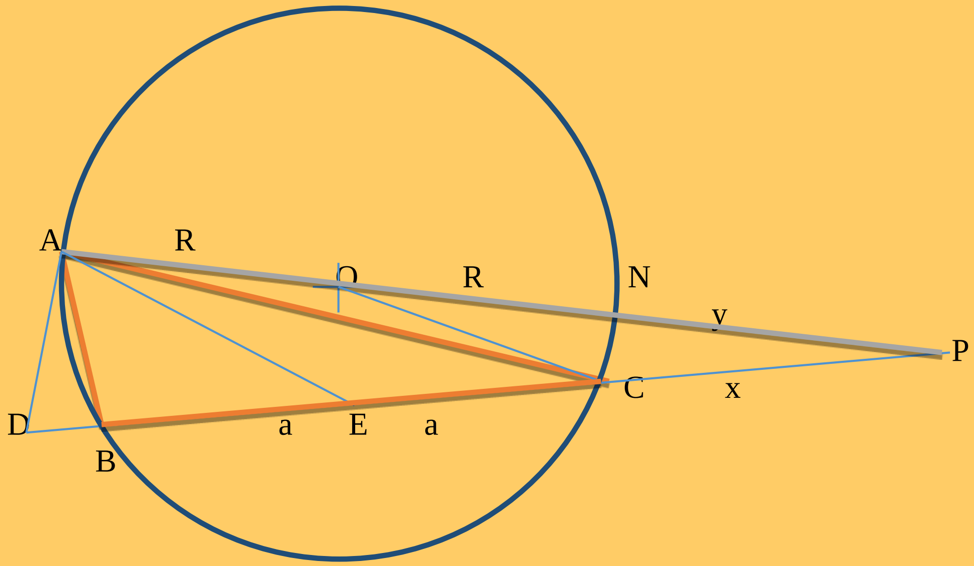
Ion Pătraşcu

Florentin Smarandache




# VARIANCE ON TOPICS OF PLANE GEOMETRY

Ion Pătraşcu     Florentin Smarandache

# VARIANCE ON TOPICS OF PLANE GEOMETRY

*Educational Publishing*

2013

Education Publishing
1313 Chesapeake Avenue
Columbus, Ohio 43212
USA
Tel. (614) 485-0721



**Peer Reviewers:**

Marius Coman, researcher, Bucharest, Romania.
Prof. Valeri Kroumov, Okayama University of Science, Japan.
Said Broumi, University of Hassan II Mohammedia, Casablanca, Morocco.
Dr. Ştefan Vlăduţescu, University of Craiova, Romania.

Many books can be downloaded from the following

**Digital Library of Science**:
http://fs.gallup.unm.edu/eBooks-otherformats.htm

***EAN: 9781599732473***
***ISBN: 978-1-59973-247-3***

*CONTENTS*

**VARIANCE ON TOPICS OF PLANE GEOMETRY**

## *Preface*

This book contains 21 papers of plane geometry.

It deals with various topics, such as: quasi-isogonal cevians, nedians, polar of a point with respect to a circle, anti-bisector, aalsonti-symmedian, anti-height and their isogonal.

A nedian is a line segment that has its origin in a triangle's vertex and divides the opposite side in $n$ equal segments.

The papers also study distances between remarkable points in the 2D-geometry, the circumscribed octagon and the inscribable octagon, the circles adjointly ex-inscribed associated to a triangle, and several classical results such as: Carnot circles, Euler's line, Desargues theorem, Sondat's theorem, Dergiades theorem, Stevanovic's theorem, Pantazi's theorem, and Newton's theorem.

Special attention is given in this book to *orthological triangles, bi-orthological triangles, ortho-homological triangles,* and *tri-homological triangles.*

The notion of "ortho-homological triangles" was introduced by the Belgium mathematician Joseph Neuberg in 1922 in the journal Mathesis and it characterizes the triangles that are simultaneously orthogonal (i.e. the sides of one triangle are perpendicular to the sides of the other triangle) and homological. We call this "ortho-homological of first type" in order to distinguish it from our next notation.

In our articles, we gave the same denomination "ortho-homological triangles" to triangles that are simultaneously orthological and homological. We call it "ortho-homological of second type."

Each paper is independent of the others. Yet, papers on the same or similar topics are listed together one after the other.

This book is a continuation of the previous book *The Geometry of Homological Triangles*, by Florentin Smarandache and Ion Pătraşcu, Educ. Publ., Ohio, USA, 244 p., 2012.

The book is intended for College and University students and instructors that prepare for mathematical competitions such as National and International Mathematical Olympiads, or the AMATYC (American Mathematical Association for Two Year Colleges) student competition, or Putnam competition, Gheorghe Ţiteica Romanian student competition, and so on.

The book is also useful for geometrical researchers.

The authors

# Quasi-Isogonal Cevians

Professor Ion Pătraşcu – National College Fraţii Buzeşti, Craiova, Romania
Professor Florentin Smarandache –University of New-Mexico, U.S.A.

In this article we will introduce the quasi-isogonal Cevians and we'll emphasize on triangles in which the height and the median are quasi-isogonal Cevians.

For beginning we'll recall:

**Definition 1**

In a triangle $ABC$ the Cevians $AD$, $AE$ are called isogonal if these are symmetric in rapport to the angle $A$ bisector.

**Observation**

In figure 1, are represented the isogonal Cevians $AD$, $AE$

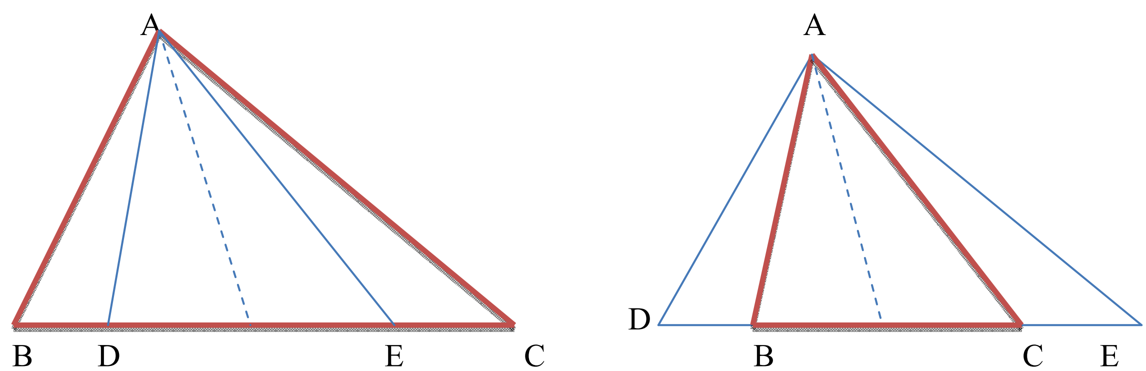



Fig. 1. Isogonal Cevians

**Proposition 1.**

In a triangle $ABC$, the height $AD$ and the radius $AO$ of the circumscribed circle are isogonal Cevians.

**Definition 2.**

We call the Cevians $AD$, $AE$ in the triangle $ABC$ quasi-isogonal if the point $B$ is between the points $D$ and $E$, the point $E$ is between the points $B$ and $C$, and $\sphericalangle DAB \equiv \sphericalangle EAC$.

**Observation**

In figure 2 we represented the quasi-isogonal Cevians $AD, AE$.

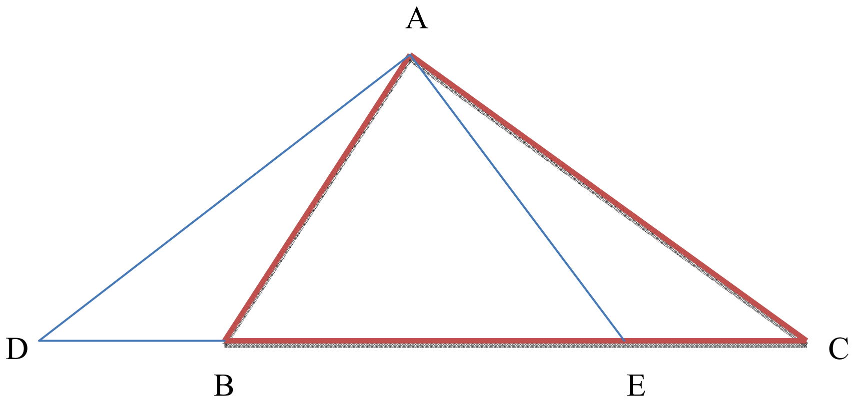



Fig. 2 quasi-isogonal Cevians

**Proposition 2**
There are triangles in which the height and the median are quasi-isogonal Cevians.

**Proof**
It is clear that if we look for triangles $ABC$ for which the height and the median from the point $A$ are quasi isogonal, then these must be obtuse-angled triangle. We'll consider such a case in which $m(\sphericalangle A) > 90°$ (see figure 3).

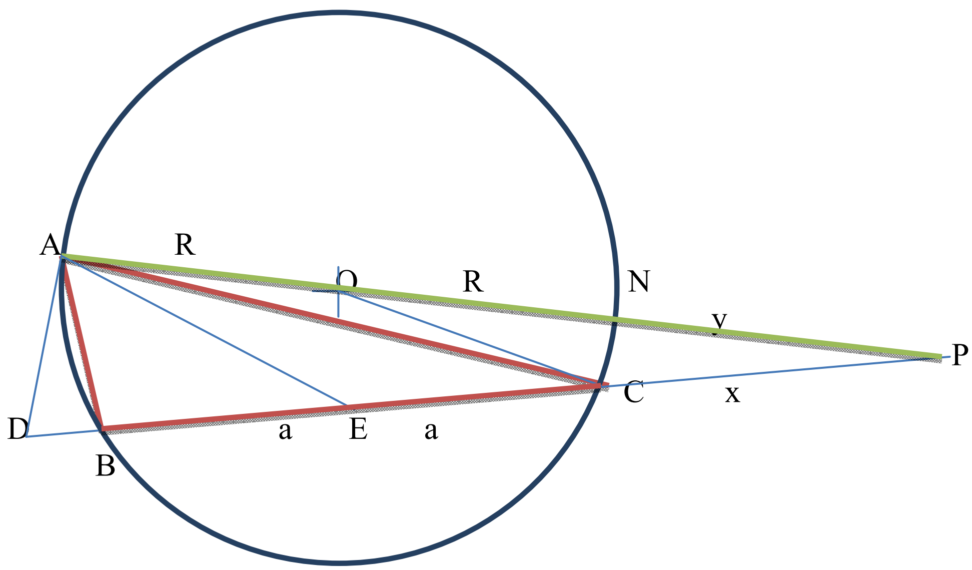



Fig. 3

Let $O$ the center of the circumscribed triangle, we note with $N$ the diametric point of $A$ and with $P$ the intersection of the line $AO$ with $BC$.

We consider known the radius $R$ of the circle and $BC = 2a$, $a < R$ and we try to construct the triangle $ABC$ in which the height $AD$ and the median $AE$ are quasi isogonal Cevians; therefore $\sphericalangle DAB \equiv \sphericalangle EAC$. This triangle can be constructed if we find the lengths $PC$ and $PN$ in function of $a$ and $R$. We note $PC = x$, $PN = y$.

We consider the power of the point $P$ in function of the circle $\mathscr{C}(O,R)$. It results that

$$x\cdot(x+2a)=y\cdot(y+2R) \qquad (1)$$

From the Property 1 we have that $\sphericalangle DAB \equiv \sphericalangle OAC$. On the other side $\sphericalangle OAC \equiv \sphericalangle OCA$ and $AD, AE$ are quasi isogonal, we obtain that $OC \parallel AE$.

The Thales' theorem implies that:

$$\frac{x}{a}=\frac{y+R}{R} \qquad (2)$$

Substituting $x$ from (2) in (1) we obtain the equation:

$$\left(a^2-R^2\right)y^2-2R\left(R^2-2a^2\right)y+3a^2R^2=0 \qquad (3)$$

The discriminant of this equation is:

$$\Delta=4R^2\left(R^4-a^2R^2+a^4\right)$$

Evidently $\Delta>0$, therefore the equation has two real solutions.

Because the product of the solutions is $\dfrac{3a^2R^2}{a^2-R^2}$ and it is negative we obtain that one of solutions is strictly positive. For this positive value of $y$ we find the value of $x$, consequently we can construct the point $P$, then the point $N$ and at the intersection of the line $PN$ we find $A$ and therefore the triangle $ABC$ is constructed.

For example, if we consider $R=\sqrt{2}$ and $a=1$, we obtain the triangle $ABC$ in which $AB=\sqrt{2}$, $BC=2$ and $AC=1+\sqrt{3}$.

We leave to our readers to verify that the height and the median from the point $A$ are quasi isogonal.

**Nedians and Triangles with the Same Coefficient of Deformation**

Ion Pătraşcu – National College Fraţii Buzeşti, Craiova, Romania
Florentin Smarandache – University of New Mexico, Gallup, NM, USA

In [1] Dr. Florentin Smarandache generalized several properties of the nedians. Here, we will continue the series of these results and will establish certain connections with the triangles which have the same coefficient of deformation.

**Definition 1**

The line segments that have their origin in the triangle's vertex and divide the opposite side in $n$ equal segments are called nedians.

We call the nedian $AA_i$ being of order $i$ ($i \in N^*$), in the triangle $ABC$, if $A_i$ divides the side $(BC)$ in the rapport $\frac{i}{n}$ ($\overrightarrow{BA_i} = \frac{i}{n} \cdot \overrightarrow{BC}$ or $\overrightarrow{CA_i} = \frac{i}{n} \cdot \overrightarrow{CB}$, $1 \le i \le n-1$)

**Observation 1**

The medians of a triangle are nedians of order 1, in the case when $n = 3$, these are called tertian.

We'll recall from [1] the following:

**Proposition 1**

Using the nedians of the same of a triangle, we can construct a triangle.

**Proposition 2**

The sum of the squares of the lengths of the nedians of order $i$ of a triangle $ABC$ is given by the following relation:

$$AA_i^2 + BB_i^2 + CC_i^2 = \frac{i^2 - in + n^2}{n^2}\left(a^2 + b^2 + c^2\right) \tag{1}$$

We'll prove

**Proposition3.**

The sum of the squares of the lengths of the sides of the triangle $A_0B_0C_0$, determined by the intersection of the nedians of order $i$ of the triangle $ABC$ is given by the following relation:

$$A_0B_0^2 + B_0C_0^2 + C_0A_0^2 = \frac{(n-2i)^2}{i^2 - in + n^2}\left(a^2 + b^2 + c^2\right) \tag{2}$$

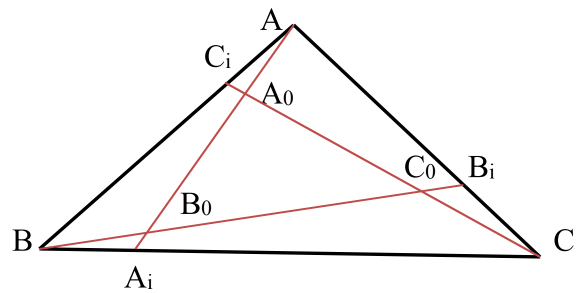



Fig. 1

We noted

$$\{A_0\} = CC_i \cap AA_i,\ \{B_0\} = AA_i \cap BB_i,\ \{C_0\} = BB_i \cap CC_i.$$

**Proof**

We'll apply the Menelaus 'theorem in the triangle $AA_iC$ for the transversals $B - B_0 - B_i$, see Fig. 1.

$$\frac{BA_i}{BC} \cdot \frac{B_iC}{B_iA} \cdot \frac{B_0A}{B_0A_i} = 1 \quad (3)$$

Because $BA_i = \frac{ia}{n}$, $B_iC = \frac{ib}{n}$, $B_iA = \frac{(n-i)b}{n}$, from (3) it results that:

$$B_0A = \frac{n(n-i)}{i^2 - in + n^2} AA_i \quad (4)$$

The Menelaus 'theorem applied in the triangle $AA_iB$ for the transversal $C - C_0 - C_i$ gives

$$\frac{CA_i}{CB} \cdot \frac{C_iB}{C_iA} \cdot \frac{A_0A}{A_0A_i} = 1 \quad (5)$$

But $CA_i = \frac{(n-i)a}{n}$, $C_iB = \frac{(n-i)c}{n}$, $C_iA = \frac{ic}{n}$, which substituted in (5), gives

$$AA_0 = \frac{in}{i^2 - in + n^2} AA_i \quad (6)$$

It is observed that $A_0B_0 = AB_0 - AA_0$ and using the relation (4) and (6) we find:

$$A_0B_0 = \frac{n\left(n-2i\right)}{i^2 - in + n^2} AA_i \quad (7)$$

Similarly, we obtain:

$$B_0C_0 = \frac{n\left(n-2i\right)}{i^2 - in + n^2} BB_i \quad (8)$$

$$C_0A_0 = \frac{n\left(n-2i\right)}{i^2 - in + n^2} CC_i \quad (9)$$

Using the relations (7), (8) and (9), after a couple of computations we obtain the relation (2).

**Observation 2.**

The triangle formed by the nedians of order $i$ as sides is similar with the triangle formed by the intersections of the nedians of order $i$.

Indeed, the relations (7), (8) and (9) show that the sides $A_0B_0$, $B_0C_0$, $C_0A_0$ are proportional with $AA_i$, $BB_i$, $CC_i$

The Russian mathematician V. V. Lebedev introduces in [2] the notion of coefficient of deformation of a triangle. To define this notion we need a couple of definitions and observations.

**Definition 2**

If $ABC$ is a triangle and in its exterior on its sides are constructed the equilateral triangles $BCA_1, CAB_1, ABC_1$, then the equilateral triangle $O_1O_2O_3$ formed by the centers of the circumscribed circles to the equilateral triangles, described above, is called the exterior triangle of Napoleon.

If the equilateral triangles $BCA_1, CAB_1, ABC_1$ intersect in the interior of the triangle $ABC$ then the equilateral triangle $O_1'O_2'O_3'$ formed by the centers of the circumscribed circles to these triangles is called the interior triangle of Napoleon.

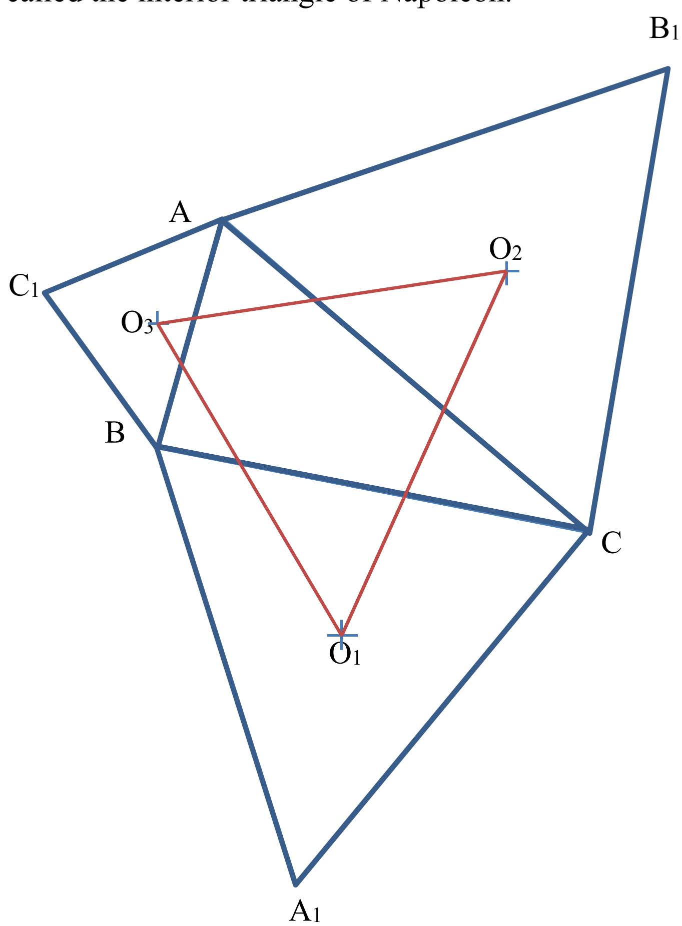



Fig. 2

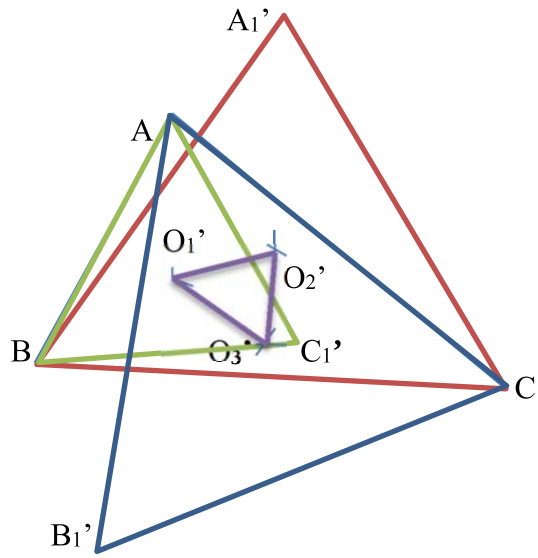


Fig. 3

**Observation 3**

In figure 2 is represented the external triangle of Napoleon and in figure 3 is represented the interior triangle of Napoleon.

**Definition 3**

A coefficient of deformation of a triangle is the rapport between the side of the interior triangle of Napoleon and the side of the exterior triangle of Napoleon corresponding to the same triangle.

**Observation 4**

The coefficient of deformation of the triangle $ABC$ is

$$k = \frac{O_1{}'O_2{}'}{O_1O_2}$$

**Proposition 4**

The coefficient of deformation $k$ of triangle $ABC$ ha the following formula:

$$k = \left( \frac{a^2 + b^2 + c^2 - 4s\sqrt{3}}{a^2 + b^2 + c^2 + 4s\sqrt{3}} \right)^{\frac{1}{2}} \tag{10}$$

where $s$ is the aria of the triangle $ABC$.

**Proof**

We’ll apply the cosine theorem in the triangle $CO_1{}'O_2{}'$ (see Fig. 3), in which

$$CO_1{}' = \frac{a\sqrt{3}}{3},\ \ CO_2{}' = \frac{b\sqrt{3}}{3} \text{ , and } m(\sphericalangle O_1CO_2{}') = C - 60^\circ .$$

We have

$$O_1'O_2'^2 = \frac{3a^2}{9} + \frac{3b^2}{9} - 2\frac{ab}{3}cos\left(C - 60°\right)$$

Because

$$cos\left(C - 60°\right) = cos\,C \cdot cos\,60° + sin\,60° \cdot sin\,C = \frac{1}{2}cos\,C + \frac{\sqrt{2}}{2}sin\,C \text{ and}$$

$$cosC = \frac{b^2 + a^2 - c^2}{2ab}, \text{ and}$$

$$ab\,sin\,C = 2s,$$

we obtain

$$O_1'O_2'^2 = \frac{a^2 + b^2 + c^2 - 4s\sqrt{3}}{6} \tag{11}$$

Similarly

$$O_1O_2^2 = \frac{a^2 + b^2 + c^2 + 4s\sqrt{3}}{6} \tag{12}$$

By dividing the relations (11) and (12) and resolving the square root we proved the proposition.

**Observation 5**

In an equilateral triangle the deformation coefficient is $k = 0$. In general, for a triangle $ABC$, $0 \le k < 1$.

**Observation 6**

From (11) it results that in a triangle is true the following inequality:

$$a^2 + b^2 + c^2 \ge 4s\sqrt{3} \tag{13}$$

which is the inequality Weitzeböck.

**Observation 7**

In a triangle there following inequality – stronger than (13) – takes also place:

$$a^2 + b^2 + c^2 \ge 4s\sqrt{3} + \left(a - b\right)^2 + \left(b - c\right)^2 + \left(c - a\right)^2 \tag{14}$$

which is the inequality of Finsher - Hadwiger.

**Observation 8**

It can be proved that in a triangle the coefficient of deformation can be defined by the

$$k = \frac{AA_1'}{AA_1} \tag{15}$$

**Definition 4**

We define the Brocard point in triangle $ABC$ the point $\Omega$ from the triangle plane, with the property:

$$\sphericalangle \Omega AB \equiv \sphericalangle \Omega BC \equiv \sphericalangle \Omega CA \tag{16}$$

The common measure of the angles from relation (16) is called the Brocard angle and is noted

$\sphericalangle \Omega AB = \omega$

**Observation 9**

A triangle $ABC$ has, in general, two points Brocard $\Omega$ and $\Omega'$ which are isogonal conjugated (see Fig. 4)

**Proposition 5**

In a triangle $ABC$ takes place the following relation:

$$ctg\omega = \frac{a^2+b^2+c^2}{4s} \qquad (17)$$

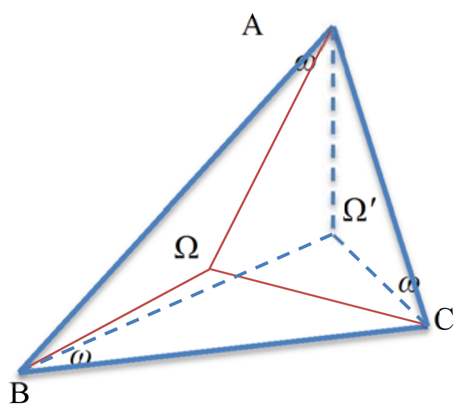



Fig. 4

**Proof**

We'll show, firstly, that in a non-rectangle triangle $ABC$ is true the following relation:

$$ctg\omega = ctgA + ctgB + ctgC \qquad (18)$$

Applying the sin theorem in triangle $A\Omega B$ and $A\Omega C$, we obtain

$$\frac{B\Omega}{\sin\omega} = \frac{c}{\sin B\Omega A} \text{ and } \frac{A\Omega}{\sin\omega} = \frac{b}{\sin A\Omega C}$$

Because $m(\sphericalangle B\Omega A) = 180° - m(\sphericalangle B)$ and $m(\sphericalangle A\Omega C) = 180° - m(\sphericalangle A)$ from the precedent relations we retain that

$$\frac{A\Omega}{B\Omega} = \frac{b}{c}\frac{\sin B}{\sin A} \qquad (19)$$

On the other side also from the sin theorem in triangle $A\Omega B$, we obtain

$$\frac{A\Omega}{B\Omega} = \frac{\sin(B-\omega)}{\sin\omega} \qquad (20)$$

Working out $\sin(B-\omega)$, taking into account that $\frac{b}{c} = \frac{\sin B}{\sin C}$ and that $\sin B = \sin(A+C)$, we obtain (18).

In a triangle $ABC$ is true the relation $ctgA = \frac{a^2+b^2+c^2}{4s}$ (19) and the analogues.

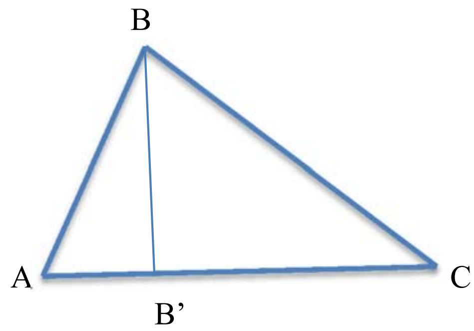



Fig. 5

Indeed, if $m(\sphericalangle A) < 90°$ and $B'$ is the orthogonal projection of $B$ on $AC$ (see Fig. 5), then

$$ctgA = \frac{AB'}{BB'} = \frac{c \cdot \cos A}{BB'}$$

Because $BB' = \frac{2s}{b}$ it results that $ctgA = \frac{2bc\cos A}{4s}$

From the cosine theorem we get

$$2bc\cos A = b^2 + c^2 - a^2$$

Replacing in (18) the $ctgA,\ ctgB,\ ctgC$, we obtain (17)

**Observation 10**

The coefficient of deformation $k$ of triangle $ABC$ is given by

$$k = \left(\frac{ctg\omega - \sqrt{3}}{ctg\omega + \sqrt{3}}\right)^{\frac{1}{2}} \qquad (21)$$

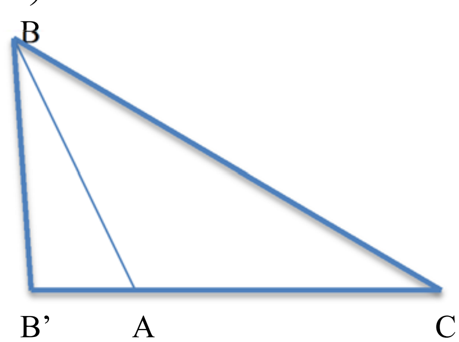



Fig. 6

Indeed, from (10) and (17), it results, without difficulties (21)

**Proposition 6** (V.V. Lebedev)

The necessary and sufficient condition for two triangles to have the same coefficient of deformation is to have the same Brocard angle.

**Proof**

If the triangles $ABC$ and $A_1B_1C_1$ have equal coefficients of deformation $k = k_1$ then from relation 21 it results

$$\frac{ctg\omega-\sqrt{3}}{ctg\omega+\sqrt{3}}=\frac{ctg\omega_1-\sqrt{3}}{ctg\omega_1+\sqrt{3}}$$

Which leads to $ctg\omega = ctg\omega_1$ with the consequence that $\omega = \omega_1$.

Reciprocal, if $\omega = \omega_1$, immediately results, using (21), that takes place $k = k_1$.

**Proposition 7**

Two triangles $ABC$ and $A_1B_1C_1$ have the same coefficient of deformation if and only if

$$\frac{s_1}{s}=\frac{a_1^2+b_1^2+c_1^2}{a^2+b^2+c^2} \qquad (22)$$

($s_1$ being the aria of triangle $A_1B_1C_1$, with the sides $a_1, b_1, c_1$)

**Proof**

If $\omega, \omega_1$ are the Brocard angles of triangles $ABC$ and $A_1B_1C_1$ then, taking into consideration (17) and Proposition 6, we'll obtain (22). Also from (22) taking into consideration of (17) and Proposition 6, we'll get $k = k_1$.

**Proposition 8**

Triangle $A_iB_iC_i$ formed by the legs of the nediands of order $i$ of triangle $ABC$ and triangle $ABC$ have the same coefficient of deformation.

