When $n = 1,\ m = 1$, (2) is $\bigcirc_1 h_1 = h_0 = 1$. We apply $\bigcirc_1^{-1}$ to this, and get $h_1 = \bigcirc_1^{-1} 1 = p_1 + A$, where $\bigcirc_1 A = 0$. From the fact that h_1 is homogeneous, we can see that $A = 0$. Hence, we have $h_1 = p_1$.

Next, let us evaluate h_2. When $n = 1,\ m = 2$, (2) is $\bigcirc_1 h_2 = h_1 = p_1$. We apply $\bigcirc_1^{-1}$ to this, and get $h_2 = \bigcirc_1^{-1} p_1 = \frac{1}{2}p_1^2 + A$, where $\bigcirc_1 A = 0$. Furthermore, we apply $\bigcirc_2$ to (2), we get $\bigcirc_2 h_2 = \bigcirc_2(\frac{1}{2}p_1^2 + A) = \frac{1}{2} + \bigcirc_2 A = h_0 = 1$. $\bigcirc_2 A = \frac{1}{2}$. We apply $\bigcirc_2^{-1}$ to this, and get $A = \frac{1}{2}p_2 + B$, where $\bigcirc_1 B = \bigcirc_2 B = 0$. As the above, from the fact that h_2 is homogeneous, we can see that $B = 0$. Therefore, we finally obtain $h_2 = \frac{1}{2}p_2 + \frac{1}{2}p_1^2$.

Lemma 7. *The elementary symmetric function has equations below*

$$\begin{aligned} e_0 &= 1 \\ \bigcirc_n e_m &= \begin{cases} e_{m-n} & n = 0, 1 \\ 0 & \textit{otherwise} \end{cases} \\ e_n(0) &= 0 \quad \textit{if } n \neq 0 \end{aligned} \tag{3}$$

Proof. It suffices to prove that the known relation $ne_n = \sum_{m=1}^{n} (-1)^{m+1} p_m e_{n-m}$ satisfies (3). $\square$

Lemma 8. *The monomial symmetric function has equations below*

$$\begin{aligned} m_\phi &= 1 \\ \bigcirc_n m_\lambda &= \begin{cases} m_{\lambda - \{n\}} & n \in \lambda \\ 0 & n \notin \lambda \end{cases} \quad \textit{if } n \neq 0 \\ m_\lambda(0) &= 0 \quad \textit{if } \lambda \neq \phi \end{aligned} \tag{4}$$

Example 9.

$$\begin{aligned} \bigcirc_2\, m_{(3,2,2)} &= m_{(3,2)} \\ \bigcirc_1\, m_{(3,2,2)} &= 0 \end{aligned}$$

Lemma 10. *The skew schur symmetric function has equations below*

$$s_{\lambda/\mu} = \begin{cases} 1 & \lambda = \mu \\ 0 & \textit{otherwise} \end{cases} \quad \textit{if } |\lambda| = |\mu| \tag{5}$$

$$\bigcirc_n\, s_{\lambda/\mu} = \sum_{\nu} s_{\lambda/\nu} \tag{6}$$

$$s_{\lambda/\mu}(0) = 0 \quad \textit{if } \lambda \neq \mu$$

here ν runs over all $|\mu| + n$ brick young diagrams that are obtained from μ by adding at most one brick on each column.

Example 11.

$$\bigcirc_3\, s_{\lambda/(1)} = s_{\lambda/(4)} + s_{\lambda/(3,1)}$$

Lemma 12. *We introduce a new symmetric function as follows*

$$\begin{aligned} I_0 &= 1 \\ \bigcirc_n I_m &= \begin{cases} \sum_{l=1}^{m} I_{l-1} I_{m-l} & \text{if } n = 1 \\ \sum_{l=1}^{m-1} I_l \left(\bigcirc_{n-1} I_{m-l-1}\right) & \text{if } n > 1 \end{cases} \\ I_n(0) &= 0 \quad \text{if } n \neq 0 \end{aligned} \tag{7}$$

Let us call it 'path-graph colouring symmetric function'. Because if $t \in \mathbb{N}$, then $I_n(t)$ equals chromatic polynomial of path-graph of n vertexes. One can check that $h_n = I_n(e)$, $e_n = I_n(h)$.

Lemma 13. *The identity*

$$\begin{aligned} \bigcirc_n \zeta &= \zeta \\ \zeta(0) &= 1 \end{aligned} \tag{8}$$

It is easily checked that $\zeta = h_0 + h_1 + h_2 + \cdots$. This symmetric function will play an important role in the application to enumerative combinatorics.