**Proof**

We'll use Proposition 7, applying the cosine theorem in triangle $A_iB_iC_i$, we'll obtain

$$B_iC_i^2 = AC_i^2 + AB_i^2 - 2AC_iAB_i \cos A$$

Because

$$AC_i = \frac{ic}{n},\ AB_i = \frac{(n-i)b}{n}$$

it results

$$B_iC_i^2 = \frac{i^2c^2}{n^2} + \frac{(n-i)^2b^2}{n^2} - \frac{2i(n-i)bc\cos A}{n^2}$$

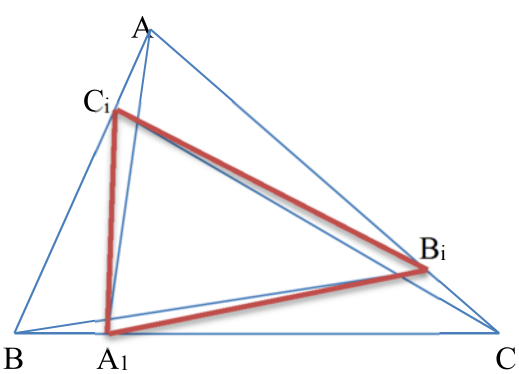



Fig. 7

The cosin theorem in the triangle $ABC$ gives

$$2bc\cos A = b^2 + c^2 - a^2$$
which substituted above gives
$$B_iC_i^2 = \frac{i^2c^2 + (n-i)^2 b^2 + i(n-i)(a^2 - b^2 - c^2)}{n^2}$$
$$B_iC_i^2 = \frac{a^2(in - i^2) + b^2(n^2 - 3in + 2i^2) + c^2(2i^2 - in)}{n^2}$$
Similarly we'll compute $C_iA_i^2$ and $A_iB_i^2$

It results
$$\frac{A_iB_i^2 + B_iC_i^2 + C_iA_i^2}{a^2 + b^2 + c^2} = \frac{n^2 - 2in + 3i^2}{n^2} \tag{23}$$
If we note
$$s_i = Aria_{\triangle} A_iB_iC_i$$
We obtain
$$s_i = s - \left(Aria_{\triangle} AB_iC_i + Aria_{\triangle} BA_iC_i + Aria_{\triangle} CA_iB_i\right) \tag{24}$$
But
$$Aria_{\triangle} AB_iC_i = \frac{1}{2} AC_i \cdot AB_i \sin A$$
$$Aria_{\triangle} AB_iC_i = \frac{1}{2} \frac{i(n-i) b \cdot c}{n^2} \sin A = \frac{i(n-i) \cdot s}{n^2}$$
Similarly, we find that
$$Aria_{\triangle} BA_iC_i = Aria_{\triangle} CA_iB_i = \frac{i(n-i) \cdot s}{n^2}$$
Revisiting (23) we get that
$$s_i = \frac{sn^2 - 3in + 3i^2}{n^2}$$
therefore,
$$\frac{s_i}{s} = \frac{n^2 - 3in + 3i^2}{n^2} \tag{25}$$
The relations (23), (25) and Proposition 7 will imply the conclusion.

**Proposition 9**

The triangle formed by the medians of a given triangle, as sides, and the given triangle have the same coefficient of deformation.

**Proof**

The medians are nedians of order I. Using (1), it results
$$AA_i^2 + BB_i^2 + CC_i^2 = \frac{3}{4}\left(a^2 + b^2 + c^2\right) \tag{26}$$
The proposition will be proved if we'll show that the rapport between the aria of the formed triangle with the medians of the given triangle and the aria of the given triangle is $\frac{3}{4}$.

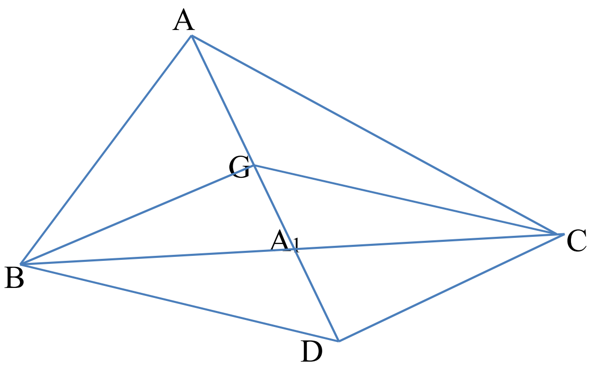


Fig. 9

If in triangle $ABC$ we prolong the median $AA_1$ such that $A_1D = GA_1$ ($G$ being the center of gravity of the triangle $ABC$), then the quadrilateral $BGCD$ is a parallelogram (see Fig. 9). Therefore $CD = BG$. It is known that $BG = \frac{2}{3}BB_1$, $CG = \frac{2}{3}CC_1$ and from construction we have that $GD = \frac{2}{3}AA_1$. Triangle $GDC$ has the sides equal to $\frac{2}{3}$ from the length of the medians of the triangle $ABC$. Because the median of a triangle divides the triangle in two equivalent triangles and the gravity center of the triangle forms with the vertexes of the triangle three equivalent triangle, it results that $Aria_{\vartriangle}GDC = \frac{1}{3}s$. On the other side the rapport of the arias of two similar triangles is equal with the squared of their similarity rapport, therefore, if we note $s_1$ the aria of the triangle formed by the medians, we have $\frac{Aria_{\vartriangle}GDC}{s_1} = \left(\frac{2}{3}\right)^2$.

We find that $\frac{s_1}{s} = \frac{3}{4}$, which proves the proposition.

**Proposition 10**

The triangle formed by the intersections of the tertianes of a given triangle and the given triangle have the same coefficient of deformation.

**Proof**

If $A_0B_0C_0$ is the triangle formed by the intersections of the tertianes, from relation (2) we'll find

$$\frac{A_0B_0^2 + B_0C_0^2 + C_0A_0^2}{a^2 + b^2 + c^2} = \frac{1}{7}$$

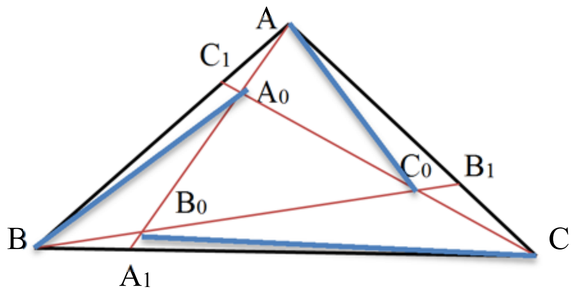



Fig 10

We note $s_0$ the aria of triangle $A_0B_0C_0$, we'll prove that $\frac{s_0}{s} = \frac{1}{7}$.

From the formulae (6) and (7), it is observed that $A_0 = A_0B_0$ and $CC_0 = C_0A_0$.

Using the median's theorem in a triangle to determine that in that triangle two triangle are equivalent, we have that:

$$Aria_{\triangle}AA_0C_0 = Aria_{\triangle}AC_0C = Aria_{\triangle}A_0B_0C_0 =$$
$$= Aria_{\triangle}CB_0C_0 = Aria_{\triangle}CBB_0 = Aria_{\triangle}BB_0A_0 = Aria_{\triangle}ABA_0$$

Because the sum of the aria of these triangles is $s$, it results that $s_0 = \frac{1}{7}s$, which shows what we had to prove.

**Proposition 11**

We made the observation that the triangle $A_0B_0C_0$ and the triangle formed by the tertianes $AA_1, BB_1, CC_1$ as sides are similar. Two similar triangles have the same Brocard angle, therefore the same coefficient of deformation. Taking into account Proposition 10, we obtain the proof of the statement

**Observation 11**

From the precedent observations it results that being given a triangle, the triangles formed by the tertianes intersections with the triangle as sides, the intersections of the tertianes of the triangle have the same coefficient of deformation.

**References**


[1] F. Smarandache – Généralisations et généralites – Edition Nouvelle, 1984, Fès, Maroc

[2] V. V. Lebedev - Nekatoroie svoistva trengolnikov Napoleone – Matematika v schole, no.6/1962

[3] V. Gh. Vodă – Triunghiul – ringul cu trei colțuri – Editura Albatros, Bucureşti, 1979

[4] R. A. Johnson – Advanced Euclidean Geometry – Dover Publications, Inc. Mineola, New York, 2007.

# From a problem of geometrical construction to the Carnot circles

Prof. Ion Pătraşcu – The Fraţii Buzeşti College, Craiova - Romania
Prof. Dr. Florentin Smarandache – University of New Mexico – U.S.A.

In this article we'll give solution to a problem of geometrical construction and we'll show the connection between this problem and the theorem relative to Carnot's circles.

Let $ABC$ a given random triangle. Using only a compass and a measuring line, construct a point $M$ in the interior of this triangle such that the circumscribed circles to the triangles $MAB$ and $MAC$ are congruent.

**Construction**

We'll start by assuming, as in many situations when we have geometrical constructions, that the construction problem is resolved.

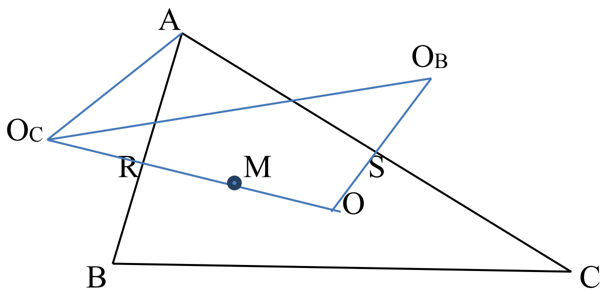



Fig. 1

Let $M$ a point in the interior of the triangle $ABC$ such that the circumscribed circles to the triangles $MAB$ and $MAC$ are congruent.

We'll note $O_C$ and $O_B$ the centers of these triangles, these are the intersections between the mediator of the segments $[AB]$ and $[AC]$. The quadrilateral $AO_CMO_B$ is a rhomb (therefore $M$ is the symmetrical of the point $A$ in rapport to $O_BO_C$ (see Fig. 1).

A. **Step by step construction**

We'll construct the mediators of the segments $[AB]$ and $[AC]$, let $R,S$ be their intersection points with $[AB]$ respectively $[AC]$. (We suppose that $AB < AC$, therefore $AR < AS$.) With the compass in $A$ and with the radius larger than $AS$ we construct a circle which intersects $OR$ in $O_C$ and $O_{C'}$ respectively $OS$ in $O_B$ and $O_{B'}$ - $O$ being the circumscribed circle to the triangle $ABC$.

Now we construct the symmetric of the point $A$ in rapport to $O_CO_B$; this will be the point $M$, and if we construct the symmetric of the point $A$ in rapport to $O_{C'}O_{B'}$ we obtain the point $M'$

*Lazare Carnot (1753 – 1823)*, French mathematician, mechanical engineer and political personality (Paris).

B. **Proof of the construction**

Because $AO_C = AO_B$ and $M$ is the symmetric of the point $A$ in rapport of $O_CO_B$, it results that the quadrilateral $AO_CMO_B$ will be a rhombus, therefore $O_CA = O_CM$ and $O_BA = O_BM$. On the other hand, $O_C$ and $O_B$ being perpendicular points of $AB$ respectively $AC$, we have $O_CA = O_CB$ and $O_BA = O_BC$, consequently

$$O_CA = O_CM = O_BA = O_BM = O_BC,$$

which shows that the circumscribed circles to the triangles $MAB$ and $MAC$ are congruent.

Similarly, it results that the circumscribed circles to the triangles $ABM'$ and $ACM'$ are congruent, more so, all the circumscribed circles to the triangles $MAB, MAC, M'AB, M'AC$ are congruent.

As it can be in the Fig. 2, the point $M'$ is in the exterior of the triangle $ABC$.

**Discussion**

We can obtain, using the method of construction shown above, an infinity of pairs of points $M$ and $M'$, such that the circumscribed circles to the triangles $MAB, MAC, M'AB, M'AC$ will be congruent. It seems that the point $M'$ is in the exterior of the triangle $ABC$

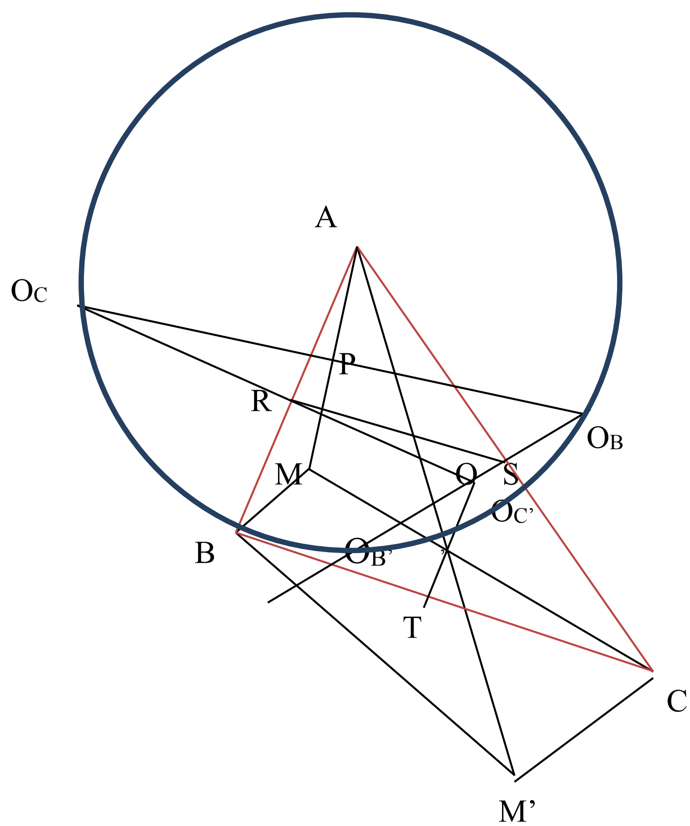



Fig. 2

**Observation**

The points $M$ from the exterior of the triangle $ABC$ with the property described in the hypothesis are those that belong to the arch $\overset{\frown}{BC}$, which does not contain the vertex $A$ from the circumscribed circle of the triangle $ABC$.

Now, we'll try to answer to the following:

**Questions**

1. Can the circumscribed circles to the triangles $MAB, MAC$ with $M$ in the interior of the triangle $ABC$ be congruent with the circumscribed circle of the triangle $ABC$
2. If yes, then, what can we say about the point $M$ ?

**Answers**

1. The answer is positive. In this hypothesis we have $OA = AO_B = AO_C$ and it results also that $O_C$ and $O_B$ are the symmetrical of $O$ in rapport to $AB$ respectively $AC$ The point $M$ will be, as we showed, the symmetric of the point $A$ in rapport to $O_C O_B$.

The point $M$ will be also the orthocenter of the triangle $ABC$. Indeed, we prove that the symmetric of the point $A$ in rapport to $O_C O_B$ is $H$ which is the orthocenter of the triangle $ABC$ Let $RS$ the middle line of the triangle $ABC$. We observe that $RS$ is also middle line in the triangle $OO_B O_C$, therefore $O_B O_C$ is parallel and congruent with $BC$, therefore it results that $M$ belongs to the height constructed from $A$ in the triangle $ABC$. We'll note $T$ the middle of $[BC]$, and let $R$ the radius of the circumscribed circle to the triangle $ABC$; we have

$$OT = \sqrt{R^2 - \frac{a^2}{4}} \text{, where } a = BC.$$

If $P$ is the middle of thesegment $[AM]$, we have

$$AP = \sqrt{R^2 - PO_B{}^2} = \sqrt{R^2 - \frac{a^2}{4}}.$$

From the relation $AM = 2 \cdot OT$ it results that $M$ is the orthocenter of the triangle $ABC$, ( $AH = 2OT$ ).

The answers to the questions 1 and 2 can be grouped in the following form:

**Proposition**

There is onlyone point in the interior of the triangle $ABC$ such that the circumscribed circles to the triangles $MAB$, $MAC$ and $ABC$ are congruent. This point is the orthocenter of the triangle $ABC$.

**Remark**

From this proposition it practically results that the unique point $M$ from the interior of the right triangle $ABC$ with the property that the circumscribed circles to the triangles $MAB, MAC, MBC$ are congruent with the circumscribed circle to the triangle is the point $H$, the triangle's orthocenter.

**Definition**

If in the triangle $ABC$, $H$ is the orthocenter, then the circumscribed circles to the triangles $HAB, HAC, HBC$ are called Carnot circles.

We can prove, without difficulty the following:

**Theorem**

The Carnot circles of a triangle are congruent with the circumscribed circle to the triangle.

**References**


[1] C. Mihalescu – Geometria elementelor remarcabile - Ed. Tehnică, Bucharest, 1957.

[2] Johnson, A. R. – Advanced Euclidean Geometry – Dover Publications, Inc., New York, 2007.

[3] Smarandache F., Pătraşcu I. – The geometry of homological triangles – Columbus, Ohio, U.S.A, 2012.

# THE POLAR OF A POINT With Respect TO A CIRCLE

**Ion Pătraşcu, Fratii Buzesti National College, Craiova, Romania**
**Florentin Smarandache, University of New Mexico, Gallup, NM 87301, USA**

In this article we establish a connection between the notion of the symmedian of a triangle and the notion of polar of a point in rapport to a circle

We'll prove for beginning two properties of the symmedians.

**Lemma 1**

If in triangle ABC inscribed in a circle, the tangents to this circle in the points B and C intersect in a point S, then AS is symmedian in the triangle ABC.

**Proof**

We'll note L the intersection point of the line AS with BC (see fig. 1).

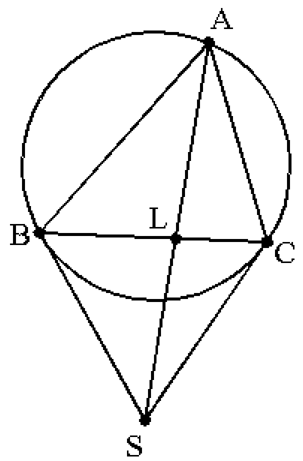



**Fig. 1**

We have

$$\frac{\text{Aria}\,\Delta ABL}{\text{Aria}\,\Delta ACL}=\frac{BL}{LC}=\frac{\text{Aria}\,\Delta BSL}{\text{Aria}\,\Delta CSL}$$

It result

$$\frac{\text{Aria}\,\Delta ABS}{\text{Aria}\,\Delta ACS}=\frac{BL}{LC} \tag{1}$$

We observe that

$$m(\angle ABS)=m\left(\hat{B}\right)+m\left(\hat{A}\right) \text{ and } m(\angle ACS)=m\left(\hat{C}\right)+m\left(\hat{A}\right)$$

We obtain that

$$\sin(\angle ABS)=\sin C \text{ and } \sin(\angle ACS)=\sin B$$

We have also

$$\frac{\text{Aria}\,\Delta ABS}{\text{Aria}\,\Delta ACS}=\frac{AB\cdot SB\cdot \sin C}{AC\cdot SC\cdot \sin B}=\frac{BL}{LC} \tag{2}$$

From the sinus' theorem it results

$$\frac{\sin C}{\sin B} = \frac{AB}{AC} \tag{3}$$

The relations (2) and lead us to the relation

$$\frac{BL}{LC} = \left(\frac{AB}{AC}\right)^2,$$

which shows that AS is symmedian in the triangle ABC.

**Observations**

*1.* The proof is similar if the triangle ABC is obtuse.

*2.* If ABC is right triangle in A, the tangents in B and C are parallel, and the symmedian from A is the height from A, and, therefore, it is also parallel with the tangents constructed in B and C to the circumscribed circle.

**Definition 1**

The points A, B, C, D placed, in this order, on a line d form a harmonic division if and only if

$$\frac{AB}{AD} = \frac{CB}{CD}$$

**Lemma 2**

If in the triangle ABC, AL is the interior symmedian $L \in BC$, and AP is the external median $P \in BC$, then the points P, B, L, C form a harmonic division.

**Proof**

It is known that the external symmedian AP in the triangle ABC is tangent in A to the circumscribed circle (see fig. 2), also, it can be proved that:

$$\frac{PB}{PC} = \left(\frac{AB}{AC}\right)^2 \tag{1}$$

but

$$\frac{LB}{LC} = \left(\frac{AB}{AC}\right)^2 \tag{2}$$

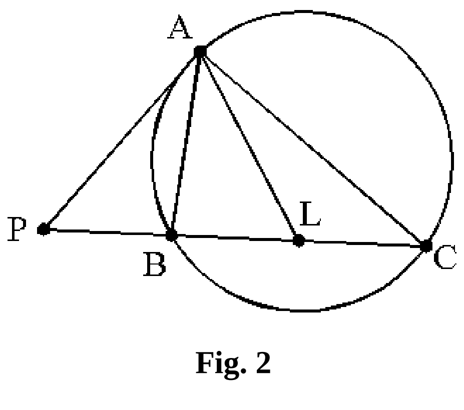



**Fig. 2**

From the relations (1) and (2) it results

$$\frac{PB}{PC} = \frac{LB}{LC},$$

Which shows that the points P, B, L, C form a harmonic division.

**Definition 2**

If P is a point exterior to circle $\mathcal{C}(0,r)$ and B, C are the intersection points of the circle with a secant constructed through the point P, we will say about the point Q∈(BC) with the property $\frac{PB}{PC}=\frac{QB}{QC}$ that it is the harmonic conjugate of the point P in rapport to the circle $\mathcal{C}(0,r)$.

**Observation**

In the same conjunction, the point P is also the conjugate of the point Q in rapport to the circle (see fig. 3).

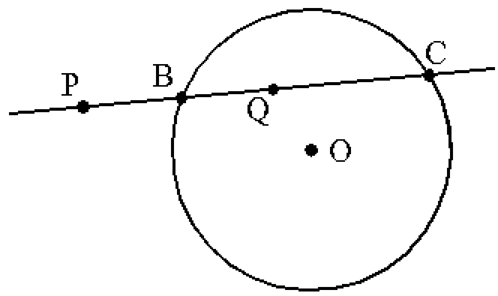



**Fig. 3**

**Definition 3**

The set of the harmonic conjugates of a point in rapport with a given circle is called the **polar** of that point in rapport to the circle.

**Theorem**

The polar of an exterior point to the circle is the circle's cord determined by the points of tangency with the circle of the tangents constructed from that point to the circle.

**Proof**

Let P an exterior point of the circle $\mathcal{C}(0,r)$ and M, N the intersections of the line PO with the circle (see fig. 4).

We note T and V the tangent points with the circle of the tangents constructed from the point P and let Q be the intersection between MN and TV.

Obviously, the triangle MTN is a right triangle in T, TQ is its height (therefore the interior symmedian, and TP is the exterior symmedian, and therefore the points P, M, Q, N form a harmonic division, (Lemma 2)). Consequently, Q is the harmonic conjugate of P in rapport to the circle and it belongs to the polar of P in rapport to the circle.

We'll prove that (TV) is the polar of P in rapport with the circle. Let M′N′ be the intersections of a random secant constructed through the point P with the circle, and X the intersection of the tangents constructed in M′ and N′ to the circle.

In conformity to Lemma 1, the line XT is for the triangle M′TN′ the interior symmedian, also TP is for the same triangle the exterior symmedian.

If we note Q′ the intersection point between XT and M′N′ it results that the point Q′ is the harmonic conjugate of the point P in rapport with the circle, and consequently, the point Q′ belongs to the polar P in rapport to the circle.

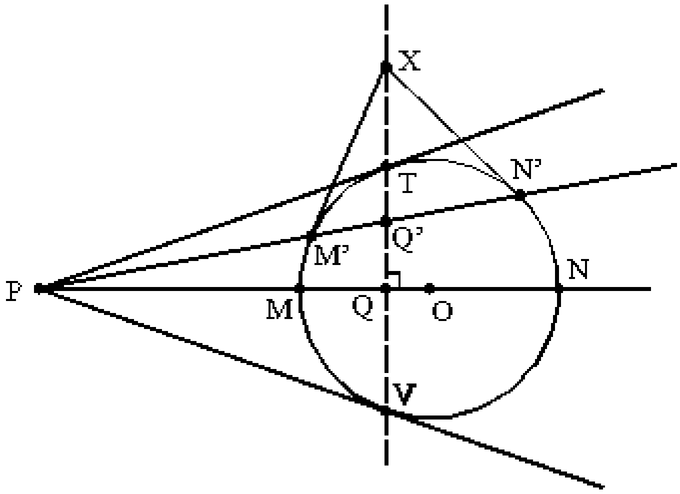



**Fig. 4**

For the triangle $VM'N'$, according to Lemma 1, the line $VX$ is the interior symmedian and $VP$ is for the same triangle the external symmedian. It will result, according to Lemma 2, that if $\{Q''\} = VX \cap M'N'$, the point $Q''$ is the harmonic conjugate of the point $P$ in rapport to the circle. Because the harmonic conjugate of a point in rapport with a circle is a unique point, it results that $Q'=Q''$. Therefore the points $V$, $T$, $X$ are collinear and the point $Q'$ belongs to the segment $(TV)$.

**Reciprocal**

If $Q_1 \in (TV)$ and $PQ_1$ intersect the circle in $M_1$ and $N_1$, we much prove that the point $Q_1$ is the harmonic conjugate of the point $P$ in rapport to the circle.

Let $X_1$ the intersection point of the tangents constructed from $M_1$ and $N_1$ to the circle. In the triangle $M_1TN_1$ the line $X_1T$ is interior symmedian, and the line $TP$ is exterior symmedian. If $\{Q_1'\} = X_1T \cap M_1N_1$ then $P$, $M_1$, $Q_1'$, $N_1$ form a harmonic division.

Similarly, in the triangle $M_1VN_1$ the line $VX_1$ is interior symmedian, and $VP$ exterior symmedian. If we note $\{Q_1''\} = VX_1 \cap M_1N_1$, it results that the point $Q_1''$ is the harmonic conjugate of the point $P$ in rapport to $M_1$ and $N_1$. Therefore, we obtain $Q_1' = Q_1''$. On the other side, $X_1$, $T$, $Q_1'$ and $V$, $X_1$, $Q_1''$ are collinear, but $Q_1' = Q_1''$, it result that $X_1$, $T$, $Q_1'$, $V$ are collinear, and then $Q_1'=Q_1$, therefore $Q_1$ is the conjugate of $P$ in rapport with the circle.

# Several Metrical Relations Regarding the Anti-Bisector, the Anti-Symmedian, the Anti-Height and their Isogonal

Professor Ion Pătraşcu – Fraţii Buzeşti National College, Craiova
Professor Florentin Smarandache –University of New Mexico, U.S.A.

We suppose known the definitions of the isogonal cevian and isometric cevian; we remind that the anti-bisector, the anti-symmedian, and the anti-height are the isometrics of the bisector, of the symmedian and of the height in a triangle.

It is also known the following Steiner (1828) relation for the isogonal cevians $AA_1$ and $AA_1'$:

$$\frac{BA_1}{CA_1}\cdot\frac{BA_1'}{CA_1'}=\left(\frac{AB}{AC}\right)^2$$

We'll prove now that there is a similar relation for the isometric cevians

**Proposition**

In the triangle $ABC$ let consider $AA_1$ and $AA_1'$ two isometric cevians, then there exists the following relation:

$$\frac{\sin\left(\widehat{BAA_1}\right)}{\sin\left(\widehat{CAA_1}\right)}\cdot\frac{\sin\left(\widehat{BAA_1'}\right)}{\sin\left(\widehat{CAA_1'}\right)}=\left(\frac{\sin B}{\sin C}\right)^2 \qquad (*)$$

**Proof**

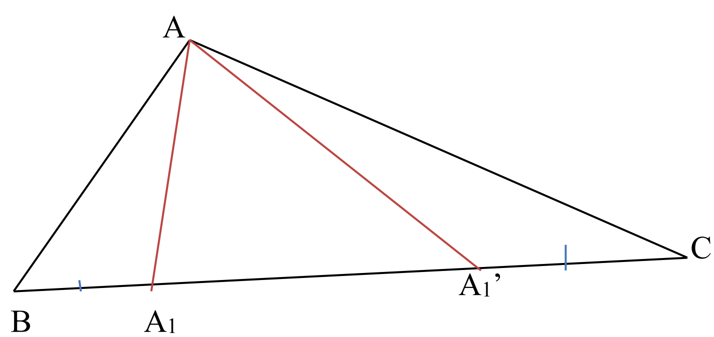



*Fig. 1*

The sinus theorem applied in the triangles $ABA_1, ACA_1$ implies (see above figure)

$$\frac{\sin\left(\widehat{BAA_1}\right)}{BA_1}=\frac{\sin B}{AA_1} \qquad (1)$$

$$\frac{\sin\left(\widehat{CAA_1}\right)}{CA_1}=\frac{\sin C}{AA_1} \qquad (2)$$

From the relations (1) and (2) we retain

$$\frac{\sin\left(\widehat{BAA_1}\right)}{\sin\left(\widehat{CAA_1}\right)}=\frac{\sin B}{\sin C}\cdot\frac{BA_1}{CA_1} \tag{3}$$

The sinus theorem applied in the triangles $ACA_1^{'}, ABA_1^{'}$ leads to

$$\frac{\sin\left(\widehat{CAA_1^{'}}\right)}{A_1^{'}C}=\frac{\sin C}{AA_1^{'}} \tag{4}$$

$$\frac{\sin\left(\widehat{BAA_1^{'}}\right)}{BA_1^{'}}=\frac{\sin B}{AA_1^{'}} \tag{5}$$

From the relations (4) and (5) we obtain:

$$\frac{\sin\left(\widehat{BAA_1^{'}}\right)}{\sin\left(\widehat{CAA_1^{'}}\right)}=\frac{\sin B}{\sin C}\cdot\frac{BA_1^{'}}{CA_1^{'}} \tag{6}$$

Because $BA_1 = CA_1^{'}$ and $A_1C = BA_1^{'}$ ) the cevians being isometric), from the relations (3) and (6) we obtain relation (*) from the proposition's enouncement.

**Applications**

1. If $AA_1$ is the bisector in the triangle $ABC$ and $AA_1^{'}$ is its isometric, that is an anti-bisector, then from (*) we obtain

$$\frac{\sin\left(\widehat{BAA_1^{'}}\right)}{\sin\left(\widehat{CAA_1^{'}}\right)}=\left(\frac{\sin B}{\sin C}\right)^2 \tag{7}$$

Taking into account of the sinus theorem in the triangle $ABC$ we obtain

$$\frac{\sin\left(\widehat{BAA_1^{'}}\right)}{\sin\left(\widehat{CAA_1^{'}}\right)}=\left(\frac{AC}{AB}\right)^2 \tag{8}$$

2. If $AA_1$ is symmedian and $AA_1^{'}$ is an anti-symmedian, from (*) we obtain

$$\frac{\sin\left(\widehat{BAA_1^{'}}\right)}{\sin\left(\widehat{CAA_1^{'}}\right)}=\left(\frac{AC}{AB}\right)^3$$

Indeed, $AA_1$ being symmedian it is the isogonal of the median $AM$ and

$$\frac{\sin\left(\widehat{MAB}\right)}{\sin\left(\widehat{MAC}\right)}=\frac{\sin B}{\sin C} \quad \text{and}$$

$$\frac{\sin\left(\widehat{BAA_1^{'}}\right)}{\sin\left(\widehat{CAA_1^{'}}\right)}=\frac{\sin\left(\widehat{MAC}\right)}{\sin\left(\widehat{MAB}\right)}=\frac{\sin C}{\sin B}=\frac{AB}{AC}$$

3. If $AA_1$ is a height in the triangle $ABC$, $A_1 \in (BC)$ and $AA_1'$ is its isometric (anti-height), the relation (*) becomes.

$$\frac{\sin\left(\widehat{BAA_1'}\right)}{\sin\left(\widehat{CAA_1'}\right)} = \left(\frac{AC}{AB}\right)^2 \cdot \frac{\cos C}{\cos B}$$

Indeed

$$\sin\left(\widehat{BAA_1'}\right) = \frac{BA_1}{AB};\ \sin\left(\widehat{CAA_1'}\right) = \frac{CA_1}{AC}$$

therefore

$$\frac{\sin\left(\widehat{BAA_1'}\right)}{\sin\left(\widehat{CAA_1'}\right)} = \frac{AC}{AB} \cdot \frac{BA_1}{CA_1}$$

From (*) it results

$$\frac{\sin\left(\widehat{BAA_1'}\right)}{\sin\left(\widehat{CAA_1'}\right)} = \frac{AC}{AB} \cdot \frac{CA_1}{BA_1}$$

or

$$CA_1 = AC \cdot \cos C \text{ and } BA_1 = AB \cdot \cos B$$

therefore

$$\frac{\sin\left(\widehat{BAA_1'}\right)}{\sin\left(\widehat{CAA_1'}\right)} = \left(\frac{AC}{AB}\right)^2 \cdot \frac{\cos C}{\cos B}$$

4. If $AA_1''$ is the isogonal of the anti-bisector $AA_1'$ then

$$\frac{BA_1''}{A_1''C} = \left(\frac{AB}{AC}\right)^3 \quad \text{(Maurice D'Ocagne, 1883)}$$

**Proof**

The Steiner's relation for $AA_1''$ and $AA_1'$ is

$$\frac{BA_1''}{A_1''C} \cdot \frac{BA_1'}{A_1'C} = \left(\frac{AB}{AC}\right)^2$$

But $AA_1$ is the bisector and according to the bisector theorem $\frac{BA_1}{CA_1} = \frac{AB}{AC}$ but $BA_1' = CA_1$ and $A_1'C = BA_1$ therefore

$$\frac{CA_1'}{BA_1'} = \frac{AB}{AC}$$

and we obtain the D'Ocagne relation

5. If in the triangle $ABC$ the cevian $AA_1''$ is isogonal to the symmedian $AA_1'$ then

$$\frac{BA_1''}{A_1''C} = \left(\frac{AB}{AC}\right)^4$$

**Proof**

Because $AA_1$ is a symmedian, from the Steiner's relation we deduct that

$$\frac{BA_1}{CA_1} = \left(\frac{AB}{AC}\right)^2$$

The Steiner's relation for $AA_1''$, $AA_1'$ gives us

$$\frac{BA_1''}{A_1''C} \cdot \frac{BA_1'}{CA_1'} = \left(\frac{AB}{AC}\right)^2$$

Taking into account the precedent relation, we obtain

$$\frac{BA_1''}{A_1''C} = \left(\frac{AB}{AC}\right)^4$$

6.

If $AA_1''$ is the isogonal of the anti-height $AA_1'$ in the triangle $ABC$ in which the height $AA_1$ has $A_1 \in (BC)$ then

$$\frac{BA_1''}{A_1''C} = \left(\frac{AB}{AC}\right)^3 \cdot \frac{\cos B}{\cos C}$$

**Proof**

If $AA_1$ is height in triangle $ABC$ $A_1 \in (BC)$ then

$$\frac{BA_1}{A_1C} = \frac{AB}{AC} \cdot \frac{\cos B}{\cos C}$$

Because $AA_1'$ is anti-median, we have $BA_1 = CA_1'$ and $A_1C = BA_1'$ then

$$\frac{BA_1''}{A_1''C} = \frac{AC}{AB} \cdot \frac{\cos C}{\cos B}$$

**Observation**

The precedent results can be generalized for the anti-cevians of rang k and for their isogonal.

# An Important Application of the Computation of the Distances between Remarkable Points in the Triangle Geometry

**Prof. Ion Pătraşcu, The Fraţii Buzeşti College, Craiova, Romania**
**Prof. Florentin Smarandache, University of New Mexico, U.S.A.**

In this article we'll prove through computation the Feuerbach's theorem relative to the tangent to the nine points circle, the inscribed circle, and the ex-inscribed circles of a given triangle.

Let $ABC$ a given random triangle in which we denote with $O$ the center of the circumscribed circle, with $I$ the center of the inscribed circle, with $H$ the orthocenter, with $I_a$ the center of the $A$ ex-inscribed circle, with $O_9$ the center of the nine points circle, with $p=\dfrac{a+b+c}{2}$ the semi-perimeter, with $R$ the radius of the circumscribed circle, with $r$ the radius of the inscribed circle, and with $r_a$ the radius of the $A$ ex-inscribed circle.

**Proposition**

In a triangle $ABC$ are true the following relations:

(i) $OI^2 = R^2 - 2Rr$ Euler's relation

(ii) $OI_a^2 = R^2 + 2Rr_a$ Feuerbach's relation

(iii) $OH^2 = 2r^2 - 2p^2 + 9R^2 + 8Rr$

(iv) $IH^2 = 3r^2 - p^2 + 4R^2 + 4Rr$

(v) $I_aH^2 = r^2 - p^2 + 2r_a^{\ 2} + 4R^2 + 4Rr$

**Proof**

(i) The positional vector of the center $I$ of the inscribed circle of the given triangle $ABC$ is

$$\overrightarrow{PI} = \frac{1}{2p}\left(a\overrightarrow{PA} + b\overrightarrow{PB} + c\overrightarrow{PC}\right)$$

For any point $P$ in the plane of the triangle $ABC$.
We have

$$\overrightarrow{OI} = \frac{1}{2p}\left(a\overrightarrow{OA} + b\overrightarrow{OB} + c\overrightarrow{OC}\right)$$

We compute $\overrightarrow{OI} \times \overrightarrow{OI}$, and we obtain:

$$OI^2 = \frac{1}{4p^2}\left(a^2OA^2 + b^2OB^2 + c^2OC^2 + 2ab\overrightarrow{OA}\times\overrightarrow{OB} + 2bc\overrightarrow{OB}\times\overrightarrow{OC} + 2ca\overrightarrow{OC}\times\overrightarrow{OA}\right)$$

From the cosin's theorem applied in the triangle $OBC$ we get

$$\overrightarrow{OB}\times\overrightarrow{OC}=R^2-\frac{a^2}{2}$$

and the similar relations, which substituted in the relation for $OI^2$ we find

$$OI^2=\frac{1}{4p^2}\left(R^2\cdot 4p^2-abc\cdot 2p\right)$$

Because $abc=4Rs$ and $s=pr$ it results (i)

(ii) The position vector of the center $I_a$ of the A ex-inscribed circle is give by:

$$\overrightarrow{PI_a}=\frac{1}{2(p-a)}\left(-a\overrightarrow{PA}+b\overrightarrow{PB}+c\overrightarrow{PC}\right)$$

We have:

$$\overrightarrow{OI_a}=\frac{1}{2(p-a)}\left(-a\overrightarrow{OA}+b\overrightarrow{OB}+c\overrightarrow{OC}\right)$$

Computing $\overrightarrow{OI_a}\cdot\overrightarrow{OI_a}$ we obtain

$$\overrightarrow{OI_a}^2=R^2\cdot\frac{a^2+b^2+c^2}{2(p-a)^2}-\frac{ab}{2(p-a)^2}\overrightarrow{OA}\times\overrightarrow{OB}+\frac{bc}{2(p-a)^2}\overrightarrow{OB}\times\overrightarrow{OC}-\frac{ac}{2(p-a)^2}\overrightarrow{OA}\times\overrightarrow{OC}$$

Because $\overrightarrow{OB}\times\overrightarrow{OC}=R^2-\frac{a^2}{2}$ and $s=r_a(p-a)$, executing a simple computation we obtain the Feuerbach's relation.

(iii) In a triangle it is true the following relation

$$\overrightarrow{OH}=\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}$$

This is the Sylvester's relation.

We evaluate $\overrightarrow{OH}\times\overrightarrow{OH}$ and we obtain:

$$OH^2 = 9R^2-(a^2+b^2+c^2).$$

We'll prove that in a triangle we have:

$$ab+bc+ca=p^2+r^2+4Rr$$

and

$$a^2+b^2+c^2=2p^2-2r^2-8Rr$$

We obtain

$$\frac{s^2}{p}=(p-a)(p-b)(p-c)=-p^3+p(ab+bc+ca)-abc$$

Therefore

$$\frac{s^2}{p^2}=-p^2+ab+bc+ca-\frac{4Rs}{p}$$

We find that

$$ab+bc+ca=p^2+r^2+4Rr$$

Because

$$a^2+b^2+c^2=(a+b+c)^2-2(ab+bc+ca)$$

it results that

$$a^2+b^2+c^2=2p^2-2r^2-8Rr$$

which leads to (iii).

(iv) In the triangle $ABC$ we have

$$\overrightarrow{IH}=\overrightarrow{OH}-\overrightarrow{OI}$$

We compute $IH^2$, and we obtain:

$$IH^2=OH^2+OI^2-2\overrightarrow{OH}\cdot\overrightarrow{OI}$$

$$\overrightarrow{OH}\times\overrightarrow{OI}=\left(\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}\right)\cdot\frac{1}{2p}\left(a\overrightarrow{OA}+b\overrightarrow{OB}+c\overrightarrow{OC}\right)$$

$$\overrightarrow{OH}\times\overrightarrow{OI}=\frac{1}{2p}\left[R^2(a+b+c)+(a+b)\times\overrightarrow{OA}\times\overrightarrow{OB}+(b+c)\times\overrightarrow{OB}\times\overrightarrow{OC}+(c+a)\times\overrightarrow{OC}\times\overrightarrow{OA}\right]=$$

$$=3R^2-\frac{a^3+b^3+c^3}{2(a+b+c)}-\frac{a^2+b^2+c^2}{2}.$$

$$IH^2=4R^2-2Rr-\frac{a^3+b^3+c^3}{a+b+c}$$

To express $a^3+b^3+c^3$ in function of $p,r,R$ we'll use the identity:

$$a^3+b^3+c^3-3abc=(a+b+c)\left(a^2+b^2+c^2-ab-bc-ca\right).$$

and we obtain

$$a^3+b^3+c^3=2p\left(p^2-3r^2-6Rr\right)$$

Substituting in the expression of $IH^2$, we'll obtain the relation (iv)

(v) We have

$$\overrightarrow{HI_a}=\frac{1}{2(p-a)}\left(-a\overrightarrow{HA}+b\overrightarrow{HB}+c\overrightarrow{HC}\right)$$

We'll compute $\overrightarrow{HI}_a\times\overrightarrow{HI}_a$

$$HI_a{}^2=\frac{1}{4(p-a)^2}\left(a^2HA^2+b^2HB^2+c^2HC^2-2ab\overrightarrow{HA}\times\overrightarrow{HB}-2ac\overrightarrow{HA}\times\overrightarrow{HC}+2bc\overrightarrow{HB}\times\overrightarrow{HC}\right)$$

If $A_1$ is the middle point of $(BC)$ it is known that $\overrightarrow{AH}=2\overrightarrow{OA_1}$, therefore

$$AH^2=4R^2-a^2$$

also,

$$\overrightarrow{HA}\times\overrightarrow{HB}=\left(\overrightarrow{OB}+\overrightarrow{OC}\right)\left(\overrightarrow{OC}+\overrightarrow{OA}\right)$$

We obtain:

$$\overrightarrow{HA}\times\overrightarrow{HB}=4R^2-\frac{1}{2}\left(a^2+b^2+c^2\right)$$

Therefore

$$a^2+b^2+c^2=2\left(p^2-r^2-4Rr\right)$$

It results

$$\overrightarrow{HA}\times\overrightarrow{HB}=r^2-p^2+4R^2+4Rr$$

Similarly,

$$\overrightarrow{HB}\times\overrightarrow{HC}=\overrightarrow{HC}\times\overrightarrow{HA}=r^2-p^2+4R^2+4Rr$$

$$HI_a^{\;2}=\frac{1}{4(p-a)^2}\left[4R^2\left(a^2+b^2+c^2\right)-\left(a^4+b^4+c^4\right)+\left(r^2-p^2+4R^2+4Rr\right)\left(2bc-2ab-2ac\right)\right]$$

Because $b+c-a=2(p-a)$, it results

$$2bc-2ab-2ac=4(p-a)^2-\left(a^2+b^2+c^2\right)$$

$$HI_a^{\;2}=\frac{1}{4(p-a)^2}\left[\left(a^2+b^2+c^2\right)\left(p^2-r^2-4Rr\right)+4(p-a)^2\left(r^2-p^2+4R^2+4Rr\right)-\left(a^4+b^4+c^4\right)\right]$$

It is known that

$$16s^2=2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4$$

From which we find

$$a^2b^2+b^2c^2+c^2a^2=\left(ab+bc+ca\right)^2-2abc\left(a+b+c\right)=\left(r^2+p^2+4Rr\right)^2-4pabc$$

Substituting, and after several computations we obtain (v).

**Theorem (K. Feuerbach)**

In a given triangle the circle of the nine points is tangent to the inscribed circle and to the ex-inscribed circles of the triangle.

**Proof**

We apply the median's theorem in the triangle $OIH$ and we obtain

$$4IO_9^2=2\left(OI^2+IH^2\right)-OH^2$$

We substitute $OI^2, IH^2, OH^2$ with the obtained formulae in function of $r, R, p$ and after several simple computations we'll obtain

$$IO_9=\frac{R}{2}-r$$

This relation shows that the circle of the nine points (which has the radius $\frac{R}{2}$) is tangent to inscribed circle.

We apply the median's theorem for the triangle $OI_aH$, and we obtain

$$4I_aO_9^2=2\left(OI_a^{\;2}+I_aH^2\right)-OH^2$$

We substitute $OI_a, I_aH, OH$ and we'll obtain

$$I_aO_9 = \frac{R}{2} + r_a$$

This relation shows that the circle of the nine points and the A- ex-inscribed circle are tangent in exterior.

**Note**

In an article published in the Gazeta Matematică, no. 4, from 1982, the late Romanian Professor Laurențiu Panaitopol asked for the finding of the strongest inequality of the type $kR^2 + hr^2 \geq a^2 + b^2 + c^2$ and proves that this inequality is

$$8R^2 + 4r^2 \geq a^2 + b^2 + c^2.$$

Taking into consideration that

$$IH^2 = 4R^2 + 2r^2 - \frac{a^2 + b^2 + c^2}{2}$$

and that $IH^2 \geq 0$ we re-find this inequality and its geometrical interpretation.

**References**

[1] Claudiu Coandă, Geometrie analitică în coordonate baricentrice, Editura Reprograf, Craiova, 1997.

[2] Dan Sachelarie, Geometria triunghiului, Anul 2000, Editura Matrix Rom, Bucureşti , 2000.

# The Duality and the Euler's Line

**Prof. Ion Pătraşcu – The Fraţii Buzeşti National College, Craiova - Romania**
**Prof. Florentin Smarandache – University of New Mexico – U.S.A.**

In this article we'll discuss about a theorem which results from a duality transformation of a theorem and the configuration in relation to the Euler's line.

**Theorem**

Let $ABC$ a given random triangle, $I$ the center of its inscribed circle, and $A'B'C'$ its triangle of contact. The perpendiculars constructed in $I$ on $AI, BI, CI$ intersect $BC, CA, AB$ respectively in the points $A_1, B_1, C_1$. The medians of the triangle of contact intersect the second time the inscribed circle in the points $A_1', B_1', C_1'$, and the tangents in these points to the inscribed circle intersect the lines $BC, CA, AB$ in the points $A_2, B_2, C_2$ respectively.

Then:

i) The points $A_1, B_1, C_1$ are collinear;

ii) The points $A_2, B_2, C_2$ are collinear;

iii) The lines $A_1B_1, A_2B_2$ are parallel.

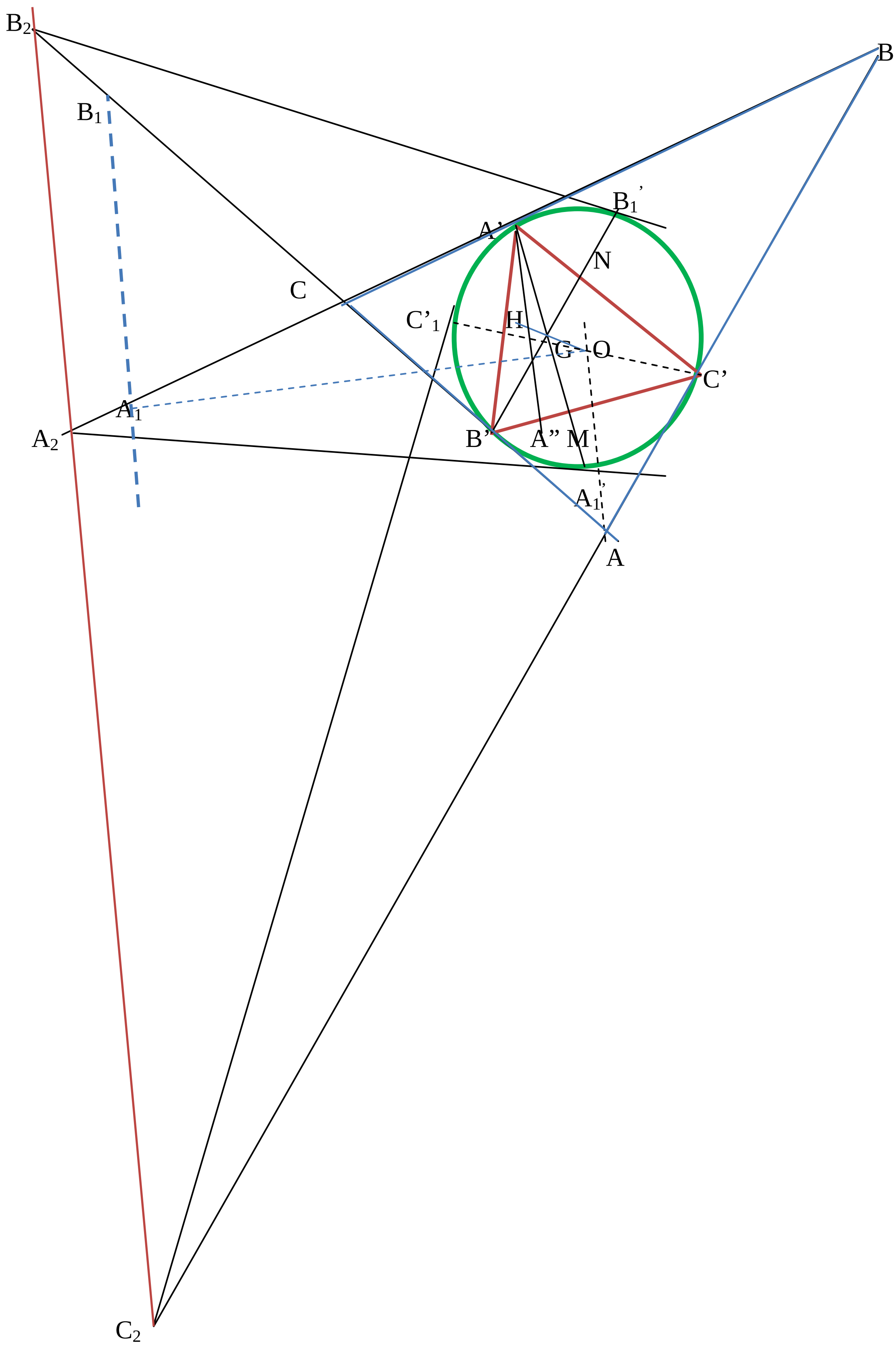
**Proof**

We'll consider a triangle $A'B'C'$ circumscribed to the circle of center $O$. Let $A'A'', B'B'', C'C''$ its heights concurrent in a point $H$ and $A'M, B'N, C'P$ its medians concurrent in the weight center $G$. It is known that the points $O, H, G$ are collinear; these are situated on the Euler's line of the triangle $A'B'C'$.

We'll transform this configuration (see the figure) through a duality in rapport to the circumscribed circle to the triangle $A'B'C'$.

To the points $A', B', C'$ correspond the tangents in $A', B', C'$ to the given circle, we'll note $A, B, C$ the points of intersection of these tangents. For triangle $ABC$ the circle $A'B'C'$ becomes inscribed circle, and $A'B'C'$ is the triangle of contact of $ABC$.

To the mediators $A'M, B'N, C'P$ will correspond through the considered duality, their pols, that is the points $A_2, B_2, C_2$ obtained as the intersections of the lines $BC, CA, AB$ with the tangents in the points $A_1', B_1', C_1'$ respectively to the circle $A'B'C'$ ($A_1', B_1', C_1'$ are the intersection points with the circle $A'B'C'$ of the lines $(A'M, (B'N, (C'P)$. To the height $A'M$ corresponds its pole noted $A_1$ situated on $BC$ such that $m\left(\widehat{AOA_1}\right) = 90°$ (indeed the pole of $B'C'$ is the point $A$ and because $A'M \perp B'C'$ we have $m\left(\widehat{AOA_1}\right) = 90°$), similarly to the height $B'N$ we'll correspond the point $B_1$ on $AC$ such that $m\left(\widehat{BOB_1}\right) = 90°$, and to the height $C'N$ will correspond the point $C_1$ on $AB$ such that $m\left(\widehat{COC_1}\right) = 90°$.



B2
B
B1
B1'
A'
N
C
C'1
H
G
O
C'
A1
A2
B'
A"
M
A1'
A
C2

Because the heights are concurrent in $H$ it means that their poles, that is the points $A_1, B_1, C_1$ are collinear.

Because the medians are concurrent in the point $G$ it means that their poles, that is the points $A_2, B_2, C_2$ are collinear.

The lines $A_1B_1C_1$ and $A_2B_2C_2$ are respectively the poles of the points $H$ and $G$, because $H, G$ are collinear with the point $O$; this means that these poles are perpendicular lines on $OG$ respectively on $OH$; consequently these are parallel lines.

By re-denoting the point $O$ with $I$ we will be in the conditions of the propose theorem and therefore the proof is completed.

**Note**

This theorem can be proven also using an elementary method. We'll leave this task for the readers.

# Two Applications of Desargues' Theorem

Prof. Florentin Smarandache, University of New Mexico, U.S.A.
Prof. Ion Pătraşcu, The National College "Fraţii Buzeşti", Craiova, Romania

In this article we will use the Desargues' theorem and its reciprocal to solve two problems.

For beginning we will enunciate and prove Desargues' theorem:

**Theorem 1** (G.Desargues, 1636, the famous "perspective theorem": When two triangles are in perspective, the points where the corresponding sides meet are collinear.)

Let two triangle $ABC$ and $A_1B_1C_1$ be in a plane such that $AA_1 \cap BB_1 \cap CC_1 = \{O\}$,

$$AB \cap A_1B_1 = \{N\}$$

$$BC \cap B_1C_1 = \{M\}$$

$$CA \cap C_1A_1 = \{P\}$$

then the points $N$, $M$, $P$ are collinear.

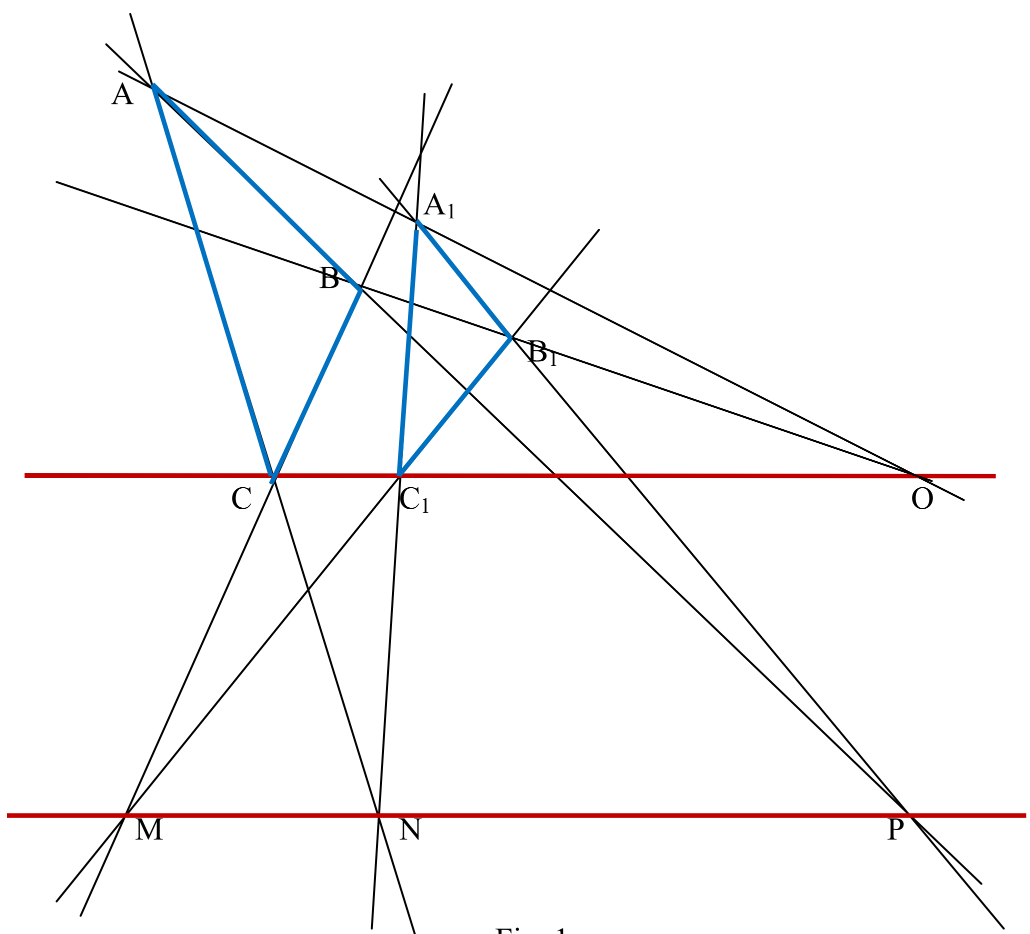



Fig. 1

**Proof**

Let $\{O\} = AA_1 \cap BB_1 \cap CC_1$, see Fig.1.. We'll apply the Menelaus' theorem in the triangles $OAC$; $OBC$; $OAB$ for the transversals $N, A_1, C_1$; $M, B_1, C_1$; $P, B_1, A_1$, and we obtain

$$\frac{NA}{NC} \cdot \frac{C_1C}{C_1O} \cdot \frac{A_1O}{A_1A} = 1 \quad (1)$$

$$\frac{MC}{MB} \cdot \frac{B_1B}{B_1O} \cdot \frac{C_1O}{C_1C} = 1 \quad (2)$$

$$\frac{PB}{PA} \cdot \frac{B_1O}{B_1B} \cdot \frac{A_1A}{A_1O} = 1 \quad (3)$$

By multiplying the relations (1), (2), and (3) side by side we obtain

$$\frac{NA}{NC} \cdot \frac{MC}{MB} \cdot \frac{PB}{PA} = 1.$$

This relation, shows that $N$, $M$, $P$ are collinear (in accordance to the Menealaus' theorem in the triangle $ABC$ ).

**Remark 1**

The triangles $ABC$ and $A_1B_1C_1$ with the property that $AA_1, BB_1, CC_1$ are concurrent are called homological triangles. The point of concurrency point is called the homological point of the triangles. The line constructed through the intersection points of the homological sides in the homological triangles is called the triangles' axes of homology.

**Theorem 2** (The reciprocal of the Desargues' theorem)

` If two triangles $ABC$ and $A_1B_1C_1$ are such that

$$AB \cap A_1B_1 = \{N\}$$

$$BC \cap B_1C_1 = \{M\}$$

$$CA \cap C_1A_1 = \{P\}$$

And the points $N$, $M$, $P$ are collinear, then the triangles $ABC$ and $A_1B_1C_1$ are homological.

**Proof**

We'll use the reduction ad absurdum method .

Let

$$AA_1 \cap BB_1 = \{O\}$$

$$AA_1 \cap CC_1 = \{O_1\}$$

$$BB_1 \cap CC_1 = \{O_2\}$$

We suppose that $O \neq O_1 \neq O_2 \neq O_3$.

The Menelaus' theorem applied in the triangles $OAB$, $O_1AC$, $O_2BC$ for the transversals $N, A_1, B_1$; $P, A_1, C_1$; $M, B_1, C_1$, gives us the relations

$$\frac{NB}{NA} \cdot \frac{B_1O}{B_1B} \cdot \frac{AA_1}{A_1O} = 1 \quad (4)$$

$$\frac{PA}{PC}\cdot\frac{A_1O_1}{A_1O}\cdot\frac{C_1C}{C_1O_1}=1 \quad (5)$$

$$\frac{MC}{MB}\cdot\frac{B_1B}{B_1O}\cdot\frac{C_1O_2}{C_1C}=1 \quad (6)$$

Multiplying the relations (4), (5), and (6) side by side, and taking into account that the points $N$, $M$, $P$ are collinear, therefore

$$\frac{PA}{PC}\cdot\frac{MC}{MB}\cdot\frac{NB}{NA}=1 \quad (7)$$

We obtain that

$$\frac{A_1O_1}{A_1O}\cdot\frac{B_1O}{B_1O_2}\cdot\frac{C_1O_2}{C_1O_2}=1 \quad (8)$$

The relation (8) relative to the triangle $A_1B_1C_1$ shows, in conformity with Menelaus' theorem, that the points $O, O_1, O_2$ are collinear. On the other hand the points $O, O_1$ belong to the line $AA_1$, it results that $O_2$ belongs to the line $AA_1$. Because $BB_1 \cap CC_1 = \{O_2\}$, it results that $\{O_2\} = AA_1 \cap BB_1 \cap CC_1$, and therefore $O_2 = O_1 = O$, which contradicts the initial supposition.

**Remark 2**

The Desargues' theorem is also known as the theorem of the homological triangles.

**Problem 1**

If $ABCD$ is a parallelogram, $A_1 \in (AB), B_1 \in (BC), C_1 \in (CD), D_1 \in (DA)$ such that the lines $A_1D_1, BD, B_1C_1$ are concurrent, then:

a) The lines $AC, A_1C_1$ and $B_1D_1$ are concurrent

b) The lines $A_1B_1, C_1D_1$ and $AC$ are concurrent.

**Solution**

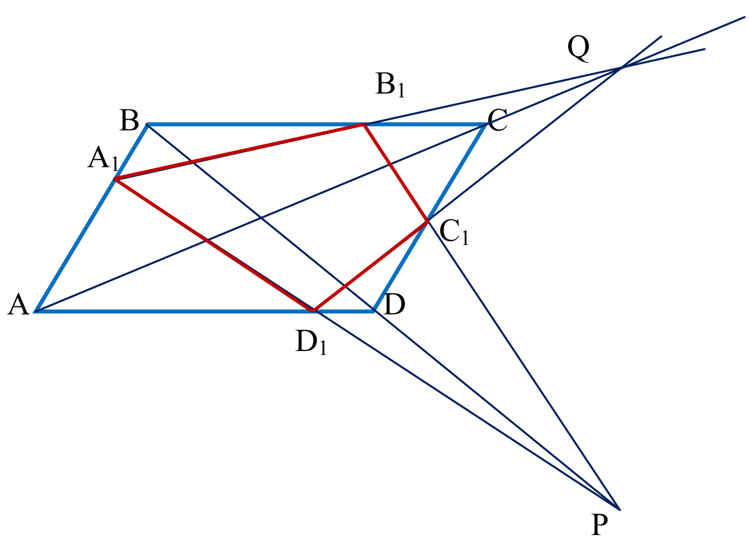



Fig. 2

Let $\{P\} = A_1D_1 \cap B_1C_1 \cap BD$ see Fig. 2. We observe that the sides $A_1D_1$ and $B_1C_1$; $CC_1$ and $AD_1$; $A_1A$ and $CB_1$ of triangles $AA_1D_1$ and $CB_1C_1$ intersect in the collinear points $P, B, D$. Applying the reciprocal theorem of Desargues it results that these triangles are homological, that is, the lines: $AC, A_1C_1$ and $B_1D_1$ are collinear.

Because $\{P\} = A_1D_1 \cap B_1C_1 \cap BD$ it results that the triangles $DC_1D_1$ and $BB_1A_1$ are homological. From the theorem of the of homological triangles we obtain that the homological lines
$DC_1$ and $BB_1$**;** $DD_1$ and $BA_1$**;** $D_1C_1$ and $A_1B_1$ intersect in three collinear points, these are $C, A, Q$, where $\{Q\} = D_1C_1 \cap A_1B_1$. Because $Q$ is situated on $AC$ it results that $A_1B_1, C_1D_1$ and $AC$ are collinear.