Example 14. The twelvefold way can be expressed in terms of my calculus. If n balls and m boxes exist, the the twelvefold way is

ball	box	any f	surjective f	injective f
dist.	dist.	$\bigcirc_1^n \zeta^m\big\|_{p\to 0}$	$\bigcirc_1^n (\zeta-1)^m\big\|_{p\to 0}$	$\bigcirc_1^n p_1^m\big\|_{p\to 1}$
dist.	indist.	$\bigcirc_1^n h_m(\zeta)\big\|_{p\to 0}$	$\bigcirc_1^n h_m(\zeta-1)\big\|_{p\to 0}$	$\bigcirc_1^n h_m\big\|_{p\to 1}$
indist.	dist.	$\bigcirc_n \zeta^m\big\|_{p\to 0}$	$\bigcirc_n (\zeta-1)^m\big\|_{p\to 0}$	$\bigcirc_n p_1^m\big\|_{p\to 1}$
indist.	indist.	$\bigcirc_n h_m(\zeta)\big\|_{p\to 0}$	$\bigcirc_n h_m(\zeta-1)\big\|_{p\to 0}$	$\bigcirc_n h_m\big\|_{p\to 1}$

The information about the twelvefold way is available in [1].

3 Series Expansion Of Symmetric Function

Let $A \in \Lambda_{\mathbb{Q}}$, and B_λ be a basis of $\Lambda_{\mathbb{Q}}^{|\lambda|}$. A can be expanded in terms of B_λ.

$$\begin{aligned} A &= A|_{p\to q} + \left((\bigcirc_1 A)|_{p\to q}\right)\left(\bigcirc_1 B_{(1)}\right)^{-1}\left(B_{(1)}\big|_{p\to q}\right) \\ &+ \left((\bigcirc_2 A)|_{p\to q} \quad (\bigcirc_1^2 A)|_{p\to q}\right)\begin{pmatrix} \bigcirc_2 B_{(2)} & \bigcirc_1^2 B_{(2)} \\ \bigcirc_2 B_{(1,1)} & \bigcirc_1^2 B_{(1,1)} \end{pmatrix}^{-1}\begin{pmatrix} B_{(2)}\big|_{p\to q} \\ B_{(1,1)}\big|_{p\to q} \end{pmatrix} + \cdots \end{aligned}$$

Let us call this formula 'a Taylor series expansion of A by B around q'. As a special case, we obtain for $p \to 0$

$$\begin{aligned} A &= A|_{p\to 0} + \left((\bigcirc_1 A)|_{p\to 0} \right) \left(\bigcirc_1 B_{(1)} \right)^{-1} \left(B_{(1)} \right) \\ &+ \left((\bigcirc_2 A)|_{p\to 0} \quad (\bigcirc_1^2 A)|_{p\to 0} \right) \begin{pmatrix} \bigcirc_2 B_{(2)} & \bigcirc_1^2 B_{(2)} \\ \bigcirc_2 B_{(1,1)} & \bigcirc_1^2 B_{(1,1)} \end{pmatrix}^{-1} \begin{pmatrix} B_{(2)} \\ B_{(1,1)} \end{pmatrix} + \cdots \end{aligned}$$

Let us call this formula 'a Maclaurin series expansion of A by B'. These formulas are useful for evaluating transition matrix between bases of $\Lambda_{\mathbb{Q}}$.

Corollary 15. *One expands the complete homogeneous symmetric function h_n by the power sum symmetric function p around A and substitute $p \to B$, then one gets*

$$h_n(A+B) = \sum_{m=0}^{n} h_m(A) h_{n-m}(B)$$

Example 16. When n = 3, we have

$$h_3(A+B) = \sum_{m=0}^{3} h_m(A) h_{3-m}(B) =$$

$$\begin{aligned} \frac{1}{3}(A_3+B_3) &+ \frac{1}{2}(A_2+B_2)(A_1+B_1) + \frac{1}{6}(A_1+B_1)^3 \\ &= 1\left(\frac{1}{3}B_3 + \frac{1}{2}B_2B_1 + \frac{1}{6}B_1^3\right) + A_1\left(\frac{1}{2}B_2 + \frac{1}{2}B_1^2\right) + \\ \left(\frac{1}{2}A_2 + \frac{1}{2}A_1^2\right)B_1 &+ \left(\frac{1}{3}A_3 + \frac{1}{2}A_2A_1 + \frac{1}{6}A_1^3\right)1 \end{aligned}$$

Corollary 17. *If A_λ, B_λ are the bases of $\Lambda_{\mathbb{Q}}^n$, the transition matrix between those two bases is easily obtained from a Maclaurin series expansion of A by B.*

Example 18. the transition matrix from the monomial symmetric functions to the schur symmetric function of size 2 is

$$\begin{aligned} \begin{pmatrix} s_{(2)} \\ s_{(1,1)} \end{pmatrix} &= \begin{pmatrix} \bigcirc_2\, s_{(2)} & \bigcirc_1^2\, s_{(2)} \\ \bigcirc_2\, s_{(1,1)} & \bigcirc_1^2\, s_{(1,1)} \end{pmatrix} \begin{pmatrix} \bigcirc_2\, m_{(2)} & \bigcirc_1^2\, m_{(2)} \\ \bigcirc_2\, m_{(1,1)} & \bigcirc_1^2\, m_{(1,1)} \end{pmatrix} \begin{pmatrix} m_{(2)} \\ m_{(1,1)} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} m_{(2)} \\ m_{(1,1)} \end{pmatrix} \end{aligned}$$

The information about transition matrix is available in [2]

References

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