**Problem 2**

Let $ABCD$ a convex quadrilateral such that

$$AB \cap CB = \{E\}$$
$$BC \cap AD = \{F\}$$
$$BD \cap EF = \{P\}$$
$$AC \cap EF = \{R\}$$
$$AC \cap BD = \{O\}$$

We note with $G, H, I, J, K, L, M, N, Q, U, V, T$ respectively the middle points of the segments: $(AB),(BF),(AF),(AD),(AE),(DE),(CE),(BE),(BC),(CF),(DF),(DC)$. Prove that

i) The triangle $POR$ is homological with each of the triangles: $GHI,\ JKL,\ MNQ,\ UVT$.
ii) The triangles $GHI$ and $JKL$ are homological.
iii) The triangles $MNQ$ and $UVT$ are homological.
iv) The homology centers of the triangles $GHI,\ JKL,\ POR$ are collinear.
v) The homology centers of the triangles $MNQ,\ UVT,\ POR$ are collinear.

**Solution**

i) when proving this problem we must observe that the $ABCDEF$ is a complete quadrilateral and if $O_1, O_2, O_3$ are the middle of the diagonals $(AC),(BD)$ respective $EF$, these point are collinear. The line on which the points $O_1, O_2, O_3$ are located is called the Newton-Gauss line [* for complete quadrilateral see [1]].

The considering the triangles $POR$ and $GHI$ we observe that $GI \cap OR = \{O_1\}$ because $GI$ is the middle line in the triangle $ABF$ and then it contains the also the middle of the segment $(AC)$, which is $O_1$. Then $HI \cap PR = \{O_3\}$ because $HI$ is middle line in the triangle $AFB$ and $O_3$ is evidently on the line $PR$ also. $GH \cap PO = \{O_2\}$ because $GH$ is middle line in the triangle $BAF$ and then it contains also $O_2$ the middle of the segment $(BD)$.

The triangles $GIH$ and $ORP$ have as intersections of the homological lines the collinear points $O_1, O_2, O_3$, according to the reciprocal theorem of Desargues these are homological.

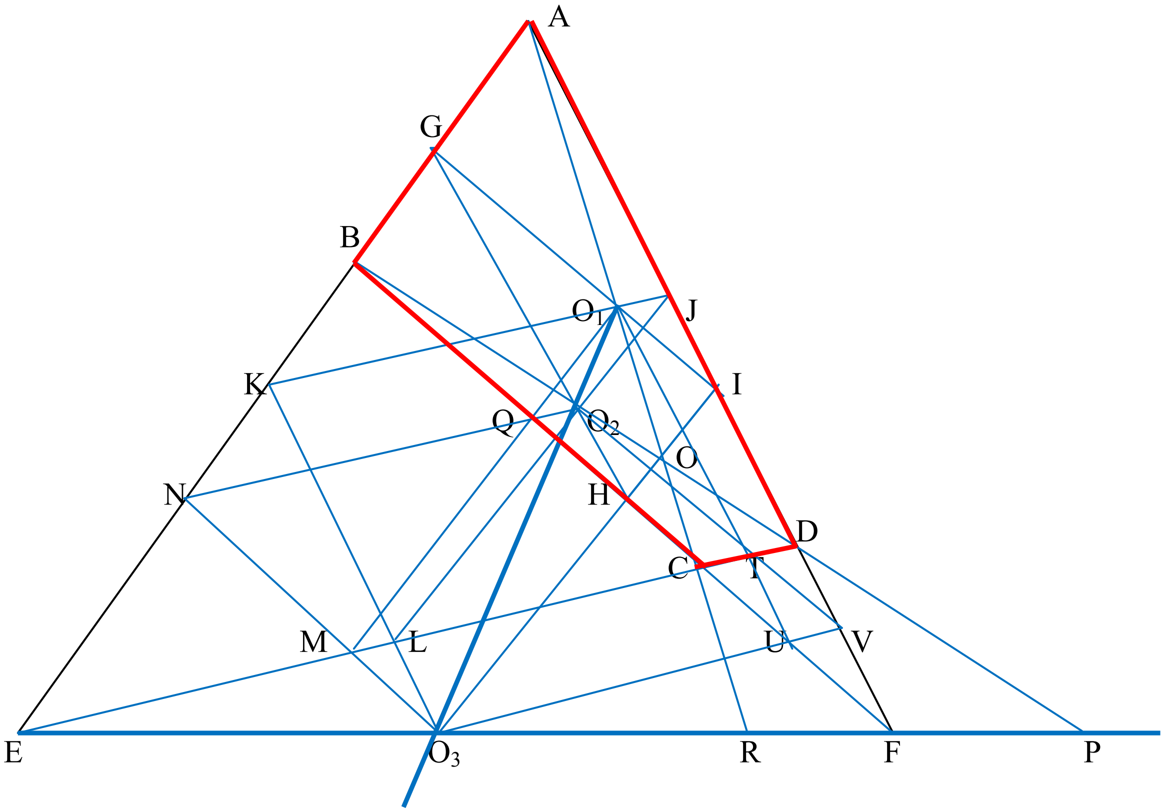



Fig. 3

Similarly, we can show that the triangle $ORP$ is homological with the triangles $JKL$, $MNQ$, and $UVT$ (the homology axes will be $O_1, O_2, O_3$).

ii) We observe that

$$GI \cap JK = \{O_1\}$$

$$GH \cap JL = \{O_2\}$$

$$HI \cap KL = \{O_3\}$$

then $O_1, O_2, O_3$ are collinear and we obtain that the triangles $GIH$ and $JKL$ are homological

iii) Analog with ii)

iv) Apply the Desargues' theorem. If three triangles are homological two by two, and have the same homological axes then their homological centers are collinear.

v) Similarly with iv).

**Remark 3**

The precedent problem could be formulates as follows:

The four medial triangles of the four triangles determined by the three sides of a given complete quadrilateral are, each of them, homological with the diagonal triangle of the complete quadrilateral and have as a common homological axes the Newton-Gauss line of the complete quadrilateral.

We mention that:

- The *medial triangle* of a given triangle is the triangle determined by the middle points of the sides of the given triangle (it is also known as the complementary triangle).
- The *diagonal triangle* of a complete quadrilateral is the triangle determined by the diagonals of the complete quadrilateral.

We could add the following comment:

Considering the four medial triangles of the four triangles determined by the three sides of a complete quadrilateral, and the diagonal triangle of the complete quadrilateral, we could select only two triplets of triangles homological two by two. Each triplet contains the diagonal triangle of the quadrilateral, and the triplets have the same homological axes, namely the Newton-Gauss line of the complete quadrilateral.

**Open problems**

1. What is the relation between the lines that contain the homology centers of the homological triangles' triplets defined above?
2. Desargues theorem was generalized in [2] in the following way: Let's consider the points $A_1,\ldots,A_n$ situated on the same plane, and $B_1,\ldots,B_n$ situated on another plane, such that the lines $A_iB_i$ are concurrent. Then if the lines $A_iA_j$ and $B_iB_j$ are concurrent, then their intersecting points are collinear.
   Is it possible to generalize Desargues Theorem for two polygons both in the same plane?
3. What about Desargues Theorem for polyhedrons?

**References**


[1] Roger A. Johnson – Advanced Euclidean Geometry – Dovos Publications, Inc. Mineola, New York, 2007.

[2] F. Smarandache, Generalizations of Desargues Theorem, in "Collected Papers", Vol. I, p. 205, Ed. Tempus, Bucharest, 1998.

# An Application of Sondat's Theorem

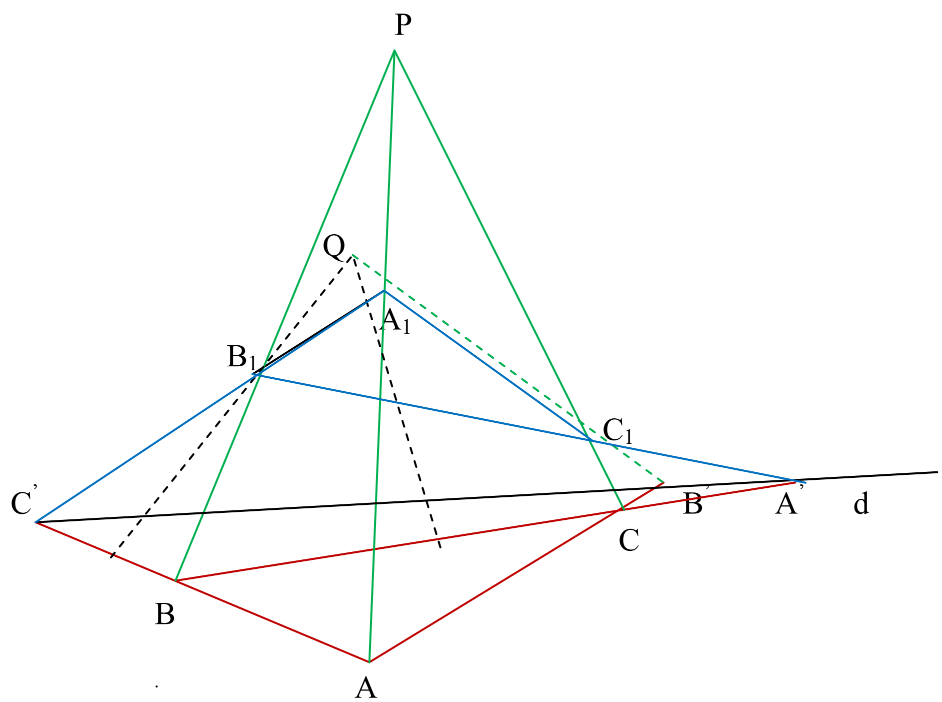
# Regarding the Ortho-homological Triangles

Ion Pătraşcu, "Fraţii Buzeşti" National College, Craiova, Romania;
Florentin Smarandache, ENSIETA, Brest, France.

In this article we prove the Sodat's theorem regarding the ortho-homogolgical triangle and then we use this theorem along with Smarandache-Pătraşcu theorem to obtain another theorem regarding the ortho-homological triangles.

**Theorem (P. Sondat)**

Consider the ortho-homological triangles $ABC$, $A_1B_1C_1$. We note $Q$, $Q_1$ their orthological centers, $P$ the homology center and $d$ their homological axes. The points $P$, $Q$, $Q_1$ belong to a line that is perpendicular on $d$



**Proof.**

Let $Q$ the orthologic center of the $ABC$ the $A_1B_1C_1$ (the intersection of the perpendiculars constructed from $A_1,\ B_1,\ C_1$ respectively on $BC,\ CA,\ AB$), and $Q_1$ the other orthologic center of the given triangle.

We note $\{B'\} = CA \cap C_1A_1$, $\{A'\} = BC \cap B_1C_1$, $\{C'\} = AB \cap A_1B_1$.

We will prove that $PQ \perp d$ which is equivalent to

$$B'P^2 - B'Q^2 = C'P^2 - C'Q^2 \tag{1}$$

We have that

$$\overrightarrow{PA_1} = \alpha\overrightarrow{A_1A},\ \overrightarrow{PB_1} = \beta\overrightarrow{B_1B},\ \overrightarrow{PC_1} = \gamma\overrightarrow{C_1C}$$

From Menelaus' theorem applied in the triangle $PAC$ relative to the transversals $B',\ C_1,\ A_1$ we obtain that

$$\frac{B'C}{B'A} = \frac{\alpha}{\gamma} \tag{2}$$

The Stewart's theorem applied in the triangle $PAB'$ implies that

$$PA^2 \cdot CB' + PB'^2 \cdot AC - PC^2 \cdot AB' = AC \cdot CB' \cdot AB' \tag{3}$$

Taking into account (2), we obtain:

$$\gamma PC^2 - \alpha PA^2 = (\gamma - \alpha)PB'^2 - \alpha B'A^2 + \gamma B'C^2 \tag{4}$$

Similarly, we obtain:

$$\gamma QC^2 - \alpha QA^2 = (\gamma - \alpha)QB'^2 + \gamma B'C^2 - \alpha B'A^2 \tag{5}$$

Subtracting the relations (4) and (5) and using the notations:

$$PA^2 - QA^2 = u,\ PB^2 - QB^2 = v,\ PC^2 - QC^2 = t$$

we obtain:

$$PB'^2 - QB'^2 = \frac{\gamma t - \alpha u}{\gamma - \alpha} \tag{6}$$

The Menelaus' theorem applied in the triangle $PAB$ for the transversal $C',\ B,\ A_1$ gives

$$\frac{C'B}{C'A} = \frac{\alpha}{\beta} \tag{7}$$

From the Stewart's theorem applied in the triangle $PC'A$ and the relation (7) we obtain:

$$\alpha PA^2 - \beta PB^2 = (\alpha - \beta)C'P^2 + \alpha C'A^2 - \beta C'B^2 \tag{8}$$

Similarly, we obtain:

$$\alpha QA^2 - \beta QB^2 = (\alpha - \beta)C'Q^2 + \alpha C'A^2 - \beta C'B^2 \tag{9}$$

From (8) and (9) it results

$$C'P^2 - C'Q^2 = \frac{\alpha u - \beta v}{\alpha - \beta} \tag{10}$$

The relation (1) is equivalent to:

$$\alpha\beta(u-v)+\beta\gamma(v-t)+\gamma\alpha(t-u)=0 \tag{11}$$

To prove relation (11) we will apply first the Stewart theorem in the triangle $CAP$, and we obtain:

$$CA^2\cdot PA_1+PC^2\cdot A_1A-CA_1^2\cdot PA=PA_1\cdot A_1A\cdot PA \tag{12}$$

Taking into account the previous notations, we obtain:

$$\alpha CA^2+PC^2-CA_1^2(1+\alpha)=PA_1^2+\alpha A_1A^2 \tag{13}$$

Similarly, we find:

$$\alpha BA^2+PB^2-BA_1^2(1+\alpha)=PA_1^2+\alpha A_1A^2 \tag{14}$$

From the relations (13) and (14) we obtain:

$$\alpha BA^2-\alpha CA^2+PB^2-PC^2-(1+\alpha)\left(BA_1^2-CA_1^2\right)=0 \tag{15}$$

Because $A_1Q\perp BC$, we have that $BA_1^2-CA_1^2=QB^2-QC^2$, which substituted in relation (15) gives:

$$BA^2-CA^2+QC^2-QB^2=\frac{t-v}{\alpha} \tag{16}$$

Similarly, we obtain the relations:

$$CB^2-AB^2+QA^2-QC^2=\frac{u-t}{\beta} \tag{17}$$

$$AC^2-BC^2+QB^2-QA^2=\frac{v-u}{\gamma} \tag{18}$$

By adding the relations (16), (17) and (18) side by side, we obtain

$$\frac{t-v}{\alpha}+\frac{u-t}{\beta}+\frac{v-u}{\gamma}=0 \tag{19}$$

The relations (19) and (11) are equivalent, and therefore, $PQ\perp d$, which proves the Sondat's theorem.

**Theorem (Smarandache – Pătraşcu)**

Consider triangle $ABC$ and the inscribed triangle $A_1B_1C_1$ ortho-homological, $Q$, $Q_1$ their centers of orthology, $P$ the homology center and $d$ their homology axes. If $A_2B_2C_2$ is the podar triangle of $Q_1$, $P_1$ is the homology center of triangles $ABC$ and $A_2B_2C_2$, and $d_1$ their homology axes, then the points $P$, $Q$, $Q_1$, $P_1$ are collinear and the lines $d$ and $d_1$ are parallel

**Proof.**

Applying the Sondat's theorem to the ortho-homological triangle $ABC$ and $A_1B_1C_1$, it results that the points $P$, $Q$, $Q_1$ are collinear and their line is perpendicular on $d$. The same theorem applied to triangles $ABC$ and $A_2B_2C_2$ shows the collinearity of the points $P_1$, $Q$, $Q_1$, and the conclusion that their line is perpendicular on $d_1$.

From these conclusions we obtain that the points $P$, $Q$, $Q_1$, $P_1$ are collinear and the parallelism of the lines $d$ and $d_1$.

**References**

[1] Cătălin Barbu – Teoreme Fundamentale din Geometria Triunghiului, Editura Unique, Bacău, 2008.
[2] Florentin Smarandache – Multispace & Multistructure Neutro-sophic Transdisciplinarity (100 Collected Papers of Sciences), Vol. IV. North-European Scientific Publishers Hanko, Finland, 2010.

## Another proof of a theorem relative to the orthological triangles

Ion Pătraşcu – National College "Fraţii Buzeşti, Craiova, Romania
Florentin Smarandache – New Mexico University, U.S.A.

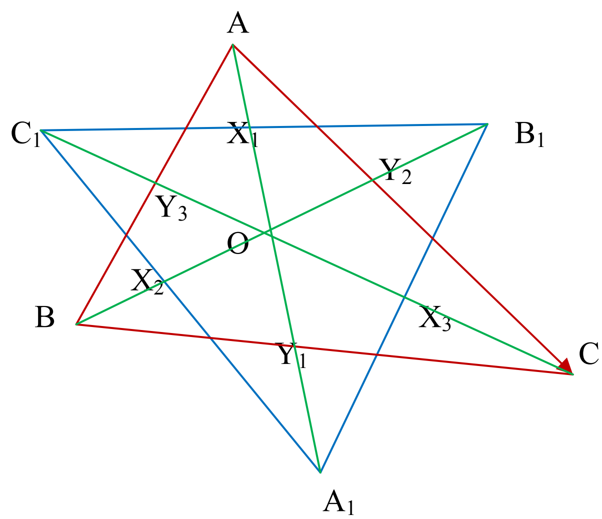
In [1] we proved, using barycentric coordinates, the following theorem:

**Theorem:** (generalization of the C. Coşniţă theorem)
If $P$ is a point in the triangle's $ABC$ plane, which is not on the circumscribed triangle, $A'B'C'$ is its pedal triangle and $A_1, B_1, C_1$ three points such that

$$\overrightarrow{PA'}\cdot\overrightarrow{PA_1} = \overrightarrow{PB'}\cdot\overrightarrow{PB_1} = \overrightarrow{PC'}\cdot\overrightarrow{PC_1} = k, \quad k \in R^*,$$

then the lines $AA_1$, $BB_1$, $CC_1$ are concurrent.
Bellow, will prove, using this theorem, the following:

**Theorem**
If the triangles $ABC$ and $A_1B_1C_1$ are orthological and their orthological centers coincide, then the lines $AA_1$, $BB_1$, $CC_1$ are concurrent (the triangles $ABC$ and $A_1B_1C_1$ are homological).
**Proof:**
Let $O$ be the unique orthological center of the triangles $ABC$ and $A_1B_1C_1$ and




$$\{X_1\} = AO \cap B_1C_1$$
$$\{X_2\} = BO \cap A_1C_1$$
$$\{X_3\} = CO \cap A_1B_1$$

We denote

$$\{Y_1\} = OA_1 \cap BC$$
$$\{Y_2\} = OB_1 \cap AC$$
$$\{Y_3\} = OC_1 \cap AB$$

We observe that $\sphericalangle OAY_3 = \sphericalangle OC_1X_1$ (angles with perpendicular sides).

Therefore:

$$\sin OAY_3 = \frac{OY_3}{OA}$$
$$\sin OC_1X_1 = \frac{OX_1}{OC_1},$$

then

$$OX_1 \cdot OA = OY_3 \cdot OC_1 \tag{1}$$

Also

$$\sphericalangle OC_1X_2 = \sphericalangle OBY_3$$

therefore

$$\sin OC_1X_2 = \frac{OX_2}{OC_1}$$
$$\sin OBY_3 = \frac{OY_3}{OB}$$

and consequently:

$$OX_2 \cdot OB = OY_3 \cdot OC_1 \tag{2}$$

Following the same path:

$$\sin OA_1X_2 = \frac{OX_2}{OC_1} = \sin OBY_1 = \frac{OY_1}{OB}$$

from which

$$OX_2 \cdot OB = OA_1 \cdot OY_1 \tag{3}$$

Finally

$$\sin OA_1X_3 = \frac{OX_3}{OA_1} = \sin OCY_1 = \frac{OY_1}{OC}$$

from which:

$$OX_3 \cdot OC = OA_1 \cdot OY_1 \tag{4}$$

The relations (1), (2), (3), (4) lead to

$$OX_1 \cdot OA = OX_2 \cdot OB = OX_3 \cdot OC \tag{5}$$

From (5) using the Coşniță's generalized theorem, it results that $A_1A,\ B_1B,\ C_1C$ are concurrent.

**<u>Observation:</u>**

If we denote $P$ the homology center of the triangles $ABC$ and $A_1B_1C_1$ and $d$ is the intersection of their homology axes, them in conformity with the Sondat's theorem, it results that $OP \perp d$.

**References:**


[1] Ion Pătraşcu – Generalizarea teoremei lui Coşniță – Recreații Matematice, An XII, nr. 2/2010, Iaşi, Romania

[2] Florentin Smarandache – Multispace & Multistructure, Neutrosophic Transdisciliniarity, 100 Collected papers of science, Vol. IV, 800 p., North-European Scientific Publishers, Honka, Finland, 2010.

# Two Triangles with the Same Orthocenter and a Vectorial Proof of Stevanovic's Theorem

Prof. Ion Pătraşcu – The National College "Fraţii Buzeşti", Craiova, Romania
Prof. Florentin Smarandache – University of New Mexico, U.S.A.

**Abstract.** In this article we'll emphasize on two triangles and provide a vectorial proof of the fact that these triangles have the same orthocenter. This proof will, further allow us to develop a vectorial proof of the Stevanovic's theorem relative to the orthocenter of the Fuhrmann's triangle.

**Lemma 1**

Let $ABC$ an acute angle triangle, $H$ its orthocenter, and $A', B', C'$ the symmetrical points of $H$ in rapport to the sides $BC, CA, AB$.

We denote by $X, Y, Z$ the symmetrical points of $A, B, C$ in rapport to $B'C', C'A', A'B'$ The orthocenter of the triangle $XYZ$ is $H$.

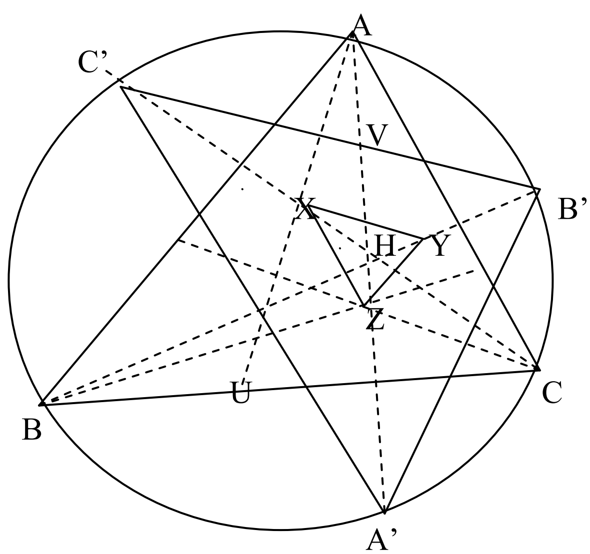



Fig. 1

**Proof**

We will prove that $XH \perp YZ$, by showing that $\overrightarrow{XH} \cdot \overrightarrow{YZ} = 0$.
We have (see Fig.1)

$$\overrightarrow{VH} = \overrightarrow{AH} - \overrightarrow{AX}$$
$$\overrightarrow{BC} = \overrightarrow{BY} + \overrightarrow{YZ} + \overrightarrow{ZC},$$

from here

$$\overrightarrow{YZ} = \overrightarrow{BC} - \overrightarrow{BY} - \overrightarrow{ZC}$$

Because $Y$ is the symmetric of $B$ in rapport to $A'C'$ and $Z$ is the symmetric of $C$ in rapport to $A'B'$, the parallelogram's rule gives us that:

$$\overrightarrow{BY} = \overrightarrow{BC'} + \overrightarrow{BA'}$$
$$\overrightarrow{CZ} = \overrightarrow{CB'} + \overrightarrow{CA'}.$$

Therefore
$$\overrightarrow{YZ} = \overrightarrow{BC} - \left(\overrightarrow{BC'} + \overrightarrow{BA'}\right) + \overrightarrow{B'C} + \overrightarrow{A'C}$$

But
$$\overrightarrow{BC'} = \overrightarrow{BH} + \overrightarrow{HC'}$$
$$\overrightarrow{BA'} = \overrightarrow{BH} + \overrightarrow{HA'}$$
$$\overrightarrow{CB'} = \overrightarrow{CH} + \overrightarrow{HB'}$$
$$\overrightarrow{CA'} = \overrightarrow{CH} + \overrightarrow{HA'}$$

By substituting these relations in the $\overrightarrow{YZ}$, we find:
$$\overrightarrow{YZ} = \overrightarrow{BC} + \overrightarrow{C'B'}$$

We compute
$$\overrightarrow{XH} \cdot \overrightarrow{YZ} = \left(\overrightarrow{AH} - \overrightarrow{AX}\right) \cdot \left(\overrightarrow{BC} + \overrightarrow{C'B'}\right) = \overrightarrow{AX} \cdot \overrightarrow{BC} + \overrightarrow{AH} \cdot \overrightarrow{C'B'} - \overrightarrow{AX} \cdot \overrightarrow{BC} - \overrightarrow{AX} \cdot \overrightarrow{C'B'}$$

Because
$$AH \perp BC$$
we have
$$\overrightarrow{AH} \cdot \overrightarrow{BC} = 0,$$
also
$$AX \perp B'C'$$
and therefore
$$\overrightarrow{AX} \cdot \overrightarrow{B'C'} = 0.$$

We need to prove also that
$$\overrightarrow{XH} \cdot \overrightarrow{YZ} = \overrightarrow{AH} \cdot \overrightarrow{C'B'} - \overrightarrow{AX} \cdot \overrightarrow{BC}$$

We note:
$$\{U\} = AX \cap BC \text{ and } \{V\} = AH \cap B'C'$$
$$\overrightarrow{AX} \cdot \overrightarrow{BC} = AX \cdot BC \cdot cox\sphericalangle\left(AX, BC\right) = AX \cdot BC \cdot cox\left(\sphericalangle AUC\right)$$
$$\overrightarrow{AH} \cdot \overrightarrow{C'B'} = AH \cdot C'B' \cdot cox\sphericalangle\left(AH, C'B'\right) = AH \cdot C'A' \cdot cox\left(\sphericalangle AVC'\right)$$

We observe that

$\sphericalangle AUC \equiv \sphericalangle AVC'$ (angles with the sides respectively perpendicular).

The point $B'$ is the symmetric of $H$ in rapport to $AC$, consequently
$$\sphericalangle HAC \equiv \sphericalangle CAB',$$
also the point $C'$ is the symmetric of the point $H$ in rapport to $AB$, and therefore
$$\sphericalangle HAB \equiv \sphericalangle BAC'.$$
From these last two relations we find that
$$\sphericalangle B'AC' = 2\sphericalangle A.$$

The sinus theorem applied in the triangles $AB'C'$ and $ABC$ gives:
$$B'C' = 2R \cdot \sin 2A$$
$$BC = 2R \sin A$$

We'll show that

$$AX \cdot BC = AH \cdot C'B',$$

and from here

$$AX \cdot 2R\sin A = AH \cdot 2R \cdot \sin 2A$$

which is equivalent to

$$AX = 2AH\cos A$$

We noticed that

$$\sphericalangle B'AC' = 2A,$$

Because

$$AX \perp B'C',$$

it results that

$$\sphericalangle TAB \equiv \sphericalangle A,$$

we noted $\{T\} = AX \cap B'C'$.

On the other side

$$AC' = AH,\ \ AT = \frac{1}{2}AY\ ,$$

and

$$AT = AC'\cos A = AH\cos A,$$

therefore

$$\overrightarrow{XH} \cdot \overrightarrow{YZ} = 0.$$

Similarly, we prove that

$$YH \perp XZ,$$

and therefore $H$ is the orthocenter of triangle $XYZ$.

**Lemma 2**

Let $ABC$ a triangle inscribed in a circle, $I$ the intersection of its bisector lines, and $A', B', C'$ the intersections of the circumscribed circle with the bisectors $AI, BI, CI$ respectively.

The orthocenter of the triangle $A'B'C'$ is $I$.

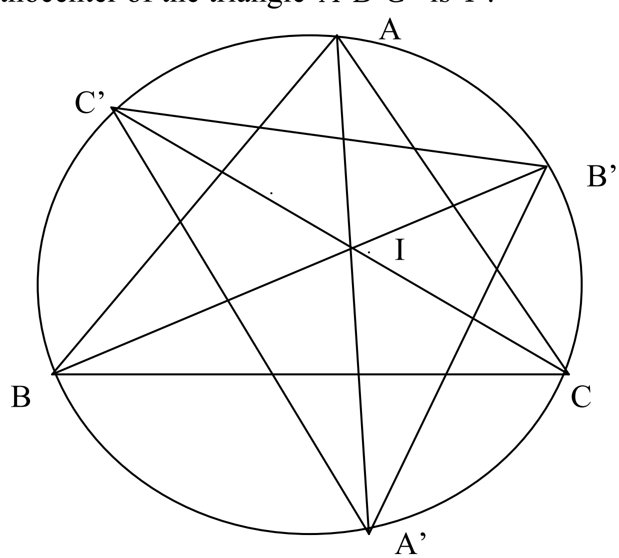



Fig. 2

**Proof**

We'll prove that $A'I \perp B'C'$.

Let

$$\alpha = m\left(\overset{\frown}{A'C}\right) = m\left(\overset{\frown}{A'B}\right),$$

$$\beta = m\left(\overset{\frown}{B'C}\right) = m\left(\overset{\frown}{B'A}\right)$$

$$\gamma = m\left(\overset{\frown}{C'A}\right) = m\left(\overset{\frown}{C'B}\right)$$

Then

$$m\sphericalangle\left(A'IC'\right) = \frac{1}{2}(\alpha + \beta + \gamma)$$

Because

$$2(\alpha + \beta + \gamma) = 360°$$

it results

$$m\sphericalangle\left(A'IC'\right) = 90°,$$

therefore

$$A'I \perp B'C'.$$

Similarly, we prove that

$$B'I \perp A'C',$$

and consequently the orthocenter of the triangle $A'B'C'$ is $I$, the center of the circumscribed circle of the triangle $ABC$.

**Definition**

Let $ABC$ a triangle inscribed in a circle with the center in $O$ and $A', B', C'$ the middle of the arcs $\overset{\frown}{BC}, \overset{\frown}{CA}, \overset{\frown}{AB}$ respectively. The triangle $XYZ$ formed by the symmetric of the points $A', B', C'$ respectively in rapport to $BC, CA, AB$ is called the Fuhrmann triangle of the triangle $ABC$.

**Note**

In 2002 the mathematician Milorad Stevanovic proved the following theorem:

**Theorem** (M. Stevanovic)

In an acute angle triangle the orthocenter of the Fuhrmann's triangle coincides with the center of the circle inscribed in the given triangle.

**Proof**

We note $A'B'C'$ the given triangle and let $A, B, C$ respectively the middle of the arcs $\overset{\frown}{B'C'}, \overset{\frown}{C'A'}, \overset{\frown}{A'B'}$ (see Fig. 1). The lines $AA', BB', CC'$ being bisectors in the triangle $A'B'C'$ are concurrent in the center of the circle inscribed in this triangle, which will note $H$, and which, in conformity with Lemma 2 is the orthocenter of the triangle $ABC$. Let $XYZ$ the Fuhrmann triangle of the triangle $A'B'C'$, in conformity with Lemma 1, the orthocenter of $XYZ$ coincides with $H$ the orthocenter of $ABC$, therefore with the center of the inscribed circle in the given triangle $A'B'C'$.

**References**

1. Fuhrmann W., *Synthetische Beweise Planimetrischer Sätze*, Berlin (1890), http://jl.ayme.pagesperso-orange.fr/Docs/L'ortocentre du triangle de Fuhrmann
2. Stevanovic M., *Orthocentrer of Fuhrmann triangle*, Message Hyacinthos # du 20/09/2002, http://group.yahoo.com/group/Hyacinthos
3. Barbu, Cătălin, *Teoreme fundamentale din geometria triunghiului*, Editura Unique, Bacău, Romania, 2008.

# Two Remarkable Ortho-Homological Triangles

Prof. Ion Pătraşcu - The National College “Fraţii Buzeşti”, Craiova, Romania
Prof. Florentin Smarandache – University of New Mexico, U.S.A.

In a previous paper [5] we have introduced the ortho-homological triangles, which are triangles that are orthological and homological simultaneously.

In this article we call attention to two remarkable ortho-homological triangles (the given triangle $ABC$ and its first Brocard’s triangle), and using the Sondat’s theorem relative to orthological triangles, we emphasize on four important collinear points in the geometry of the triangle. Orthological / homological / orthohomological triangles in the 2D-space are generalized to orthological / homological / orthohomological polygons in 2D-space, and even more to orthological / homological / orthohomological triangles, polygons, and polyhedrons in 3D-space.

**Definition 1**

The first Brocard triangle of a given triangle $ABC$ is the triangle formed by the projections of the symmedian center of the triangle $ABC$ on its perpendicular bisectors.

**Observation**

In figure 1 we note with $K$ the symmedian center, $OA'$, $OB'$, $OC'$ the perpendicular bisectors of the triangle $ABC$ and $A_1B_1C_1$ the first Brocard’s triangle.

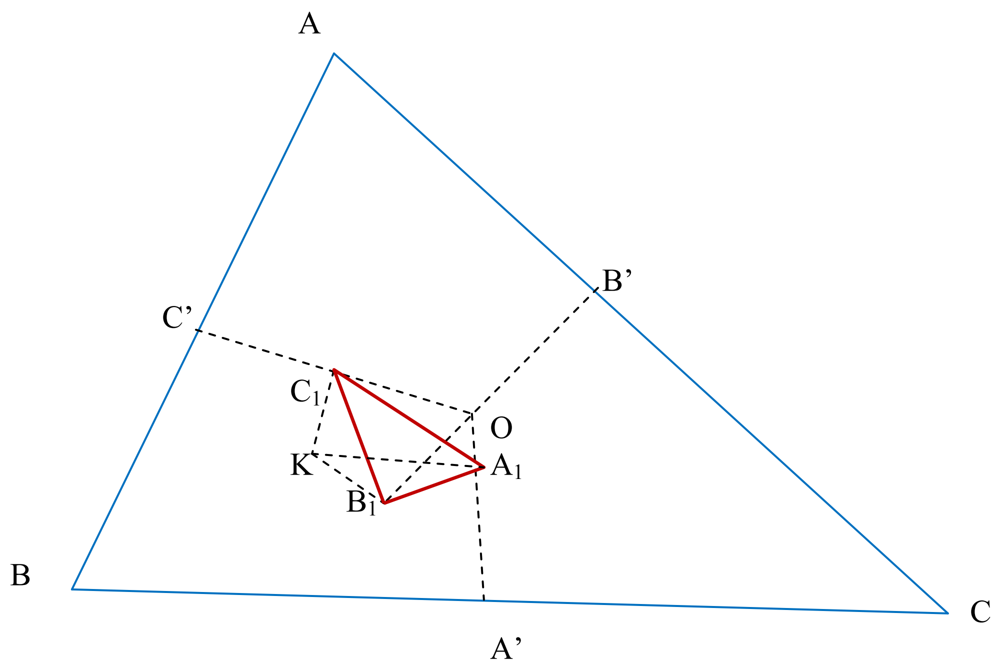



Fig. 1

**Theorem 1**

If $ABC$ is a given triangle and $A_1B_1C_1$ is its first triangle Brocard, then the triangles $ABC$ and $A_1B_1C_1$ are ortho-homological.

We'll perform the proof of this theorem in two stages.

I. We prove that the triangles $A_1B_1C_1$ and $ABC$ are orthological.

The perpendiculars from $A_1$, $B_1$, $C_1$ on $BC$, $CA$ respective $AB$ are perpendicular bisectors in the triangle $ABC$, therefore are concurrent in $O$, the center of the circumscribed circle of triangle $ABC$ which is the orthological center for triangles $A_1B_1C_1$ and $ABC$.

II. We prove that the triangles $A_1B_1C_1$ and $ABC$ are homological, that is the lines $AA_1$, $BB_1$, $CC_1$ are concurrent.

To continue with these proves we need to refresh some knowledge and some helpful results.

**Definition 2**

In any triangle $ABC$ there exist the points $\Omega$ and $\Omega'$ and the angle $\omega$ such that:

$$m(\sphericalangle \Omega AB) = \sphericalangle \Omega BC = \sphericalangle \Omega CA = \omega$$

$$m(\sphericalangle \Omega' BA) = \sphericalangle \Omega' CA = \sphericalangle \Omega' AB = \omega$$

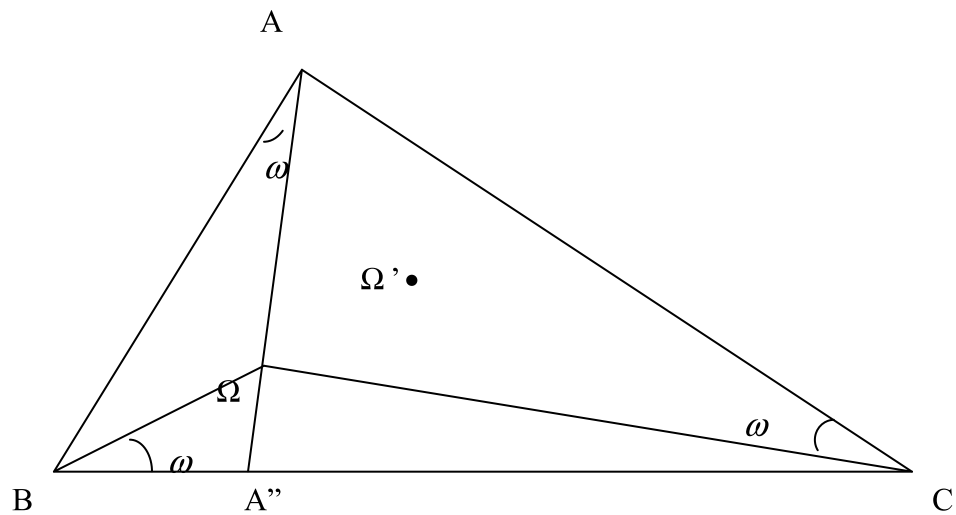



Fig. 2

The points $\Omega$ and $\Omega'$ are called the first, respectively the second point of Brocard and $\omega$ is called the Brocard's angle.

**Lemma 1**

In the triangle $ABC$ let $\Omega$ the first point of Brocard and $\{A''\}$ $\{A''\} = A\Omega \cap BC$, then:

$$\frac{BA"}{CA"} = \frac{c^2}{a^2}$$

**Proof**

$$Aria_{\triangle} ABA" = \frac{1}{2} AB \cdot AA" \sin \omega \quad (1)$$

$$Aria_{\triangle} ACA" = \frac{1}{2} AC \cdot AA" \sin(A - \omega) \quad (2)$$

From (1) and (2) we find:

$$\frac{Aria_{\triangle} ABA"}{Aria_{\triangle} ACA"} = \frac{AB \cdot \sin \omega}{AC \cdot \sin(A - \omega)} \quad (3)$$

On the other side, the mentioned triangles have the same height built from $A$, therefore:

$$\frac{Aria_{\triangle} ABA"}{Aria_{\triangle} ACA"} = \frac{BA"}{CA"} \quad (4)$$

From (3) and (4) we have:

$$\frac{BA"}{CA"} = \frac{AB \cdot \sin \omega}{AC \cdot \sin(A - \omega)} \quad (5)$$

Applying the sinus theorem in the triangle $A\Omega C$ and in the triangle $B\Omega C$, it results:

$$\frac{C\Omega}{\sin(A - \omega)} = \frac{AC}{\sin A\Omega C} \quad (6)$$

$$\frac{C\Omega}{\sin \omega} = \frac{BC}{\sin B\Omega C} \quad (7)$$

Because

$$m(\sphericalangle A\Omega C) = 180^o - A$$

$$m(\sphericalangle B\Omega C) = 180^o - C$$

From the relations (6) and (7) we find:

$$\frac{\sin \omega}{\sin(A - \omega)} = \frac{AC}{BC} \cdot \frac{\sin C}{\sin A} \quad (8)$$

Applying the sinus theorem in the triangle $ABC$ leads to:

$$\frac{\sin C}{\sin A} = \frac{AB}{BC} \quad (9)$$

The relations (5), (8), (9) provide us the relation:

$$\frac{BA"}{CA"} = \frac{c^2}{a^2}$$

**Remark 1**

By making the notations: $\{B"\} = B\Omega C \cap AC$ and $\{C"\} = C\Omega A \cap AB$ we obtain also the relations:

$$\frac{CB"}{AB"} = \frac{a^2}{b^2} \text{ and } \frac{AC"}{BC"} = \frac{b^2}{c^2}$$

**Lemma 2**

In a triangle $ABC$, the Brocard's Cevian $B\Omega$, symmedian from $C$ and the median from $A$ are concurrent.

**Proof**

It is known that the symmedian $CK$ of triangle $ABC$ intersects $AB$ in the point $C_2$ such that $\frac{AC_2}{BC_2} = \frac{b^2}{c^2}$. We had that the Cevian $B\Omega$ intersects $AC$ in $B"$ such that $\frac{BC"}{B"A} = \frac{a^2}{b^2}$. The median from $A$ intersects $BC$ in $A'$ and $BA' = CA'$.

Because $\frac{A'B}{A'C} \cdot \frac{B"C}{B"A} \cdot \frac{C_2A}{C_2B} = 1$, the reciprocal of Ceva's theorem ensures the concurrency of the lines $B\Omega$, $CK$ and $AA'$.

**Lemma 3**

Give a triangle $ABC$ and $\omega$ the Brocard's angle, then

$$ctg\omega = ctgA + ctgB + ctgC \tag{9}$$

**Proof**

From the relation (8) we find:

$$\sin(A-\omega) = \frac{a}{b} \cdot \frac{\sin A}{\sin C} \cdot \sin\omega \tag{10}$$

From the sinus' theorem in the triangle $ABC$ we have that

$$\frac{a}{b} = \frac{\sin A}{\sin B}$$

Substituting it in (10) it results: $\sin(A-\omega) = \frac{\sin^2 A \cdot \sin\omega}{\sin B \cdot \sin C}$

Furthermore we have:

$$\sin(A-\omega) = \sin A \cdot \cos\omega - \sin\omega \cdot \cos A$$

$$\sin A \cdot \cos\omega - \sin\omega \cdot \cos A = \frac{\sin^2 A \cdot \sin\omega}{\sin B \cdot \sin C} \tag{11}$$

Dividing relation (11) by $\sin A \cdot \sin\omega$ and taking into account that $\sin A = \sin(B+C)$, and $\sin(B+C) = \sin B \cdot \cos C + \sin C \cdot \cos B$ we obtain relation (5)

**Lemma 4**

If in the triangle $ABC$, $K$ is the symmedian center and $K_1, K_2, K_3$ are its projections on the sides $BC$, $CA$, $AB$, then:

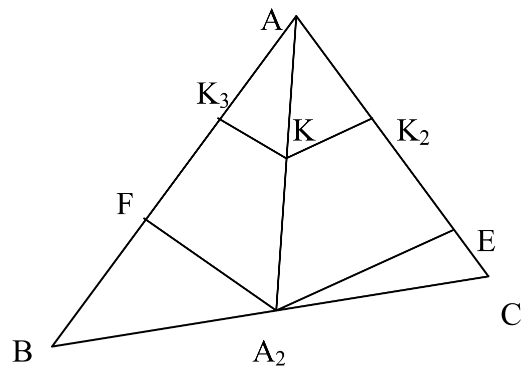



Fig. 3

$$\frac{KK_1}{a}=\frac{KK_2}{b}=\frac{KK_3}{c}=\frac{1}{2}tg\omega$$

**Proof:**

Let $AA_2$ the symmedian in the triangle $ABC$, we have:

$$\frac{BA_2}{CA_2}=\frac{Aria\vartriangle BAA_2}{Aria\vartriangle CAA_2},$$

where $E$ and $F$ are the projection of $A_2$ on $AC$ respectively $AB$.

It results that $\frac{A_2F}{A_2E}=\frac{c}{b}$

From the fact that $\vartriangle AKK_3 \sim \vartriangle AA_2F$ and $\vartriangle AKK_2 \sim \vartriangle AA_2E$ we find that $\frac{KK_3}{KK_2}=\frac{A_2F}{A_2E}$

Also: $\frac{KK_2}{b}=\frac{KK_3}{c}$, and similarly: $\frac{KK_1}{a}=\frac{KK_2}{b}$, consequently:

$$\frac{KK_1}{a}=\frac{KK_2}{b}=\frac{KK_3}{c} \tag{12}$$

The relation (12) is equivalent to:

$$\frac{aKK_1}{a^2}=\frac{bKK_2}{b^2}=\frac{cKK_3}{c^2}=\frac{aKK_1+bKK_2+cKK_3}{a^2+b^2+c^2}$$

Because

$$aKK_1+bKK_2+cKK_3=2Aria\vartriangle ABC=2S,$$

we have:

$$\frac{KK_1}{a}=\frac{KK_2}{b}=\frac{KK_3}{c}=\frac{2S}{a^2+b^2+c^2}$$

If we note $H_1, H_2, H_3$ the projections of $A, B, C$ on $BC, CA, AB$, we have

$$ctgA=\frac{H_2A}{BH_2}=\frac{bc\cos A}{2S}$$

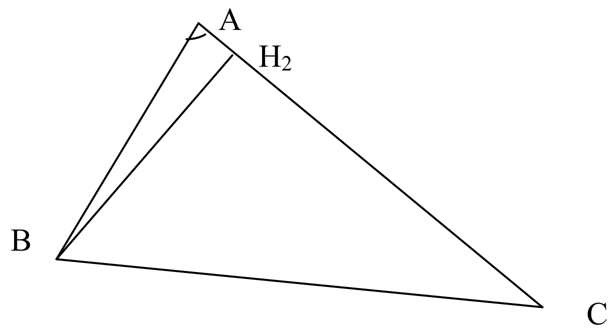



Fig. 4

From the cosine's theorem it results that : $b \cdot c \cdot \cos A = \dfrac{b^2 + c^2 - a^2}{2}$, and therefore

$$ctgA = \frac{b^2 + c^2 - a^2}{4S}$$

Taking into account the relation (9), we find:

$$ctg\omega = \frac{a^2 + b^2 + c^2}{4S},$$

then

$$tg\omega = \frac{4S}{a^2 + b^2 + c^2}$$

and then

$$\frac{KK_1}{a} = \frac{2S}{a^2 + b^2 + c^2} = \frac{1}{2} tg\omega .$$

**Lemma 5**

The Cevians $AA_1$, $BB_1$, $CC_1$ are the isotomics of the symmedians $AA_2$, $BB_2$, $CC_2$ in the triangle $ABC$.

**Proof:**

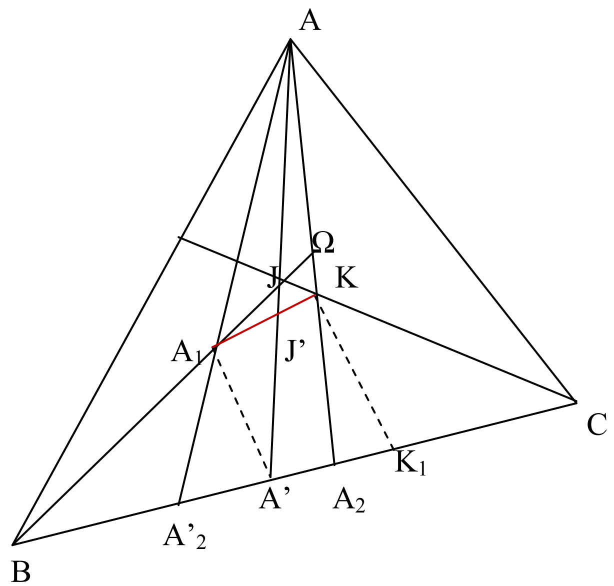



Fig. 5

In figure 5 we note J the intersection point of the Cevians from the Lemma 2.

Because $KA_1 \parallel BC$, we have that $A_1A' = KK_1 = \frac{1}{2} atg\omega$. On the other side from the right triangle $A'A_1B$ we have: $tg\sphericalangle A_1BA' = \frac{A_1A'}{BA'} = tg\omega$, consequently the point $A_1$, the vertex of the first triangle of Brocard belongs to the Cevians $B\Omega$.

We note $\{J'\} = A_1K \cap AA'$, and evidently from $A_1K \parallel BC$ it results that $JJ'$ is the median in the triangle $JA_1K$, therefore $A_1J' = J'K$.

We note with $A'_2$ the intersection of the Cevians $AA_1$ with $BC$, because $A_1K \parallel A'_2A_2$ and $AJ'$ is a median in the triangle $AA_1K$ it results that $AA'$ is a median in triangle $AA'_2A_2$ therefore the points $A'_2$ and $A_2$ are isometric.

Similarly it can be shown that $BB_2'$ and $CC_2'$ are the isometrics of the symmedians $BB_2$ and $CC_2$.

The second part of this proof: Indeed it is known that the isometric Cevians of certain concurrent Cevians are concurrent and from Lemma 5 along with the fact that the symmedians of a triangle are concurrent, it results the concurrency of the Cevians $AA_1$, $BB_1$, $CC_1$ and therefore the triangle $ABC$ and the first triangle of Brocard are homological. The homology's center (the concurrency point) of these Cevians is marked in some works with $\Omega''$ with and it is called the third point of Brocard.

From the previous proof, it results that $\Omega''$ is the isotomic conjugate of the symmedian center $K$.

**Remark 2**

The triangles ABC and $A_1B_1C_1$ (first Brocard triangle) are *triple-homological*, since first time the Cevians $AB_1$, $BC_1$, $CA_1$ are concurrent (in a Brocard point), second time the Cevians

$AC_1$, $BA_1$, $CB_1$ are also concurrent (in the second Brocard point), and third time the Cevians $AA_1$, $BB_1$, $CC_1$ are concurrent as well (in the third point of Brocard).

**Definition 3**

It is called the Tarry point of a triangle $ABC$, the concurrency point of the perpendiculars from $A$, $B$, $C$ on the sides $B_1C_1$, $C_1A_1$, $A_1B_1$ of the Brocard's first triangle.

**Remark 3**

The fact that the perpendiculars from the above definition are concurrent results from the theorem 1 and from the theorem that states that the relation of triangles' orthology is symmetric.

We continue to prove the concurrency using another approach that will introduce supplementary information about the Tarry's point.

We'll use the following:

**Lemma 6:**

The first triangle Brocard of a triangle and the triangle itself are similar.

**Proof**

From $KA_1 \parallel BC$ and $OA' \perp BC$ it results that

$$m\left(\sphericalangle KA_1O\right) = 90°$$

(see Fig. 1), similarly

$$m\left(\sphericalangle KB_1O\right) = m\left(\sphericalangle KC_1O\right) = 90°$$

and therefore the first triangle of Brocard is inscribed in the circle with $OK$ as diameter (this circle is called the Brocard circle).

Because

$$m\left(\sphericalangle A_1OC_1\right) = 180° - B$$

and $A_1, B_1, C_1, O$ are concyclic, it results that $\sphericalangle A_1B_1C_1 = \sphericalangle B$, similarly

$$m\left(\sphericalangle B'OC'\right) = 180° - A,$$

it results that

$$m\left(\sphericalangle B_1OC_1\right) = m(A)$$

but

$$\sphericalangle B_1OC_1 \equiv \sphericalangle B_1A_1C_1,$$

therefore

$$\sphericalangle B_1A_1C_1 = \sphericalangle A$$

and the triangle $A_1B_1C_1$ is similar wit the triangle $ABC$.

**Theorem 2**

The orthology center of the triangle $ABC$ and of the first triangle of Brocard is the Tarry's point $T$ of the triangle $ABC$, and $T$ belongs to the circumscribed circle of the triangle $ABC$.

**Proof**

We mark with $T$ the intersection of the perpendicular raised from $B$ on $A_1C_1$ with the perpendicular raised from $C$ on $A_1B_1$ and let

$$\{B_1'\} = BT \cap A_1C_1,\ A_1\{C_1'\} = A_1B_1 \cap CT.$$

We have

$$m(\triangle B_1'TC_1') = 180° - m(\triangle C_1A_1B_1)$$

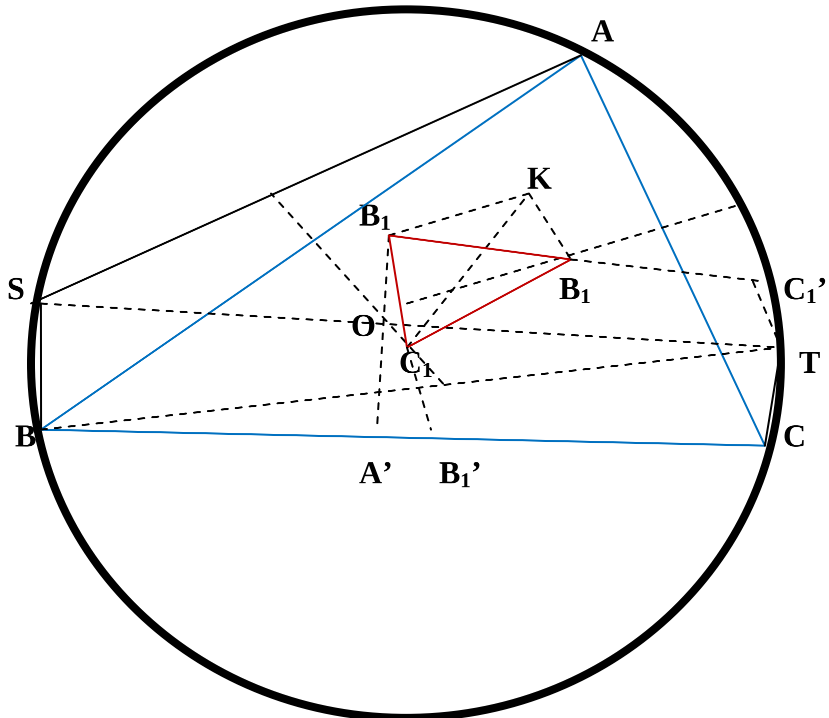



Fig 6

But because of Lemma 6 $\sphericalangle C_1A_1B_1 = \sphericalangle A$.

It results that $m(\sphericalangle B_1'TC_1') = 180° - A$, therefore

$$m(\sphericalangle BTC') + m(\sphericalangle BAC) = 180°$$

Therefore $T$ belongs to the circumscribed circle of triangle $ABC$

If $\{A_1'\} = B_1C_1 \cap AT$ and if we note with $T'$ the intersection of the perpendicular raised from $A$ on $B_1C_1$ with the perpendicular raised from $B$ on $A_1C_1$, we observe that

$$m(\sphericalangle B_1'T'A_1') = m(\sphericalangle A_1C_1B_1)$$

therefore

$$m(\sphericalangle BT'A) + m(\sphericalangle BCA)$$

and it results that $T'$ belongs to the circumscribed triangle $ABC$.

Therefore $T = T'$ and the theorem is proved.

**Theorem 3**

If through the vertexes $A, B, C$ of a triangle are constructed the parallels to the sides $B_1C_1, C_1A_1$ respectively $A_1B_1$ of the first triangle of Brocard of this triangle, then these lines are concurrent in a point S (the Steiner point of the triangle)

**Proof**

We note with $S$ the polar intersection constructed through $A$ to $B_1C_1$ with the polar constructed through $B$ to $A_1C_1$ (see Fig. 6).

We have

$$m(\sphericalangle ASB) = 180° - m(\sphericalangle A_1C_1B_1) \text{ (angles with parallel sides)}$$

because

$$m(\sphericalangle A_1C_1B_1) = m\sphericalangle C,$$

we have

$$m(\sphericalangle ASB) = 180° - m\sphericalangle C,$$

therefore $A_1SB_1C$ are concyclic.

Similarly, if we note with $S'$ the intersection of the polar constructed through $A$ to $B_1C_1$ with the parallel constructed through $C$ to $A_1B_1$ we find that the points $A_1S_1'B_1C$ are concyclic.

Because the parallels from $A$ to $B_1C_1$ contain the points $A, S, S'$ and the points $S, S', A$ are on the circumscribed circle of the triangle, it results that $S = S'$ and the theorem is proved.

**Remark 4**

Because $SA \| B_1C_1$ and $B_1C_1 \perp AT$, it results that

$$m(\sphericalangle SAT) = 90°,$$

but $S$ and $T$ belong to the circumscribed circle to the triangle $ABC$, consequently the Steiner's point and the Tarry point are diametric opposed.

**Theorem 4**

In a triangle $ABC$ the Tarry point $T$, the center of the circumscribed circle $O$, the third point of Brocard $\Omega''$ and Steiner's point $S$ are collinear points

**Proof**

The P. Sondat's theorem relative to the orthological triangles (see [4]) says that the points $T$, $O$, $\Omega''$ are collinear, therefore the points: $T$, $O$, $\Omega''$, $S$ are collinear.

**Open Questions**

1) Is it possible to have two triangles which are four times, five times, or even six times orthological? But triangles which are four times, five times, or even six times homological? What about orthohomological? What is the largest such rank?
   For two triangles $A_1B_1C_1$ and $A_2B_2C_2$, we can have (in the case of orthology, and similarity in the cases of homology and orthohomology) the following 6 possibilities:
   1) the perpendicular from $A_1$ onto $B_2C_2$, the perpendicular from $B_1$ onto $C_2A_2$, and the perpendicular from $C_1$ onto $A_2B_2$ concurrent;

2) the perpendicular from $A_1$ onto $B_2C_2$, the perpendicular from $B_1$ onto $A_2B_2$, and the perpendicular from $C_1$ onto $C_2A_2$ concurrent;
3) the perpendicular from $B_1$ onto $B_2C_2$, the perpendicular from $A_1$ onto $C_2A_2$, and the perpendicular from $C_1$ onto $A_2B_2$ concurrent;
4) the perpendicular from $B_1$ onto $B_2C_2$, the perpendicular from $A_1$ onto $A_2B_2$, and the perpendicular from $C_1$ onto $C_2A_2$ concurrent;
5) the perpendicular from $C_1$ onto $B_2C_2$, the perpendicular from $B_1$ onto $C_2A_2$, and the perpendicular from $A_1$ onto $A_2B_2$ concurrent;
6) the perpendicular from $C_1$ onto $B_2C_2$, the perpendicular from $B_1$ onto $A_2B_2$, and the perpendicular from $A_1$ onto $C_2A_2$ concurrent.

2) We generalize the orthological, homological, and orthohomological triangles to respectively orthological, homological, and orthohomological polygons and polyhedrons. Can we have double, triple, etc. orthological, homological, or orthohomological polygons and polyhedrons? What would be the largest rank for each case?

3) Let's have two triangles in a plane. Is it possible by changing their positions in the plane and to have these triangles be orthological, homological, orthohomological? What is the largest rank they may have in each case?

4) Study the orthology, homology, orthohomology of triangles and poligons in a 3D space.

5) Let's have two triangles, respectively two polygons, in a 3D space. Is it possible by changing their positions in the 3D space to have these triangles, respectively polygons, be orthological, homological, or orthohological?
Similar question for two polyhedrons?

**References**

[1] Roger A. Johnson – Advanced Euclidean Geometry, Dover Publications, Inc. Mineola, New York, 2007.
[2] F. Smarandache – Multispace & Multistructure, Neutrospheric Trandisciplinarity (100 Collected Papers of Sciences), Vol IV. North-European Scientific Publishers, Hanko, Findland, 2010.
[3] C. Barbu – Teoreme fundamentale din geometria triunghiului. Editura Unique, Bacău, 2008.
[4] I. Pătraşcu, F. Smarandache – http://www.scribd.com/doc/33733028 - An application of Sondat Theorem Regarding the Orthohomological Triangles.
[5] Ion Pătraşcu, Florentin Smarandache, A Theorem about Simulotaneously Orthological and Homological Triangles, http://arxiv.org/abs/1004.0347.

# A Generalization of Certain Remarkable Points of the Triangle Geometry

Prof. Claudiu Coandă – National College "Carol I", Craiova, Romania
Prof. Florentin Smarandache – University of New Mexico, Gallup, U.S.A.
Prof. Ion Pătrașcu – National College "Fraţii Buzeşti", Craiova, Romania

In this article we prove a theorem that will generalize the concurrence theorems that are leading to the Franke's point, Kariya's point, and to other remarkable points from the triangle geometry.

**Theorem 1:**

Let $P(\alpha,\beta,\gamma)$ and $A',B',C'$ its projections on the sides $BC$, $CA$ respectively $AB$ of the triangle $ABC$.

We consider the points $A''$, $B''$, $C''$ such that $\overrightarrow{PA''}=k\overrightarrow{PA'}$, $\overrightarrow{PB''}=k\overrightarrow{PB'}$, $\overrightarrow{PC''}=k\overrightarrow{PC'}$, where $k\in R^*$. Also we suppose that $AA', BB', CC'$ are concurrent. Then the lines $AA'', BB'', CC''$ are concurrent if and only if are satisfied simultaneously the following conditions:

$$\alpha\beta c\left(\frac{\beta}{b}\cos A-\frac{\alpha}{a}\cos B\right)+\beta\gamma a\left(\frac{\gamma}{c}\cos B-\frac{\beta}{b}\cos C\right)+\gamma\alpha b\left(\frac{\alpha}{a}\cos C-\frac{\gamma}{c}\cos A\right)=0$$

$$\frac{\alpha^2}{a^2}\cos A\left(\frac{\gamma}{c}\cos B-\frac{\beta}{b}\cos C\right)+\frac{\beta^2}{b^2}\cos B\left(\frac{\alpha}{a}\cos C-\frac{\gamma}{c}\cos A\right)+\frac{\gamma^2}{c^2}\cos C\left(\frac{\beta}{b}\cos A-\frac{\alpha}{a}\cos B\right)=0$$

**Proof:**

We find that

$$A'\left(0,\frac{\alpha}{2a^2}\left(a^2+b^2-c^2\right)+\beta,\ \frac{\alpha}{2a^2}\left(a^2-b^2+c^2\right)+\gamma\right)$$

$$\overrightarrow{PA''}=k\overrightarrow{PA'}=k\left[-\alpha\overrightarrow{r_A}+\frac{\alpha}{2a^2}\left(a^2+b^2-c^2\right)\overrightarrow{r_B}+\frac{\alpha}{2a^2}\left(a^2-b^2+c^2\right)\overrightarrow{r_C}\right]$$

$$\overrightarrow{PA''}=\left(\alpha''-\alpha\right)\overrightarrow{r_A}+\left(\beta''-\beta\right)\overrightarrow{r_B}+\left(\gamma''-\gamma\right)\overrightarrow{r_C}$$

We have:

$$\begin{cases}\alpha''-\alpha=-k\alpha\\ \beta''-\beta=\dfrac{k\alpha}{2a^2}\left(a^2+b^2-c^2\right),\\ \gamma''-\gamma=\dfrac{k\alpha}{2a^2}\left(a^2-b^2+c^2\right)\end{cases}$$

Therefore:

$$\begin{cases} \alpha'' = (1-k)\alpha \\ \beta'' = \dfrac{k\alpha}{2a^2}\left(a^2+b^2-c^2\right)+\beta \\ \gamma'' = \dfrac{k\alpha}{2a^2}\left(a^2-b^2+c^2\right)+\gamma \end{cases}$$

Hence:

$$A''\left((1-k)\alpha,\ \frac{k\alpha}{2a^2}\left(a^2+b^2-c^2\right)+\beta,\ \frac{k\alpha}{2a^2}\left(a^2-b^2+c^2\right)+\gamma\right)$$

Similarly:

$$B'\left(-\frac{\beta}{2b^2}\left(-a^2-b^2+c^2\right)+\alpha,\ 0,\ -\frac{\beta}{2b^2}\left(a^2-b^2-c^2\right)+\gamma\right)$$

$$B''\left(-\frac{k\beta}{2b^2}\left(-a^2-b^2+c^2\right)+\alpha,\ (1\text{-}k)\beta,\ -\frac{k\beta}{2b^2}\left(a^2-b^2-c^2\right)+\gamma\right)$$

$$C'\left(-\frac{\gamma}{2c^2}\left(-a^2+b^2-c^2\right)+\alpha,\ -\frac{\gamma}{2c^2}\left(a^2-b^2-c^2\right)+\beta,\ 0\right)$$

$$C''\left(-\frac{k\gamma}{2c^2}\left(-a^2+b^2-c^2\right)+\alpha,\ -\frac{k\gamma}{2c^2}\left(a^2-b^2-c^2\right)+\beta,\ (1\text{-}k)\gamma\right)$$

Because $AA',\ BB',\ CC'$ are concurrent, we have:

$$\frac{-\dfrac{\alpha}{2a^2}\left(-a^2-b^2+c^2\right)+\beta}{-\dfrac{\alpha}{2a^2}\left(-a^2+b^2-c^2\right)+\gamma}\cdot\frac{-\dfrac{\beta}{2b^2}\left(-a^2-b^2-c^2\right)+\gamma}{-\dfrac{\beta}{2b^2}\left(-a^2-b^2+c^2\right)+\alpha}\cdot\frac{-\dfrac{\gamma}{2c^2}\left(-a^2+b^2-c^2\right)+\alpha}{-\dfrac{\gamma}{2c^2}\left(a^2-b^2-c^2\right)+\beta}=1$$

We note

$$M=\frac{\alpha}{2a^2}\left(a^2+b^2-c^2\right)=\frac{\alpha}{a}\cdot b\cos C$$

$$N=\frac{\alpha}{2a^2}\left(a^2-b^2+c^2\right)=\frac{\alpha}{a}\cdot c\cos B$$

$$P=\frac{\beta}{2b^2}\left(-a^2+b^2+c^2\right)=\frac{\beta}{b}\cdot c\cos A$$

$$Q=\frac{\beta}{2b^2}\left(a^2+b^2-c^2\right)=\frac{\beta}{b}\cdot a\cos C$$

$$R=\frac{\gamma}{2c^2}\left(a^2-b^2+c^2\right)=\frac{\gamma}{c}\cdot a\cos B$$

$$S=\frac{\gamma}{2c^2}\left(-a^2+b^2+c^2\right)=\frac{\gamma}{c}\cdot a\cos A$$

The precedent relation becomes

$$\frac{M+\beta}{N+\gamma}\cdot\frac{P+\gamma}{Q+\alpha}\cdot\frac{R+\alpha}{S+\beta}=1$$

The coefficients $M,N,P,Q,R,S$ verify the following relations:

$$M+N=\alpha$$
$$P+Q=\beta$$
$$R+S=\gamma$$
$$\frac{M}{Q}=\frac{\alpha}{\beta}\cdot\frac{b^2}{a^2}=\frac{\frac{\alpha}{a^2}}{\frac{\beta}{b^2}}$$
$$\frac{P}{S}=\frac{\beta}{\gamma}\cdot\frac{c^2}{b^2}=\frac{\frac{\beta}{b^2}}{\frac{\gamma}{c^2}}$$
$$\frac{R}{N}=\frac{\gamma}{\alpha}\cdot\frac{a^2}{c^2}=\frac{\frac{\gamma}{c^2}}{\frac{\alpha}{a^2}}$$

Therefore $\frac{M}{Q}\cdot\frac{P}{S}\cdot\frac{R}{N}=1$

$$(M+\beta)(P+\gamma)(R+\alpha)=\alpha\beta\gamma+\alpha\beta P+\beta\gamma R+\gamma\alpha M+\alpha MP+\beta PR+\gamma RM+MPR$$
$$(N+\gamma)(Q+\alpha)(S+\beta)=\alpha\beta\gamma+\alpha\beta N+\beta\gamma Q+\gamma\alpha S+\alpha NS+\beta NQ+\gamma QS+NQS\,.$$

We deduct that:

$$\alpha\beta P+\beta\gamma R+\gamma\alpha M+\alpha MP+\beta PR+\gamma RM=\alpha\beta N+\beta\gamma Q+\gamma\alpha S+\alpha NS+\beta NQ+\gamma QS+NQS \quad (1)$$

We apply the theorem:

Given the points $Q_i(a_i,b_i,c_i),\ i=\overline{1,3}$ in the plane of the triangle $ABC$, the lines $AQ_1, BQ_2, CQ_3$ are concurrent if and only if $\frac{b_1}{c_1}\cdot\frac{c_2}{a_2}\cdot\frac{a_3}{b_3}=1$.

For the lines $AA'', BB'', CC''$ we obtain

$$\frac{kM+\beta}{kN+\gamma}\cdot\frac{kP+\alpha}{kS+\beta}\cdot\frac{kR+\alpha}{kS+\beta}=1\,.$$

It result that

$$k^2(\alpha\beta P+\beta\gamma R+\gamma\alpha M)+k(\alpha MP+\beta PR+\gamma RM)=$$
$$=k^2(\alpha\beta N+\beta\gamma Q+\gamma\alpha S)+k(\alpha NS+\beta NQ+\gamma QS) \quad (2)$$

For relation (1) to imply relation (2) it is necessary that

$$\alpha\beta P+\beta\gamma R+\gamma\alpha M=\alpha\beta N+\beta\gamma Q+\gamma\alpha S$$

and

$$\alpha NS+\beta NQ+\gamma QS=\alpha MP+\beta PR+\gamma RM$$

or

$$\begin{cases} \alpha\beta c\left(\frac{\beta}{b}\cos A-\frac{\alpha}{a}\cos B\right)+\beta\gamma a\left(\frac{\gamma}{c}\cos B-\frac{\beta}{b}\cos C\right)+\gamma\alpha b\left(\frac{\alpha}{a}\cos C-\frac{\gamma}{c}\cos A\right)=0 \\ \frac{\alpha^2}{a^2}\cos A\left(\frac{\gamma}{c}\cos B-\frac{\beta}{b}\cos C\right)+\frac{\beta^2}{b^2}\cos B\left(\frac{\gamma}{c}\cos B-\frac{\beta}{b}\cos C\right)+\frac{\gamma^2}{c^2}\cos C\left(\frac{\beta}{b}\cos A-\frac{\alpha}{a}\cos B\right)=0 \end{cases}$$

As an open problem, we need to determine the set of the points from the plane of the triangle $ABC$ that verify the precedent relations.

We will show that the points $I$ and $O$ verify these relations, proving two theorems that lead to Kariya's point and Franke's point.

**Theorem 2** (Kariya -1904)

Let $I$ be the center of the circumscribe circle to triangle $ABC$ and $A', B', C'$ its projections on the sides $BC$, $CA$, $AB$. We consider the points $A'', B'', C''$ such that:

$$\overrightarrow{IA''}=k\overrightarrow{IA'},\ \overrightarrow{IB''}=k\overrightarrow{IB'},\ \overrightarrow{IC''}=k\overrightarrow{IC'},\ \ k\in R^*.$$

Then $AA''$, $BB''$, $CC''$ are concurrent (the Kariya's point)

**Proof:**

The barycentric coordinates of the point $I$ are $I\left(\frac{a}{2p},\frac{b}{2p},\frac{c}{2p}\right)$.

Evidently:

$$abc(\cos A-\cos B)+abc(\cos B-\cos C)+abc(\cos C-\cos A)=0$$

and

$$\cos A(\cos B-\cos C)+\cos B(\cos C-\cos A)+\cos C(\cos A-\cos B)=0.$$

In conclusion $AA''$, $BB''$, $CC''$ are concurrent.

**Theorem 3** (de Boutin - 1890)

Let $O$ be the center of the circumscribed circle to the triangle $ABC$ and $A', B', C'$ its projections on the sides $BC, CA, AB$. Consider the points $A''$, $B''$, $C''$ such that $\frac{OA'}{OA''}=\frac{OB'}{OB''}=\frac{OC'}{OC''}=k,\ \ k\in R^*$. Then the lines $AA''$, $BB''$, $CC''$ are concurrent (The point of Franke – 1904).

**Proof:**

$O\left(\frac{R^2}{2S}\sin 2A, \frac{R^2}{2S}\sin 2B, \frac{R^2}{2S}\sin 2C\right)$, $P=N$, because $\frac{\sin 2B\cos A}{\sin B}-\frac{\sin 2A\cos B}{\sin A}=0$.

Similarly we find that $R=Q$ and $M=S$.

Also $\alpha MP=\alpha NS$, $\beta PR=\beta NQ$, $\gamma RM=\gamma QS$. It is also verified the second relation from the theorem hypothesis. Therefore the lines $AA''$, $BB''$, $CC''$ are concurrent in a point called the Franke's point.

**Remark 1**:

It is possible to prove that the Franke's points belong to Euler's line of the triangle $ABC$.

**Theorem 4**:

Let $I_a$ be the center of the circumscribed circle to the triangle $ABC$ (tangent to the side $BC$) and $A', B', C'$ its projections on the sites $BC, CA, AB$. We consider the points $A", B", C"$ such that $\overrightarrow{IA"} = k\overrightarrow{IA'}, \overrightarrow{IB"} = k\overrightarrow{IB'}, \overrightarrow{IC"} = k\overrightarrow{IC'}, k \in R^*$. Then the lines $AA", BB", CC"$ are concurrent.

**Proof**

$$I_a\left(\frac{-a}{2(p-a)}, \frac{b}{2(p-a)}, \frac{c}{2(p-a)}\right);$$

The first condition becomes:

$$-abc(\cos A+\cos B)+abc(\cos B-\cos C)-abc(-\cos C-\cos A)=0,$$ and the second condition:

$$\cos A(\cos B-\cos C)+\cos B(-\cos C-\cos A)+\cos C(\cos A+\cos B)=0$$

Is also verified.

From this theorem it results that the lines $AA", BB", CC"$ are concurrent.

**Observation 1:**

Similarly, this theorem is proven for the case of $I_b$ and $I_c$ as centers of the ex-inscribed circles.

**References**


[1] C. Coandă –Geometrie analitică în coordonate baricentrice – Editura Reprograph, Craiova, 2005.

[2] F. Smarandache – Multispace & Multistructure, Neutrosophic Trandisciplinarity (100 Collected Papers of Sciences), Vol. IV, 800 p., North-European Scientific Publishers, Finland, 2010.

# Generalization of a Remarkable Theorem

Professor Ion Pătraşcu – National College Fraţii Buzeşti, Craiova
Florentin Smarandache, Ph. D. –University of New Mexico, U.S.A.

In [1] Professor Claudiu Coandă proved, using the barycentric coordinates, the following remarkable theorem:

**Theorem** (C. Coandă)

Let $ABC$ be a triangle, where $m(\sphericalangle A) \neq 90°$ and $Q_1, Q_2, Q_3$ are three points on the circumscribed circle of the triangle $ABC$. We'll note $BQ_i \cap AC = \{B_i\}$, $i = \overline{1,3}$. Then the lines $B_1C_1$, $B_2C_2$, $B_3C_3$ are concurrent.

We will generalize this theorem using some results from projective geometry relative to the pole and polar notions.

**Theorem** (Generalization of C. Coandă theorem)

Let $ABC$ be a triangle where $m(\sphericalangle A) \neq 90°$ and $Q_1, Q_2, ..., Q_n$ points on its circumscribed circle $(n \in N,\ n \geq 3)$, $i = \overline{1,n}$. Then the lines $B_1C_1$, $B_2C_2, ..., B_nC_n$ are concurrent in fixed point.

To prove this theorem we'll utilize the following lemmas:

**Lemma 1**

If $ABCD$ is an inscribed quadrilateral in a circle and $\{P\} = AB \cap CD$, then the polar of the point $P$ in rapport with the circle is the line $EF$, where $\{E\} = AC \cap BD$ and $\{F\} = BC \cap AD$

**Lemma 2**

The pole of a line is the intersection of the corresponding polar to any two points of the line.

The pols of concurrent lines in rapport to a given circle are collinear points and the reciprocal is also true: the polar of collinear points, in rappoer with a given circle, are concurrent lines.

**Lemma 3**

If $ABCD$ is an inscribed quadrilateral in a circle and $\{P\} = AB \cap CD$, $\{E\} = AC \cap BD$ and $\{F\} = BC \cap AD$, then the polar of point $E$ in rapport to the circle is the line $PF$.

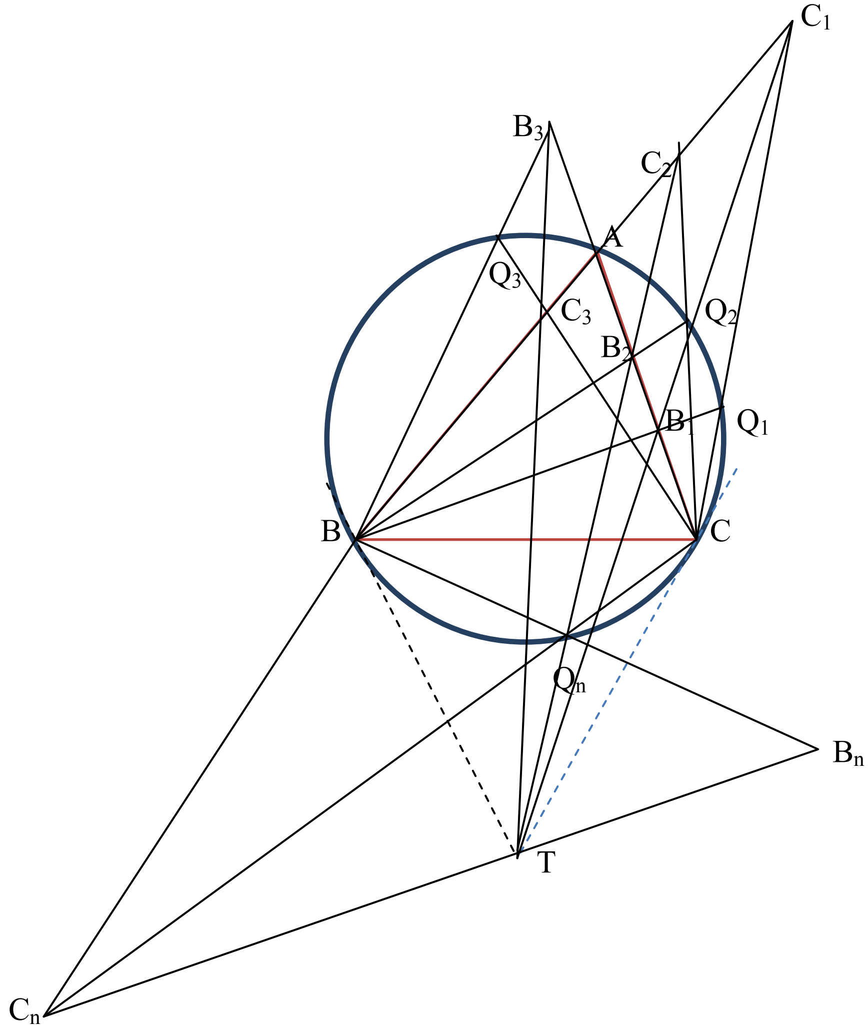
The proof for the Lemmas 1 - 3 and other information regarding the notions of pole and polar in rapport to a circle can be found in [2] or [3].

**Proof of the generalized theorem of C. Coandă**

Let $Q_1, Q_2, ..., Q_n$ points on the circumscribed circle to the triangle $ABC$ (see the figure)

We'll consider the inscribed quadrilaterals $ABCQ_n$, $i = \overline{1,n}$ and we'll note $\{T_i\} = AQ_i \cap BC$.

In accordance to Lemma 1 and Lemma 3, the lines $B_iC_i$ are the respectively polar




(in rapport with the circumscribed circle to the triangle $ABC$) to the points $T_i$.

Because the points $T_i$ are collinear (belonging to the line $BC$), from Lemma 2 we'll obtain that their polar, that is the lines $B_iC_i$, are concurrent in a point $T$.

**Remark**

The concurrency point $T$ is the harmonic conjugate in rapport with the circle of the symmedian center $K$ of the given triangle.

**References**


[1] Claudiu Coandă – Geometrie analitică în coordonate baricentrice – Editura Reprograph, Craiova, 2005.

[2] Ion Pătraşcu – O aplicație practică a unei teoreme de geometrie proiectivă – Journal: Sfera matematică, m 1b (2/2009-2010). Editura Reprograph.

[3] Roger A. Johnson – Advanced Euclidean Geometry – Dover Publications, Inc. Mineola, New York, 2007.

# Pantazi's Theorem Regarding the Bi-orthological Triangles

Prof. Ion Pătraşcu, The National College "Fraţii Buzeşti", Craiova, Romania
Prof. Florentin Smarandache, University of New Mexico, U.S.A.

In this article we'll present an elementary proof of a theorem of Alexandru Pantazi (1896-1948), Romanian mathematician, regarding the bi-orthological triangles.

## 1. Orthological triangles

**Definition**

The triangle $ABC$ is orthologic in rapport to the triangle $A_1B_1C_1$ if the perpendiculars constructed from $A, B, C$ respectively on $B_1C_1, C_1A_1$ and $A_1B_1$ are concurrent. The concurrency point is called the orthology center of the triangle $ABC$ in rapport to triangle $A_1B_1C_1$.

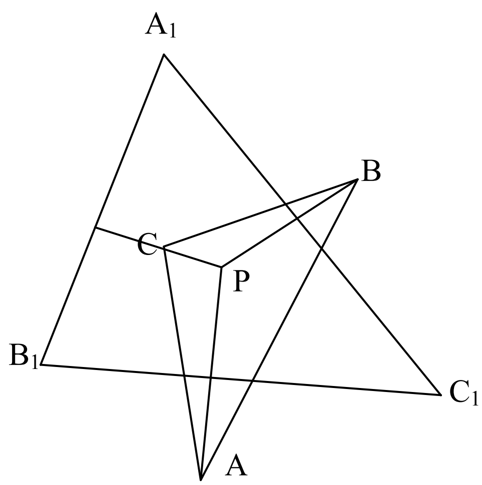



Fig. 1

In figure 1 the triangle $ABC$ is orthologic in rapport with $A_1B_1C_1$, and the orthology center is $P$.

## 2. Examples

a) The triangle $ABC$ and its complementary triangle $A_1B_1C_1$ (formed by the sides' middle) are orthological, the orthology center being the orthocenter $H$ of the triangle $ABC$.

Indeed, because $B_1C_1$ is a middle line in the triangle $ABC$, the perpendicular from $A$ on $B_1C_1$ will be the height from $A$. Similarly the perpendicular from $B$ on $C_1A_1$ and the perpendicular from $C$ on $A_1B_1$ are heights in $ABC$, therefore concurrent in $H$ (see Fig. 2)

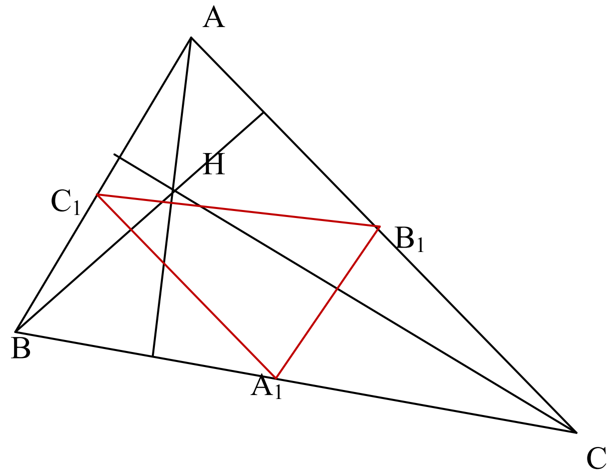



Fig. 2

b) **Definition**

Let $D$ a point in the plane of triangle $ABC$. We call the circum-pedal triangle (or meta-harmonic) of the point $D$ in rapport to the triangle $ABC$, the triangle $A_1B_1C_1$ of whose vertexes are intersection points of the Cevianes $AD, BD, CD$ with the circumscribed circle of the triangle $ABC$.

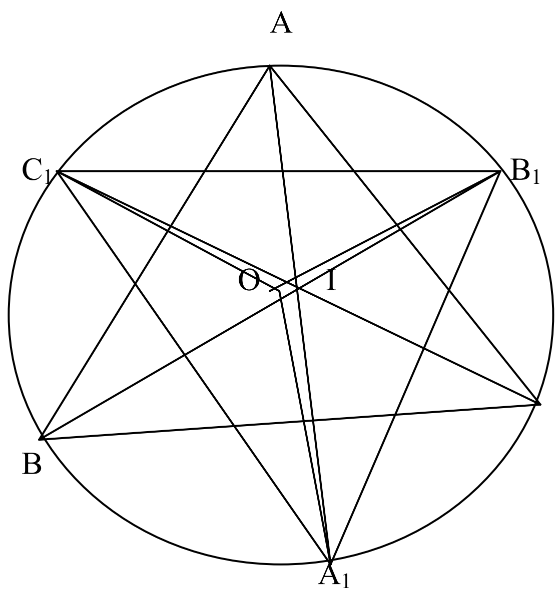



Fig. 3

The triangle circum-pedal $A_1B_1C_1$ of the center of the inscribed circle in the triangle $ABC$ and the triangle $ABC$ are orthological (Fig. 3).

The points $A_1, B_1, C_1$ are the midpoints of the arcs $\overset{\frown}{BC}, \overset{\frown}{CA}$ respectively $\overset{\frown}{AB}$. We have $\overset{\frown}{A_1B} \equiv \overset{\frown}{A_1C}$, it results that $A_1B = A_1C$, therefore $A_1$ is on the perpendicular bisector of $BC$, and therefore the perpendicular raised from $A_1$ on $BC$ passes through $O$,the center of the circumscribed circle to triangle $ABC$. Similarly the perpendiculars raised from $B_1, C_1$ on $AC$ respectively $AB$ pass through $O$. The orthology center of triangle $A_1B_1C_1$ in rapport to $ABC$ is $O$

### 3. The characteristics of the orthology property

The following Lemma gives us a necessary and sufficient condition for the triangle $ABC$ to be orthologic in rapport to the triangle $A_1B_1C_1$.

**Lemma**

The triangle $ABC$ is orthologic in rapport with the triangle $A_1B_1C_1$ if and only if:

$$\overrightarrow{MA}\cdot\overrightarrow{B_1C_1}+\overrightarrow{MB}\cdot\overrightarrow{C_1A_1}+\overrightarrow{MC}\cdot\overrightarrow{A_1B_1}=0 \qquad (1)$$

for any point $M$ from plane.

**Proof**

In a first stage we prove that the relation from the left side, which we'll note $E(M)$ is independent of the point $M$.

Let $N \neq M$ and $E(N)=\overrightarrow{NA}\cdot\overrightarrow{B_1C_1}+\overrightarrow{NB}\cdot\overrightarrow{C_1A_1}+\overrightarrow{NC}\cdot\overrightarrow{A_1B_1}$

Compute $E(M)-E(N)=\overrightarrow{MN}\cdot\left(\overrightarrow{BC}+\overrightarrow{CA}+\overrightarrow{AB}\right)$.

Because $\overrightarrow{BC}+\overrightarrow{CA}+\overrightarrow{AB}=0$ we have that $E(M)-E(N)=\overrightarrow{MN}\cdot\vec{0}=0$.

If the triangle $ABC$ is orthologic in rapport to $A_1B_1C_1$, we consider $M$ their orthologic center, it is obvious that (1) is verified. If (1) is verified for a one point, we proved that it is verified for any other point from plane.

Reciprocally, if (1) is verified for any point $M$, we consider the point $M$ as being the intersection of the perpendicular constructed from $A$ on $B_1C_1$ with the perpendicular constructed from $B$ on $C_1A_1$. Then (1) is reduced to $\overrightarrow{MC}\cdot\overrightarrow{A_1B_1}=0$, which shows that the perpendicular constructed from $C$ on $\overrightarrow{A_1B_1}$ passes through $M$. Consequently, the triangle $ABC$ is orthologic in rapport to the triangle $A_1B_1C_1$.

### 4. The symmetry of the orthology relation of triangles

It is natural to question ourselves that given the triangles $ABC$ and $A_1B_1C_1$ such that $ABC$ is orthologic in rapport to $A_1B_1C_1$, what are the conditions in which the triangle $A_1B_1C_1$ is orthologic in rapport to the triangle $ABC$.

The answer is given by the following

**Theorem** (The relation of orthology of triangles is symmetric)

If the triangle $ABC$ is othologic in rapport with the triangle $A_1B_1C_1$ then the triangle $A_1B_1C_1$ is also orthologic in rapport with the triangle $ABC$.

**Proof**

We'll use the lemma. If the triangle $ABC$ is orthologic in rapport with $A_1B_1C_1$ then

$$\overrightarrow{MA}\cdot\overrightarrow{B_1C_1}+\overrightarrow{MB}\cdot\overrightarrow{C_1A_1}+\overrightarrow{MC}\cdot\overrightarrow{A_1B_1}=0$$

for any point $M$ . We consider $M=A$, then we have

$$\overrightarrow{AA}\cdot\overrightarrow{B_1C_1}+\overrightarrow{AB}\cdot\overrightarrow{C_1A_1}+\overrightarrow{AC}\cdot\overrightarrow{A_1B_1}=0.$$

This expression is equivalent with

$$\overrightarrow{A_1A_1}\cdot\overrightarrow{BC}+\overrightarrow{A_1B_1}\cdot\overrightarrow{CA}+\overrightarrow{A_1C_1}\cdot\overrightarrow{AB}=0$$

That is with (1) in which $M=A_1$, which shows that the triangle $A_1B_1C_1$ is orthologic in rapport to triangle $ABC$ .

**Remarks**

1. We say that the triangles $ABC$ and $A_1B_1C_1$ are orthological if one of the triangle is orthologic in rapport to the other.
2. The orthology centers of two triangles are, in general, distinct points.
3. The second orthology center of the triangles from a) is the center of the circumscribed circle of triangle $ABC$ .
4. The orthology relation of triangles is reflexive. Indeed, if we consider a triangle, we can say that it is orthologic in rapport with itself because the perpendiculars constructed from $A,B,C$ respectively on $BC,CA,AB$ are its heights and these are concurrent in the orthocenter $H$ .

## 5. Bi-orthologic triangles

**Definition**

If the triangle $ABC$ is simultaneously orthologic to triangle $A_1B_1C_1$ and to triangle $B_1C_1A_1$, we say that the triangles $ABC$ and $A_1B_1C_1$ are bi-orthologic.

**Pantazi's Theorem**

If a triangle $ABC$ is simultaneously orthologic to triangle $A_1B_1C_1$ and $B_1C_1A_1$, then the triangle $ABC$ is orthologic also with the triangle $C_1A_1B_1$ .

**Proof**

Let triangle $ABC$ simultaneously orthologic to $A_1B_1C_1$ and to $B_1C_1A_1$, using lemma, it results that

$$\overrightarrow{MA}\cdot\overrightarrow{B_1C_1}+\overrightarrow{MB}\cdot\overrightarrow{C_1A_1}+\overrightarrow{MC}\cdot\overrightarrow{A_1B_1}=0 \tag{2}$$

$$\overrightarrow{MA}\cdot\overrightarrow{C_1A_1}+\overrightarrow{MB}\cdot\overrightarrow{A_1B_1}+\overrightarrow{MC}\cdot\overrightarrow{B_1C_1}=0 \tag{3}$$

For any $M$ from plane.

Adding the relations (2) and (3) side by side, we have:

$$\overrightarrow{MA}\cdot\left(\overrightarrow{B_1C_1}+\overrightarrow{C_1A_1}\right)+\overrightarrow{MB}\cdot\left(\overrightarrow{C_1A_1}+\overrightarrow{A_1B_1}\right)+\overrightarrow{MC}\cdot\left(\overrightarrow{A_1B_1}+\overrightarrow{B_1C_1}\right)=0$$

Because

$$\overrightarrow{B_1C_1}+\overrightarrow{C_1A_1}=\overrightarrow{B_1A_1},\ \overrightarrow{C_1A_1}+\overrightarrow{A_1B_1}=\overrightarrow{C_1B_1},\ \overrightarrow{A_1B_1}+\overrightarrow{B_1C_1}=\overrightarrow{A_1C_1}$$

(Chasles relation), we have:

$$\overrightarrow{MA}\cdot\overrightarrow{B_1A_1}+\overrightarrow{MB}\cdot\overrightarrow{C_1B_1}+\overrightarrow{MC}\cdot\overrightarrow{A_1C_1}=0$$

for any $M$ from plane, which shows that the triangle $ABC$ is orthologic with the triangle $C_1A_1B_1$ and the Pantazi's theorem is proved.

**Remark**

The Pantazi's theorem can be formulated also as follows: If two triangles are bi-orthologic then these are tri-orthologic.

**Open Questions**

*1)* Is it possible to extend Pantazi's Theorem (in 2D-space) in the sense that if two triangles $A_1B_1C_1$ and $A_2B_2C_2$ are bi-orthological, then they are also $k$-orthological, where $k = 4, 5,$ or $6$?

*2)* Is it true a similar theorem as Pantazi's for two bi-homological triangles and bi-orthohomological triangles (in 2D-space)? We mean, if two triangles $A_1B_1C_1$ and $A_2B_2C_2$ are bi-homological (respectively bi-orthohomological), then they are also $k$-homological (respectively $k$-orthohomological), where $k = 4, 5,$ or $6$?

3) How the Pantazi Theorem behaves if the two bi-orthological non-coplanar triangles $A_1B_1C_1$ and $A_2B_2C_2$ (if any) are in the 3D-space?

4) Is it true a similar theorem as Pantazi's for two bi-homological (respectively bi-orthohomological) non-coplanar triangles $A_1B_1C_1$ and $A_2B_2C_2$ (if any) in the 3D-space?

5) Similar questions as above for bi-orthological / bi-homological / bi-orthohomological polygons (if any) in 2D-space, and respectively in 3D-space.

6) Similar questions for bi-orthological / bi-homological / bi-orthohomological polyhedrons (if any) in 3D-space.

**References**


[1] Cătălin Barbu – Teoreme fundamentale din geometria triunghiului, Editura Unique, Bacău, 2007.

[2] http://garciacapitan.auna.com/ortologicos/ortologicos.pdf.

# A New Proof and an Application of Dergiades' Theorem

Ion Pătraşcu, Professor, The Fraţii Buzeşti College, Craiova, Romania
Florentin Smarandache, Professor, The University of New Mexico, U.S.A.

In this article we'll present a new proof of Dergiades' Theorem, and we'll use this theorem to prove that the orthological triangles with the same orthological center are homological triangles.

**Theorem 1 (Dergiades)**

Let $C_1(O_1,R_1)$, $C_2(O_2,R_2)$, $C_3(O_3,R_3)$ three circles which pass through the vertexes B and $C$, $C$ and $A$, $A$ and $B$ respectively of a given triangle $ABC$. We'll note $D,E,F$ respectively the second point of intersection between the circles $(C_1)$ and $(C_3)$, $(C_3)$ and $(C_2)$, $(C_1)$ and $(C_2)$. The perpendiculars constructed in the points $D,E,F$ on $AD$, $BE$ respectively $CF$ intersect the sides $BC$, $CA$, $AB$ in the points $X$, $Y$, $Z$. Then the points $X$, $Y$, $Z$ are collinear

**Proof**

To prove the collinearity of the points $X$, $Y$, $Z$, we will use the reciprocal of the Menelaus Theorem (see *Fig. 1*).

We have

$$\frac{XB}{XC}=\frac{Aria\Delta XDB}{Aria\Delta XDC}=\frac{DB\cdot \sin\widehat{XDB}}{DC\cdot \sin\widehat{XDC}}=\frac{DB\cdot \cos\widehat{ADB}}{DC\cdot \cos\widehat{ADC}}$$

Similarly we find

$$\frac{YC}{YA}=\frac{EC\cdot \cos\widehat{BEC}}{EA\cdot \cos\widehat{BEA}}$$

$$\frac{ZA}{ZB}=\frac{FA\cdot \cos\widehat{CFA}}{FB\cdot \cos\widehat{CFB}}$$

From the inscribed quadrilaterals $ADEB; BEFC; ADFC$, we can observe that

$$\sphericalangle ADB\equiv \sphericalangle BEA;\ \sphericalangle BEC\equiv \sphericalangle CFB;\ \sphericalangle CFA\equiv \sphericalangle ADC$$

Consequently,

$$\frac{XB}{XC}\cdot\frac{YC}{YA}\cdot\frac{ZA}{ZB}=\frac{DB}{DC}\cdot\frac{EC}{EA}\cdot\frac{FA}{FB} \qquad (1)$$

On the other side $DB=2R_3\sin\widehat{BAD}$; $EA=2R_3\sin\widehat{ABE}$; $DC=2R_2\sin\widehat{CAD}$; $FA=2R_2\sin\widehat{ACF}$; $FB=2R_1\sin\widehat{BCF}$; $EC=2R_1\sin\widehat{CBE}$.

Using these relations in (1), we obtain

$$\frac{XB}{XC}\cdot\frac{YC}{YA}\cdot\frac{ZA}{ZB}=\frac{\sin\widehat{BAD}}{\sin\widehat{CAD}}\cdot\frac{\sin\widehat{CBE}}{\sin\widehat{ABE}}\cdot\frac{\sin\widehat{ACF}}{\sin\widehat{BCF}} \qquad (2)$$

According to one of Carnot's theorem, the common strings of the circles $(C_1)$, $(C_2)$, $(C_3)$ are concurrent, that is $AD\cap BE\cap CF=\{P\}$ (the point $P$ is the radical center of the circles $(C_1)$, $(C_2)$, $(C_3)$).

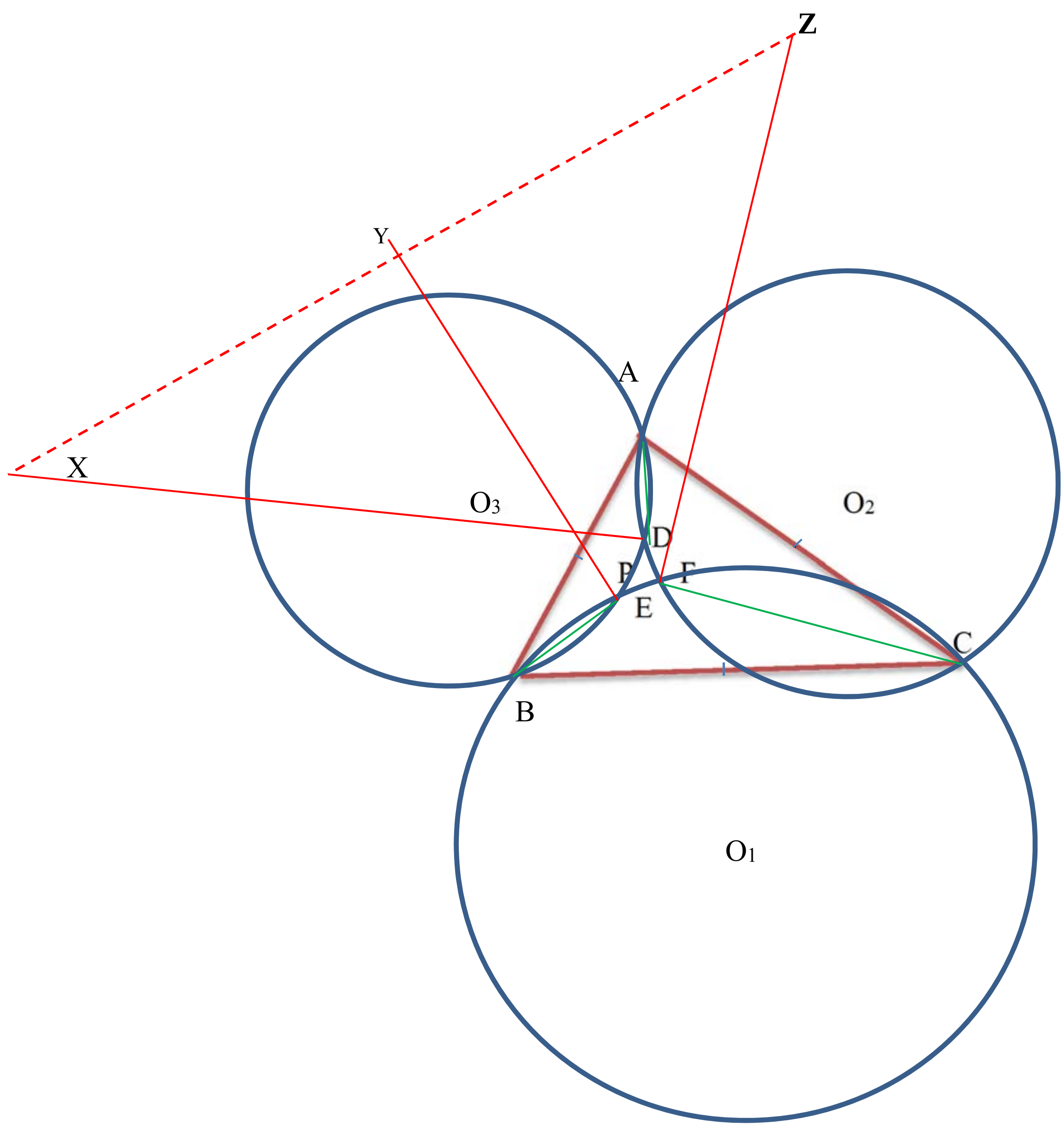



Fig 1.

In triangle $ABC$, the cevians $AD, BE, CF$ being concurrent, we can use for them the trigonometrically form of the Ceva's theorem as follows

$$\frac{\sin \widehat{BAD}}{\sin \widehat{CAD}} \cdot \frac{\sin \widehat{CBE}}{\sin \widehat{ABE}} \cdot \frac{\sin \widehat{ACF}}{\sin \widehat{BCF}} = 1 \qquad (3)$$

The relations (2) and (3) lead to

$$\frac{XB}{XC} \cdot \frac{YC}{YA} \cdot \frac{ZA}{ZB} = 1$$

Relation, which in conformity with Menelaus theorem proves the collinearity of the points $X, Y, Z$.

**Definition 1**

Two triangles $ABC$ and *A'B'C'* are called orthological if the perpendiculars constructed from *A* on *B'C'*, from $B$ on *C'A'* and from *C* on *A'B'* are concurrent. The concurrency point of these perpendiculars is called the orthological center of the triangle $ABC$ in rapport to triangle *A'B'C'*.

**Theorem 2 (The theorem of orthological triangle of J. Steiner)**

If the triangle $ABC$ is orthological with the triangle *A'B'C'*, then the triangle *A'B'C'* is also orthological in rapport to triangle $ABC$.

For the proof of this theorem we recommend [1].

**Observation**

A given triangle and its contact triangle are orthological triangles with the same orthological center. Their common orthological center is the center of the inscribed circle of the given triangle.

**Definition 3**

Two triangles $ABC$ and *A'B'C'* are called homological if and only if the lines $AA', BB', CC'$ are concurrent. The congruency point is called the homological center of the given triangles.

**Theorem 3 (Desargues – 1636)**

If $ABC$ and *A'B'C'* are two homological triangles, then the lines $(BC, B'C');\ (CA, C'A');\ (AB, A'B')$ are concurrent respectively in the points $X, Y, Z$, and these points are collinear. The line that contains the points $X, Y, Z$ is called the homological axis of the triangles $ABC$ and $A'B'C'$.

For the proof of Desargues theorem see [3].

**Theorem 4**

Two orthological triangles that have a common orthological center are homological triangles.

**Lemma 1**

Let $ABC$ and $A'B'C'$ two orthological triangles. The orthogonal projections of the vertexes $B$ and $C$ on the sides $A'C'$ respectively $A'B'$ are concyclic.

**Proof**

We note with $E, F$ the orthogonal projections f the vertexes $B$ and $C$ on $A'C'$ respectively $A'B'$ (see Fig. 2). Also, we'll note $O$ the common orthological center of the orthological triangles $ABC$ and $A'B'C'$ and $\{B''\} = EO \cap AC$, $\{C''\} = FO \cap AB$. In the triangle $A'B''C''$, $O$ being the intersection of the heights constructed from $B'', C''$, is the orthocenter of this triangle, consequently, it results that $A'O \perp B''C''$. On the other side $A'O \perp BC$; we obtain, therefore that $B''C'' \parallel BC$. Taking into consideration that $EF$ and $B''C''$ are antiparallel in rapport to $A'B'$ and $A'C'$, we obtain that $EF$ is antiparallel with $BC$, fact that shows that the quadrilateral $BCFE$ is inscribable.

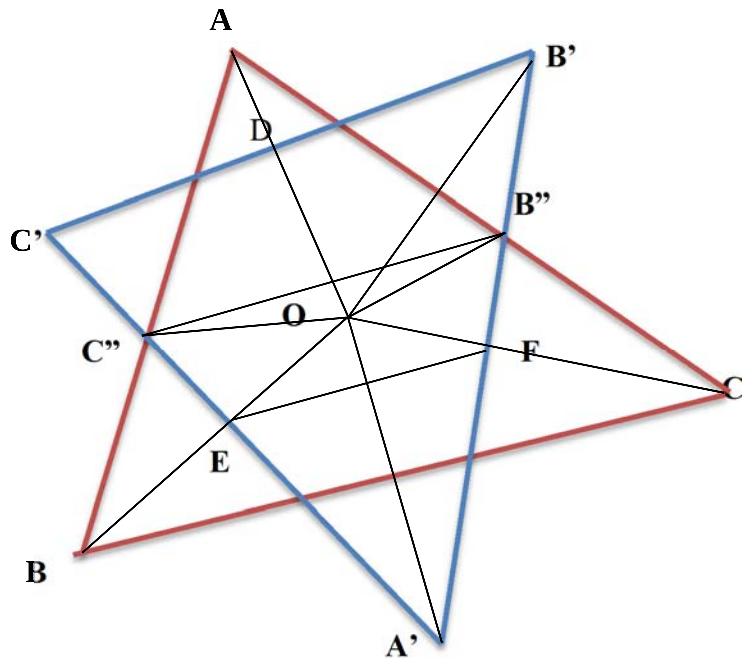


Fig. 2

**Observation**

If we denote with $D$ the projection of $A$ on $B'C'$, similarly, it will result that the points $A, D, F, C$ respectively $A, D, E, B$ are concyclic.

**Proof of Theorem 4**

The quadrilaterals $BCFE, CFDA, ADEB$ being inscribable, it result that their circumscribed circles satisfy the Dergiades theorem (Fig. 2). Applying this theorem it results that the pairs of lines $(BC, B'C');\ (CA, C'A');\ (AB, A'B')$ intersect in the collinear points $X, Y, Z$, respectively. Using the reciprocal theorem of Desargues, it result that the lines $AA', BB', CC'$ are concurrent and consequently the triangles $ABC$ and $A'B'C'$ are homological.

**Observations**

1 Triangle $O_1O_2O_3$ formed by the centers of the circumscribed circles to quadrilaterals $BCFE, CFDA, ADEB$ and the triangle $ABC$ are orthological triangles.
The orthological centers are the points $P$ - the radical center of the circles $(O_1),(O_2),(O_3)$ and $O$ - the center of the circumscribed circle of the triangle $ABC$.

2 The triangles $O_1O_2O_3$ and $DEF$ (formed by the projections of the vertexes $A, B, C$ on the sides of the triangle $A'B'C'$) are orthological. The orthological centers are the center of the circumscribed circle to triangle $DEF$ and $P$ the radical center of the circles $(O_1),(O_2),(O_3)$.

Indeed, the perpendiculars constructed from $O_1, O_2, O_3$ on $EF, FD, DA$ respectively are the mediators of these segments and, therefore, are concurrent in the center of the circumscribed circle to triangle $DEF$, and the perpendiculars constructed from $D, E, F$ on the sides of the triangle $O_1O_2O_3$ are the common strings $AD, BE, CF$, which, we observed above, are concurrent in the radical center $P$ of the circles with the centers in $O_1, O_2, O_3$.

**References**


[1] Mihalescu, C. - Geometria elementelor remarcabile – Ed. Tehnică, Bucureşti, 1957

[2] Barbu, C. – Teoreme fundamentale din geometria triunghiului – Ed. Unique, Bacău, 2008

[3] Smarandache, F. and Pătraşcu, I. – The Geometry of Homological Triangles – The Education Publisher, Inc. Columbus, Ohio, U.S.A. 2012.

# Mixt-Linear Circles Adjointly Ex-Inscribed Associated to a Triangle

**Ion Pătraşcu, Professor, The Fraţii Buzeşti College, Craiova, Romania**
**Florentin Smarandache, Professor, The University of New Mexico, U.S.A.**

**Abstract**

In [1] we introduced the mixt-linear circles adjointly inscribed associated to a triangle, with emphasizes on some of their properties. Also, we've mentioned about mixt-linear circles adjointly ex-inscribed associated to a triangle.

In this article we'll show several basic properties of the mixt-linear circles adjointly ex-inscribed associate to a triangle.

**Definition 1**

We define a mixt-linear circle adjointly ex-inscribed associated to a triangle, the circle tangent exterior to the circle circumscribed to a triangle in one of the vertexes of the triangle, and tangent to the opposite side of the vertex of that triangle.

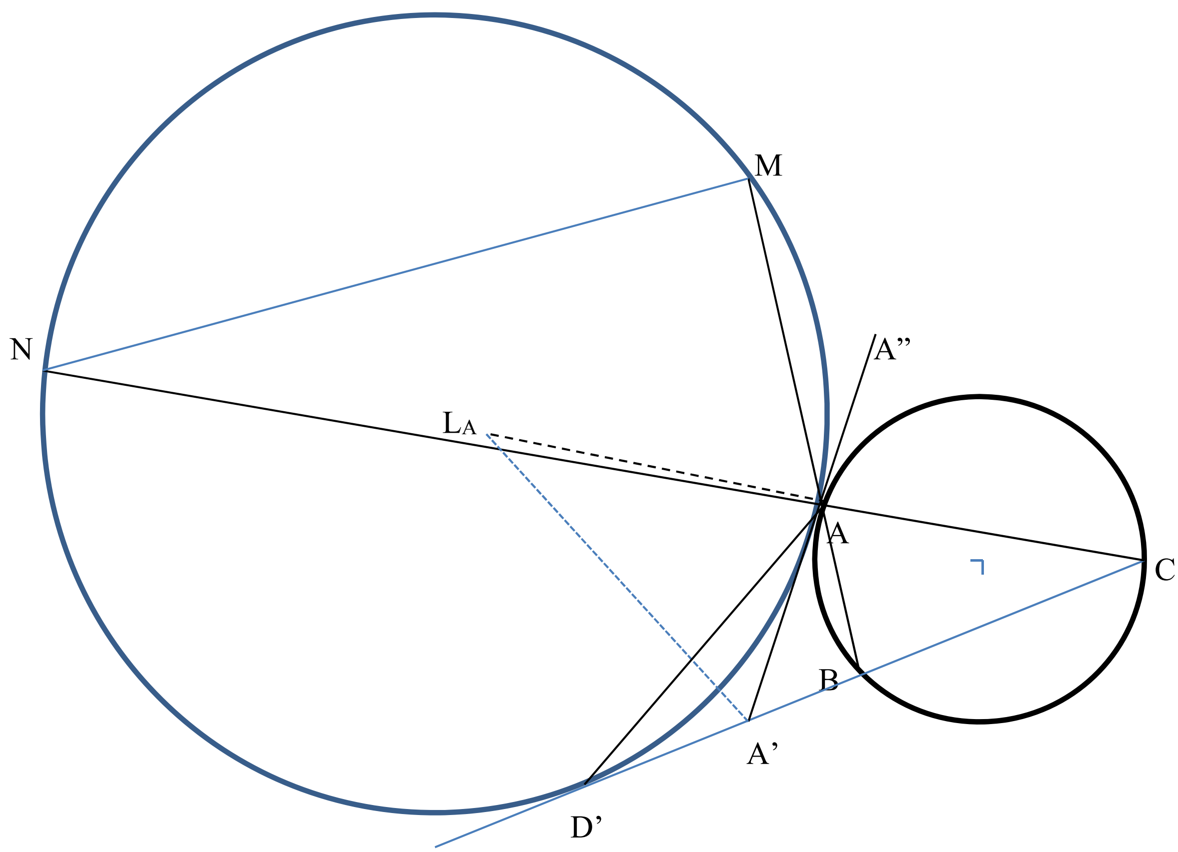



Fig. 1

**Observation**

In Fig.1 we constructed the mixt-linear circle adjointly ex-inscribed to triangle $ABC$, which is tangent in $A$ to the circumscribed circle of triangle $ABC$, and tangent to the side $BC$. Will call this the $A$-mixt-linear circle adjointly ex-inscribed to triangle $ABC$. We note $L_A$ the center of this circle.

**Remark**

In general, for a triangle exists three mixt-linear circles adjointly ex-inscribed. If the triangle $ABC$ is isosceles with the base $BC$, then we cannot talk about mixt-linear circles adjointly ex-inscribed associated to the isosceles triangle.

**Proposition 1**

The tangency point with the side $BC$ of the $A$-mixt-linear circle adjointly ex-inscribed associated to the triangle is the leg of the of the external bisectrix of the angle $BAC$

**Proof**

Let $D'$ the contact point with the side $BC$ of the $A$-mixt-linear circle adjointly ex-inscribed and let $A'$ the intersection of the tangent in the point $A$ to the circumscribed circle to the triangle $ABC$ with $BC$ (see Fig. 1)

We have

$$m\left(\sphericalangle AA'B\right)=\frac{1}{2}\left[m\left(\hat{B}\right)-m\left(\hat{C}\right)\right],$$

(we supposed that $m\left(\hat{B}\right)>m\left(\hat{C}\right)$). The tangents $AA', A'D'$ to the $A$-mixt-linear circle adjointly ex-inscribed are equal, therefore

$$m\left(\sphericalangle D'AA'\right)=\frac{1}{4}m\left(\hat{B}-\hat{C}\right).$$

Because

$$m\left(\sphericalangle A'AB\right)=\frac{1}{2}m\left(\hat{C}\right)$$

we obtain that

$$m\left(\sphericalangle D'AB\right)=\frac{1}{2}\left[m\left(\hat{B}\right)+m\left(\hat{C}\right)\right]$$

This relation shows that $D'$ is the leg of the external bisectrix of the angle $BAC$.

**Proposition 2**

The $A$-mixt-linear circle adjointly ex-inscribed to triangle $ABC$ intersects the sides $AB, AC$, respectively, in two points of a cord which is parallel to $BC$.

**Proof**

We'll note with $M, N$ the intersection points with $AB$ respectively $AC$ of the $A$-mixt-linear circle adjointly ex-inscribed. We have $\sphericalangle BCA\equiv\sphericalangle BAA'$ and $\sphericalangle A'AB\equiv\sphericalangle A''AM$ (see Fig.1).

Because $\sphericalangle A''AM=\sphericalangle ANM$, we obtain $\sphericalangle ANM\equiv\sphericalangle ACB$ which implies that $MN$ is parallel to $BC$.

**Proposition 3**

The radius $R_A$ of the $A$-mixt-linear circle adjointly ex-inscribed to triangle $ABC$ is given by the following formula

$$R_A = \frac{4(p-b)(p-c)R}{(b-c)^2}$$

**Proof**

The sinus theorem in the triangle $AMN$ implies

$$R_A = \frac{MN}{2\sin A}$$

We observe that the triangles $AMN$ and $ABC$ are similar; it results that

$$\frac{MN}{a} = \frac{AM}{c}.$$

Considering the power of the point $B$ in rapport to the $A$-mixt-linear circle adjointly ex-inscribed of triangle $ABC$, we obtain

$$BA \cdot BM = BD'^2.$$

From the theorem of the external bisectrix we have $\frac{D'B}{D'C} = \frac{c}{b}$ from which we retain $D'B = \frac{ac}{b-c}$. We obtain then $BM = \frac{a^2 c}{(b-c)^2}$, therefore

$$AM = \frac{c(a-b+c)(a+b-c)}{(b-c)^2} = \frac{4c(p-b)(p-c)}{(b-c)^2}$$

and

$$MN = \frac{4a(p-b)(p-c)}{(b-c)^2}$$

From the sinus theorem applied in the triangle $ABC$ results that $\frac{a}{2\sin A} = R$ and we obtain that

$$R_A = \frac{4(p-b)(p-c)R}{(b-c)^2}.$$

**Remark**

If we note $P \in L_A A' \cap AD'$ and $AD' = l_a'$ (the length of the exterior bisectrix constructed from $A$) in triangle $L_A P A'$, we find

$$R_A = \frac{l_a'}{2\sin\frac{B-C}{2}}.$$

We'll remind here several results needed for the remaining of this presentation.

**Definition 2**

We define an adjointly circle of triangle $ABC$ a circle which contains two vertexes of the triangle and in one of these vertexes is tangent to the respective side.

**Theorem 1**

The adjointly circles $A\overline{B}, B\overline{C}, C\overline{A}$ have a common point $\Omega$; similarly, the circles $B\overline{A}, C\overline{B}, A\overline{C}$ have a common point $\Omega'$.

The points $\Omega$ and $\Omega'$ are called the points of Brocard: $\Omega$ is the direct point of Brocard and $\Omega'$ is called the retrograde point.

The points $\Omega$ and $\Omega'$ are conjugate isogonal

$$\sphericalangle \Omega AB = \sphericalangle \Omega BC = \sphericalangle \Omega CA = \omega$$

$$\sphericalangle \Omega' AC = \sphericalangle \Omega' CB = \sphericalangle \Omega' BA = \omega$$

(see Fig. 2).

The angle $\omega$ is called the Brocard angle. More information can be found in [3].

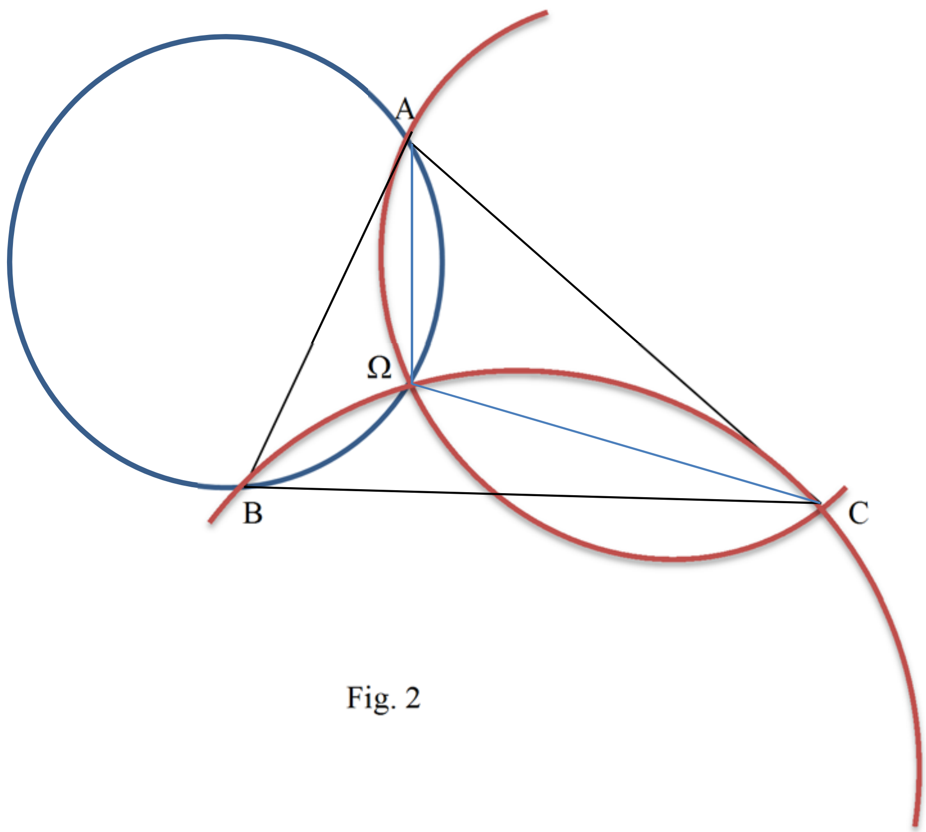



Fig. 2

**Proposition 4**

In triangle $ABC$ in which $D'$ is the leg of the external bisectrix of the angle $BAC$, the $A$-mixt-linear circle adjointly ex-inscribed to triangle $ABC$ is an adjointly circle of triangles $AD'B, AD'C$.

**Proposition 5**

In a triangle $ABC$ in which $D'$ is the leg of the external bisectrix of the angle $BAC$, the direct points of Brocard corresponding to triangles $AD'B$, $AD'C$, A, D' are concyclic.

The following theorems show remarkable properties of the mixt-linear circles adjointly ex-inscribed associated to a triangle $ABC$.

**Theorem 2**

The triangle $L_A L_B L_C$ determined by the centers of the mixt-linear circles adjointly ex-inscribed to triangle $ABC$ and the tangential triangle $T_a T_b T_c$ corresponding to $ABC$ are orthological. Their orthological centers are $O$ the center of the circumscribed circle to triangle $ABC$ and the radical center of the mixt-linear circles adjointly ex-inscribed associated to triangle $ABC$.

**Proof**

The perpendiculars constructed from $L_A, L_B, L_C$ on the corresponding sides of the tangential triangle contain the radiuses $OA$, $OB$, $OC$ respectively of the circumscribed circle.

Consequently, $O$ is the orthological center of triangles $L_A L_B L_C$ and $T_a T_b T_c$.

In accordance to the theorem of orthological triangles and the perpendiculars constructed from $T_a, T_b, T_c$ respectively on the sides of the triangle $L_A L_B L_C$ are concurrent.

The point $T_a$ belongs to the radical axis of the circumscribed circles to triangle $ABC$ and the $C$-mixt-linear circle adjointly ex-inscribed to triangle $ABC$ (belongs to the common tangent constructed in $C$ to these circles).

On the other side $T_a$ belongs to the radical axis of the $B$ and $C$-mixt-linear circle adjointly ex-inscribed, which means that the perpendicular constructed from $T_a$ on the $L_B L_C$ centers line passes through the radical center of the mixt-linear circle adjointly ex-inscribed associated to the triangle; which is the second orthological center of the considered triangles.

**Proposition 6**

The triangle $L_a L_b L_c$ (determined by the centers of the mixt-linear circles adjointly inscribed associated to the triangle $ABC$) and the triangle $L_A L_B L_C$ (determined by the centers of the mixt-linear circles adjointly ex-inscribed associated to the triangle $ABC$) are homological. The homological center is the point $O$, which is the center of the circumscribed circle of triangle $ABC$.

The proof results from the fact that the points $L_A, A, L_a, O$ are collinear. Also, $L_B, B, L_b, O$ and $L_C, C, L_c, O$ are collinear.

**Definition 3**

Given three circles of different centers, we define their Apollonius circle as each of the circles simultaneous tangent to three given circles.

**Observation**

The circumscribed circle to the triangle $ABC$ is the Apollonius circle for the mixt-linear circles adjointly ex-inscribed associated to $ABC$.

**Theorem 3**

The Apollonius circle which has in its interior the mixt-linear circles adjointly ex-inscribed to triangle $ABC$ is tangent with them in the points $T_1, T_2, T_3$ respectively. The lines $AT_1, BT_2, CT_3$ are concurrent.

**Proof**

We'll use the D'Alembert theorem: Three circles non-congruent whose centers are not collinear have their six homothetic centers placed on four lines, three on each line.

The vertex $A$ is the homothety inverse center of the circumscribed circle $(O)$ and of the $A$-mixt-linear circle adjointly ex-inscribed $(L_A)$; $T_1$ is the direct homothety center of the Apollonius circle which is tangent to the mixt-linear circles adjointly ex-inscribed and of circle $(L_A)$, and $J$ is the center of the direct homothety of the Apollonius circle and of the circumscribed circle $(O)$.

According to D'Alembert theorem, it results that the points $A, J, T_1$ are collinear. Similarly is shown that the points $B, J, T_2$ and $C, J, T_3$ are collinear.

Consequently, $J$ is the concurrency point of the lines $AT_1, BT_2, CT_3$.


[1] I. Pătraşcu, Cercuri mixtliniare adjunct inscrise associate unui triunghi, Revista Recreatii Matematice , No. 2/2013.

[2] R. A. Johnson, Advanced Euclidean Geometry, Dover Publications Inc., New York, 2007.

[3] F. Smarandache, I. Pătraşcu, The Geometry of Homological Triangles, The Education Publisher Inc., Columbus, Ohio, U.S.A., 2012.

# A PROPERTY OF THE CIRCUMSCRIBED OCTAGON

Prof. Ion Pătraşcu – "Fraţii Buzeşti" College, Craiova, Romania
Prof. Florentin Smarandache – University of New Mexico, USA

**Abstract**

In this article we'll obtain through the duality method a property in relation to the contact cords of the opposite sides of a circumscribable octagon.

In an inscribed hexagon the following theorem proved by Blaise Pascal in 1640 is true.

**Theorem 1** (Blaise Pascal)

The opposite sides of a hexagon inscribed in a circle intersect in collinear points.

To prove the Pascal theorem one may use [1].

In [2] there is a discussion that the Pascal's theorem will be also true if two or more pairs of vertexes of the hexagon coincide. In this case, for example the side $AB$ for $B \to A$ must be substituted with the tangent in $A$. For example we suppose that two pairs of vertexes coincide. The hexagon $AA'BCC'D$ for A'→ $A$, C'→ $C$ becomes the inscribed quadrilateral $ABCD$. This quadrilateral viewed as a degenerated hexagon of sides $AB, BC, CC' \to$ the tangent in $C, C'D \to CD, D'A \to DA, AA' \to$ the tangent in $A$ and the Pascal theorem leads to:

**Theorem 2**

In an inscribed quadrilateral the opposite sides and the tangents in the opposite vertexes intersect in four collinear points.

**Remark 1**

In figure 1 is presented the corresponding configuration of theorem 2.

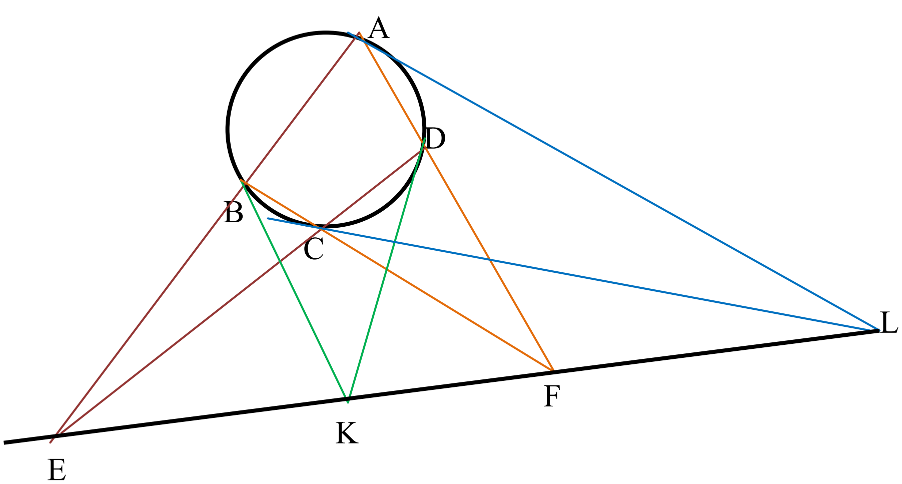



Fig. 1

For the tangents constructed in $B$ and $D$ the property is also true if we consider the $ABCD$ as a degenerated hexagon $ABB'CDD'A$.

**Theorem 3**

In an inscribed octagon the four cords determined by the contact points with the circle of the opposite sides are concurrent.

**Proof**

We'll transform through reciprocal polar the configuration from figure 1. To point $E$ will correspond, through this transformation the line determined by the tangent points with the circle of the tangents constructed from $E$ (its polar). To point $K$ corresponds the side $BD$.

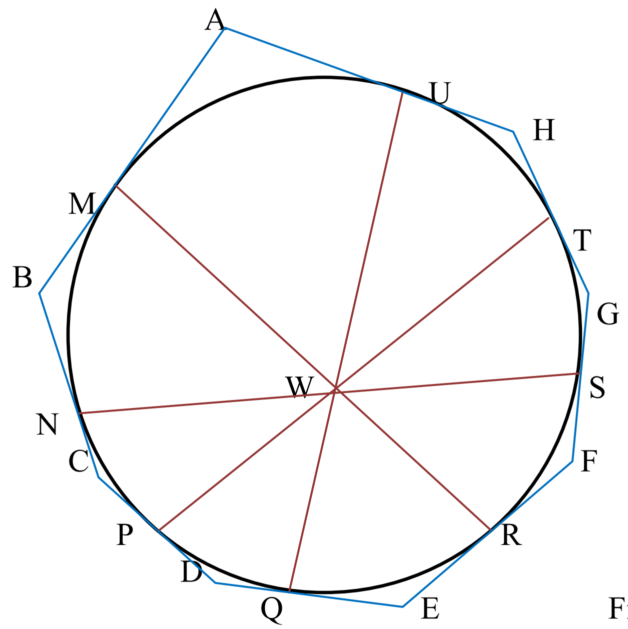



Fig.2

To point $F$ corresponds the line determined by the contact points of the tangents constructed from $F$ to the circle. To point $L$ corresponds its polar $AC$. To point $A$ corresponds, by duality, the tangent $AL$, also to points $B, C, D$ correspond the tangents $BK, CL, DK$. These four tangents together with the tangents constructed from $E$ and $F$ (also four) will contain the sides of an octagon circumscribed to the given circle.

In this octagon $(AC)$ and $(BD)$ will connect the contact points of two pairs of opposite sides with the circle; the other two lines determined by the contact points of the opposite sides of the octagon with the circle will be the polar of the points $E$ and $F$. Because the polar transformation through reciprocal polar leads to the fact that to collinear points correspond concurrent lines; the points' polar $E, K, F, L$ are concurrent; these lines are the cords to which the theorem refers to.

**Remark 2**

In figure 2 we represented an octagon circumscribed $ABCDEFGH$ . As it can be seen the cords $MR$, $NS$, $PT$, $QU$ are concurrent in the point $W$.

**References**

[1] Roger A Johnson – *Advanced Euclidean Geometry*, Dover Publications, Inc. Mineola, New-York, 2007

[2] N. Mihăileanu – *Lecţii complementare de geometrie*, Editura Didactică şi Pedagogică, Bucureşti, 1976

[3] Florentin Smarandache – *Multispace & Multistructure, Neutrosophic Trandisciplinarity (100 Collected Papers of Sciences)*, Vol IV, 800 p., North-European Scientific Publishers, Hanko, Finland, 2010

# From Newton's Theorem to a Theorem of the Inscribable Octagon

**Prof. Ion Pătraşcu, The Fraţii Buzeşti National College, Craiova, Romania**
**Prof. Florentin Smarandache, The University of New Mexico, U.S.A.**

In this article we'll prove the Newton's theorem relative to the circumscribed quadrilateral, we'll transform it through duality, and we obtain another theorem which is true for an inscribable quadrilateral, which transformed through duality, we'll obtain a theorem which is true for a circumscribable octagon.

**Theorem 1 (I. Newton)**
In a circumscribable quadrilateral its diagonals and the cords determined by the contact points of the opposite sides of the quadrilateral with the circumscribed circle are four concurrent lines.

**Proof**

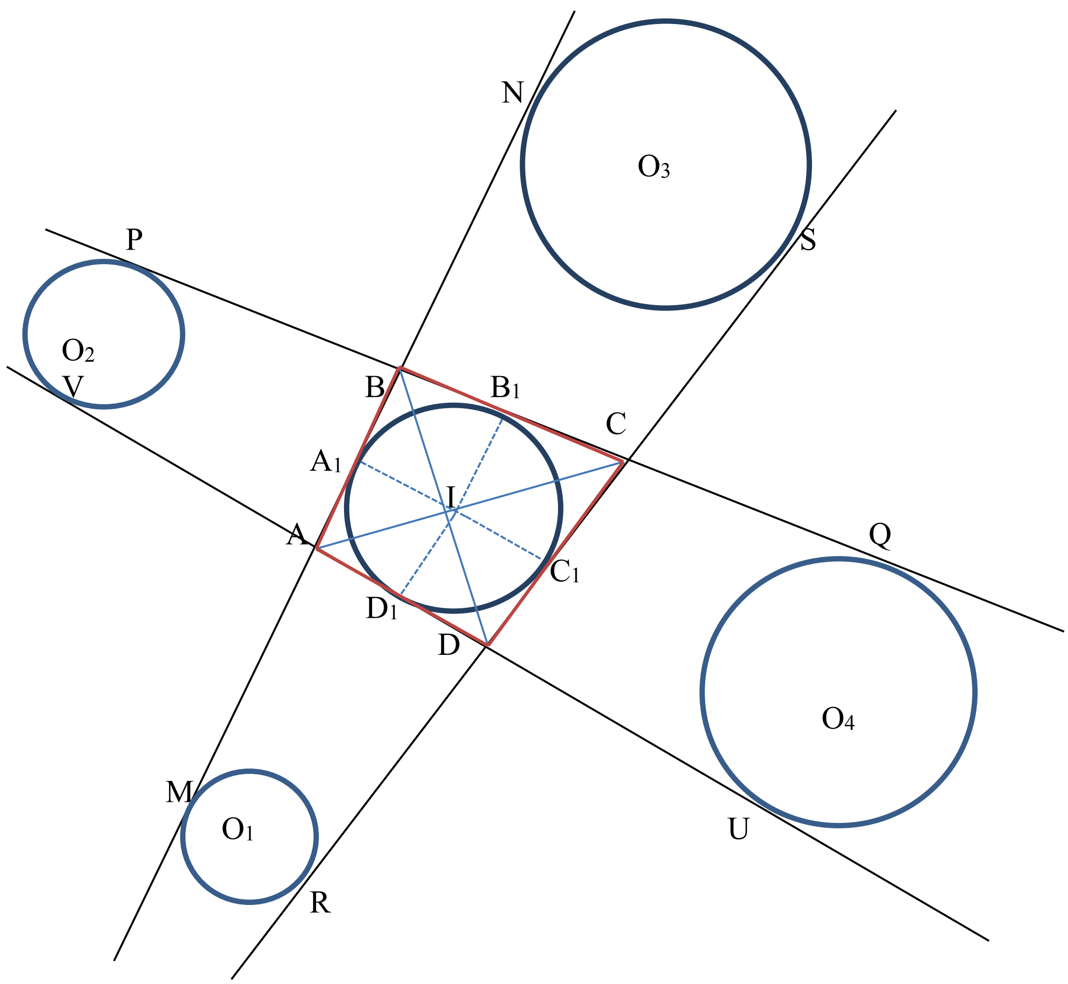



Fig. 1

We constructed the circles $O_1$, $O_2$, $O_3$, $O_4$ tangent to the extensions of the quadrilateral $ABCD$ such that

$$A_1M = A_1N = B_1P = B_1Q = C_1R = C_1S = D_1U = D_1V$$

See Fig. 1.

From $A_1M = A_1N = C_1R = C_1S$ it results that the points $A_1$ and $C_1$ have equal powers in relation to the circles $O_1$ and $O_3$, therefore $A_1C_1$ is the radical axis of these circles. Similarly $B_1D_1$ is the radical axis of the circles $O_2$ and $O_4$.

Let $I \in A_1C_1 \cap B_1D_1$. The point $I$ has equal powers in rapport to circles $O_1$, $O_2$, $O_3$, $O_4$. Because $BA_1 = BB_1$ from $B_1P = A_1N$ it results that $BP = BN$, similarly, from $DD_1 = DC_1$ and $D_1V = C_1S$ it results that $DV = DS$, therefore $B$ and $D$ have equal powers in rapport with the circles $O_3$ and $O_4$, which shows that $BD$ is the radical axis of these circles. Consequently, $I \in BD$, similarly it results that $I \in AC$, and the proof is complete.

**Theorem 2.**

In an inscribed quadrilateral in which the opposite sides intersect, the intersection points of the tangents constructed to the circumscribed circle with the opposite vertexes and the points of intersection of the opposite sides are collinear.

**Proof**

We'll prove this theorem applying the configuration from the Newton theorem, o transformation through duality in rapport with the circle inscribed in the quadrilateral. Through this transformation to the lines $AB, BC, CD, DA$ will correspond, respectively, the points $A_1$, $B_1$, $C_1$, $D_1$ their pols. Also to the lines $A_1B_1$, $B_1C_1$ ,$C_1D_1$ ,$D_1A_1$ correspond, respectively, the points $B, C, D, A$. We note $X \in AB \cap CD$ and $Y \in AD \cap BC$, these points correspond, through the considered duality, to the lines $A_1C_1$ respectively $B_1D_1$. If $I \in A_1C_1 \cap B_1D_1$ then to the point $I$ corresponds line $XY$, its polar.

To line $BD$ corresponds the point $Z \in A_1D_1 \cap C_1B_1$.

To line $AC$ corresponds the point $T \in A_1D_1 \cap C_1B_1$.

To point $\{I\} = BD \cap AC$ corresponds its polar $ZT$.

We noticed that to the point $I$ corresponds the line $XY$, consequently the points $X, Y, Z, T$ are collinear.

We obtained that the quadrilateral $A_1B_1C_1D_1$ inscribed in a circle has the property that if $A_1D_1 \cap C_1B_1 = \{Z\}$, $A_1D_1 \cap C_1B_1 = \{T\}$, the tangent in $A_1$ and the tangent in $C_1$ intersect in the point $X$; the tangent in $B_1$ and the tangent in $D_1$ intersect in $Y$, then $X$, $Y$, $Z$, $T$ are collinear (see Fig. 2).

**Theorem 3.**

In a circumscribed octagon, the four cords, determined by the octagon's contact points with the circle of the octagon opposite sides, are concurrent.

**Proof**

We'll transform through reciprocal polar the configuration in figure 3.

To point $Z$ corresponds through this transformation the line determined by the tangency points with the circle of the tangents constructed from $Z$ - its polar; to the point $Y$ it corresponds the line determined by the contact points of the tangents constructed from $T$ at the circle; to the point $X$ corresponds its polar $A_1C_1$.

To point $A_1$ corresponds through duality the tangent $A_1X$, also to the points $B_1$, $C_1$, $D_1$ correspond the tangents $B_1Y$, $C_1T$, $D_1Z$.

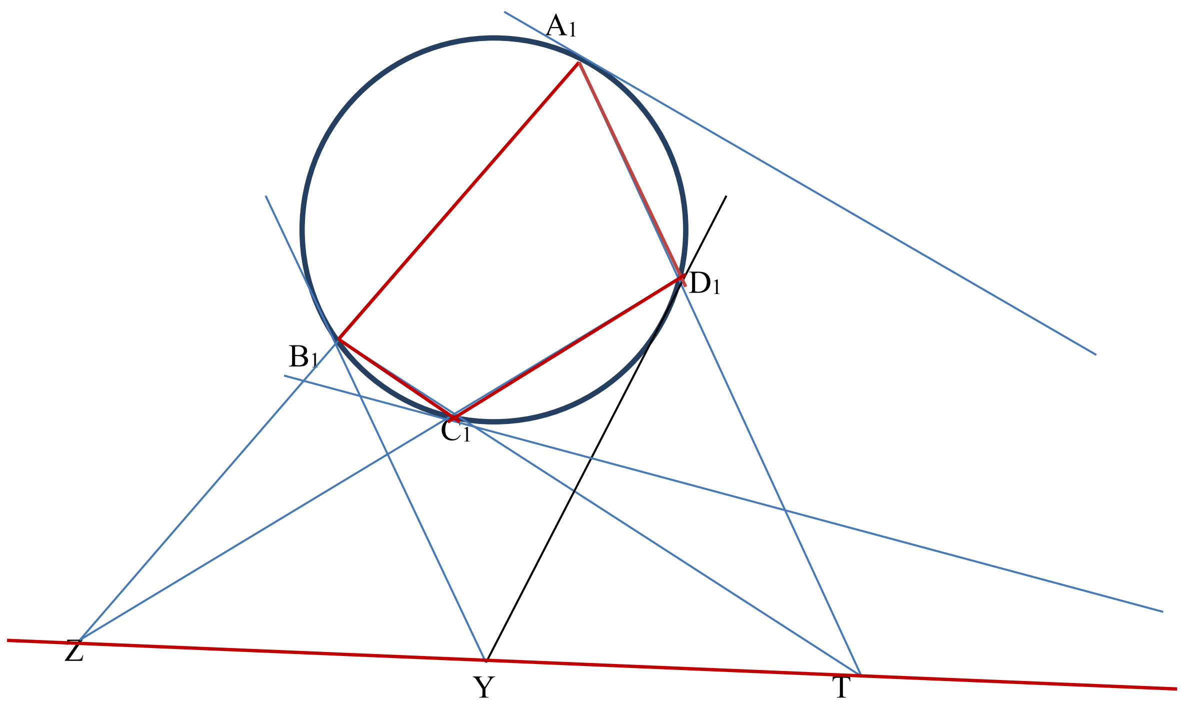



Fig. 2

These four tangents together with the tangents constructed from $X$ and $Y$ (also four) will contain the sides of an octagon circumscribed to the given circle.

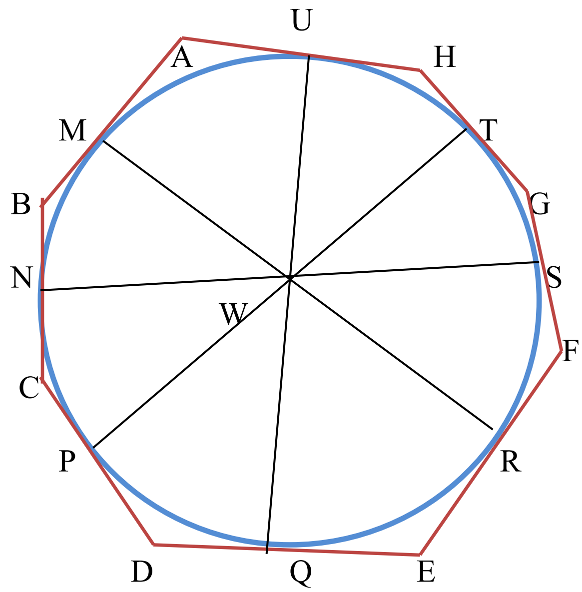



Fig .3

In this octagon $A_1C_1$ and $B_1D_1$ will connect the contact points of two pairs of sides opposed to the circle, the other two cords determined by the contact points of the opposite sides of the octagon with the circle will be the polar of the points $Z$ and $T$.

Because the transformation through reciprocal polar will make that to collinear points will correspond concurrent lines, these lines are the cords from our initial statement.

**Observation**

In figure 3 we represented an octagon $ABCDEFGH$ circumscribed to a circle.

As it can be observed the cords $MR$, $NS$, $PT$, $QU$ are concurrent in a point notated $W$

**References**

[1] Coxeter H. S. M, Greitzer S. L. –Geometry revisited – Toronto – New York, 1957 (translation in Russian, 1978)

[2] Smarandache F., Pătraşcu I. – The Geometry of Homological Triangles – The Education Publisher, Inc. Columbus, Ohio, U.S.A. -2012.

**TRIPLETS OF TRI-HOMOLOGICAL TRIANGLES**

Prof. Ion Pătraşcu, The "Fraţii Buzeşti" National College, Craiova, Romania
Prof. Florentin Smarandache, The University of New Mexico, USA

In this article will prove some theorems in relation to the triplets of homological triangles two by two. These theorems will be used later to build triplets of triangles two by two tri-homological.

**I Theorems on the triplets of homological triangles**

**Theorem 1**

Two triangles are homological two by two and have a common homological center (their homological centers coincide) then their homological axes are concurrent.

**Proof**

Let's consider the homological triangles $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3$ whose common homological center is $O$ (see figure 1.)

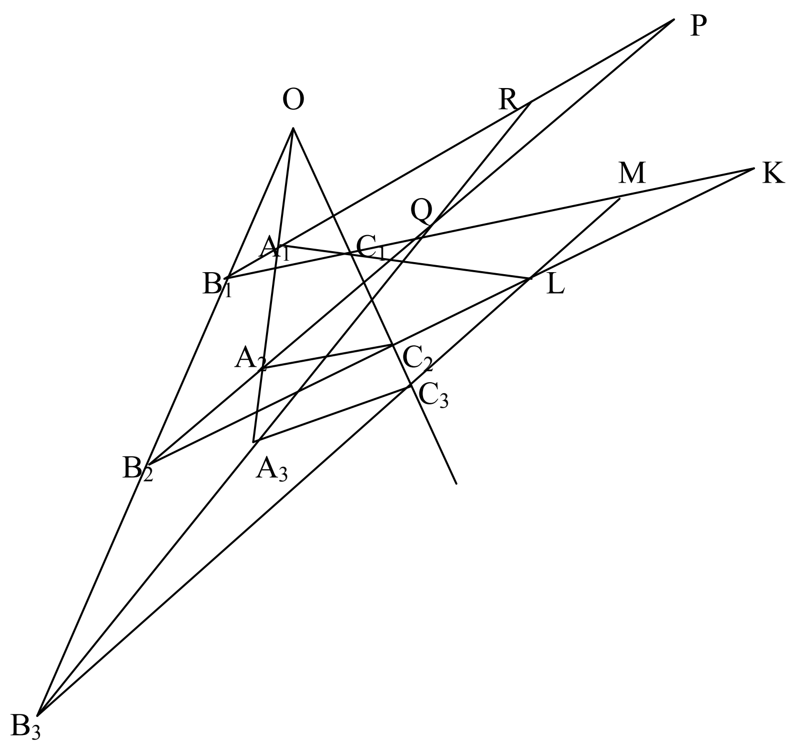



Fig. 1

We consider the triangle formed by the intersections of the lines: $A_1B_1$, $A_2B_2$, $A_3B_3$ and we note it $PQR$ and the triangle formed by the intersection of the lines $B_1C_1$, $B_2C_2$, $B_3C_3$ and we'll note it $KLM$.

We observe that $PR \cap KM = \{B_1\}, RQ \cap ML = \{B_2\}$, PQ$\cap$ KL=$\{B_3\}$ and because $B_1, B_2, B_3$ are collinear it results, according to the Desargues reciprocal theorem that the triangles $PQR$ and $KLM$ are homological, therefore $PK, RM, QL$ are concurrent lines.

The line $PK$ is the homological axes of triangles $A_1B_1C_1$ and $A_2B_2C_2$, the line $RM$ is the homological axis for triangles $A_1B_1C_1$ and $A_3B_3C_3$, and the line $QL$ is the homological axis for triangles $A_2B_2C_2$ and $A_3B_3C_3$, which proves the theorem.

**Remark 1**

Another proof of this theorem can be done using the spatial vision; if we imagine figure 1 as being the correspondent of a spatial figure, we notice that the planes $(A_1B_1C_1)$ and $(A_2B_2C_2)$ have in common the line$PK$, similarly the planes $(A_1B_1C_1)$ and $(A_3B_3C_3)$ have in common the line $QL$. If $\{O'\} = PK \cap$LQ then $O'$ will be in the plane $(A_2B_2C_2)$ and in the plane $(A_3B_3C_3)$, but these planes intersect by the line $RM$, therefore $O'$ belongs to this line as well. The lines $PK, RM, QL$ are the homological axes of the given triangles and therefore these are concurrent in $O'$.

**Theorem 2**

If three triangles are homological two by two and have the same homological axis (their homological axes coincide) then their homological axes are collinear.

**Proof**

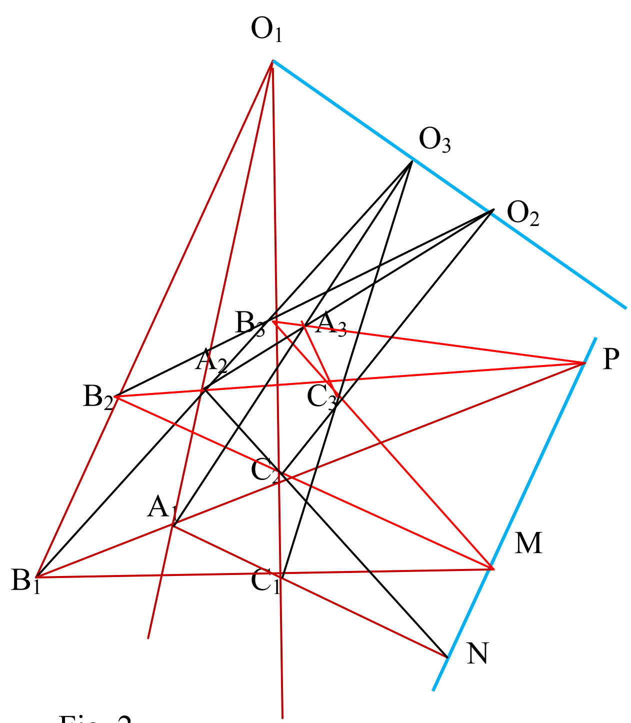



Fig. 2

Let's consider the homological triangles two by two $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3$. We note $M, N, P$ their common homological axis (see figure 2). We note $O_1$ the homological center of the triangles $A_1B_1C_1$ and $A_2B_2C_2$, with $O_2$ the homological center of the triangles $A_2B_2C_2$ and $A_3B_3C_3$ and with $O_3$ the homological center of the triangles $A_3B_3C_3$ and $A_1B_1C_1$.

We consider the triangles $A_1A_2A_3$ and $B_1B_2B_3$, and we observe that these are homological because $A_1B_1$, $A_2B_2$, $A_3B_3$ intersect in the point $P$ which is their homological center. The homological axis of these triangles is determined by the points

, $\{O_1\}=A_1A_2 \cap B_1B_2$, $\{O_2\} = A_2A_3 \cap B_2B_3$, $\{O_3\} = A_1A_3 \cap B_1B_3$

therefore the points $O_1, O_2, O_3$ are collinear and this concludes the proof of this theorem.

**Theorem 3** (The reciprocal of theorem 2)

If three triangles are homological two by two and have their homological centers collinear, then these have the same homological axis.

**Proof**

We will use the triangles from figure 2. Let therefore $O_1, O_2, O_3$ the three homological collinear points. We consider the triangles $B_1B_2B_3$ and $C_1C_2C_3$, we observe that these admit as homological axis the line $O_1O_2O_3$.

Because

$$\{O_1\} = B_1B_2 \cap C_1C_2, \{O_2\} = B_2B_3 \cap C_2C_3, \{O_3\} = B_1B_3 \cap C_1C_3,$$

It results that these have as homological center the point $\{M\} = B_1C_1 \cap B_2C_2 \cap B_3C_3$.

Similarly for the triangles $A_1A_2A_3$ and $C_1C_2C_3$ have as homological axis $O_1O_2O_3$ and the homological center $M$. We also observe that the triangles $A_1A_2A_3$ and $B_1B_2B_3$ are homological and $O_1O_2O_3$ is their homological axis, and their homological center is the point $P$. Applying the theorem 2, it results that the points $M, N, P$ are collinear, and the reciprocal theorem is then proved.

**Theorem 4** (The Veronese theorem)

If the triangles $A_1B_1C_1$, $A_2B_2C_2$ are homological and

$\{A_3\} = B_1C_2 \cap B_2C_1$, $\{B_3\} = A_1C_2 \cap A_2C_1$, $\{C_3\} = A_1B_2 \cap A_2B_1$

then the triangle $A_3B_3C_3$ is homological with each of the triangles $A_1B_1C_1$ and $A_2B_2C_2$, and their homological centers are collinear.

**Proof**

Let $O_1$ be the homological center of triangles $A_1B_1C_1$ and $A_2B_2C_2$ (see figure 3) and $A', B', C'$ their homological axis.

We observe that $O_1$ is a homological center also for the triangles $A_1B_1C_2$ and $A_2B_2C_1$. The homological axis of these triangles is $C', A_3, B_3$. Also $O_1$ is the homological center for the triangles

$B_1C_1A_2$ and $B_2C_2A_1$, it results that their homological axis is $A', B_3, C_3$

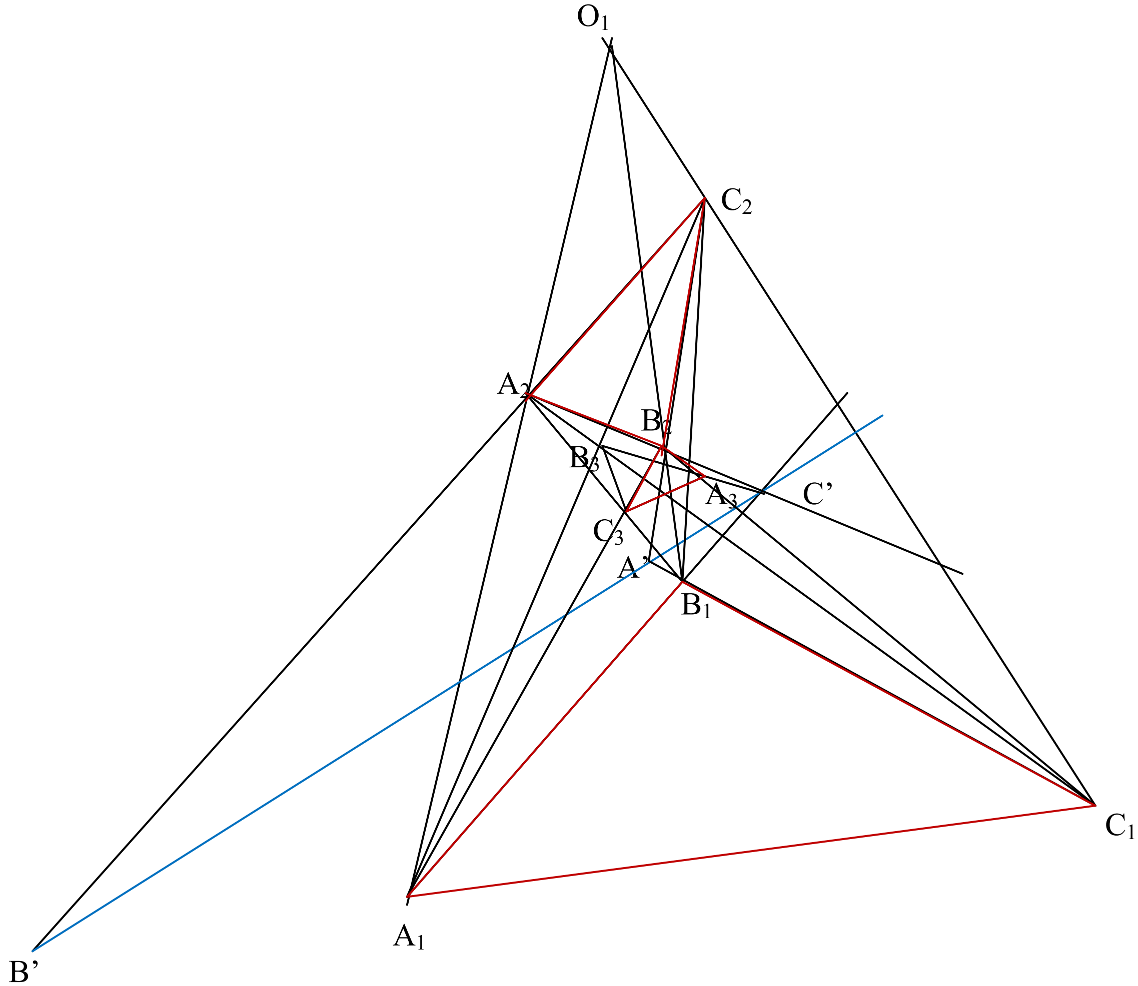



Fig. 3

Similarly, we obtain that the points $B', A_3, C_3$ are collinear, these being on a homological axis of triangle $C_1A_1B_2$ and $C_2A_2B_1$. The triplets of the collinear points $(C', A_3, B_3)$, $(B', A_3, C_3)$ and $(A', B_3, C_3)$ show that the triangle $A_3B_3C_3$ is homological with triangle $A_1B_1C_1$ and with the triangle $A_2B_2C_2$.

The triangles $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3$ are homological two by two and have the same homological axis $A', B', C'$. Using theorem 3, it results that their homological centers are collinear points.

## II. Double-homological triangles

**Definition 1**

We say that the triangles $A_1B_1C_1$ and $A_2B_2C_2$ are double-homological or bi- homological if these are homological in two modes.

**Theorem 5**

Let's consider the triangles $A_1B_1C_1$ and $A_2B_2C_2$ such that

$$B_1C_1 \cap B_2C_2 = \{P_1\}, B_1C_1 \cap A_2C_2 = \{Q_1\}, B_1C_1 \cap A_2B_2 = \{R_1\}$$

$$A_1C_1 \cap A_2C_2 = \{P_2\}, A_1C_1 \cap A_2B_2 = \{Q_2\}, A_1C_1 \cap B_2C_2 = \{R_2\}$$

$$A_1B_1 \cap A_2B_2 = \{P_3\}, A_1B_1 \cap B_2C_2 = \{Q_3\}, A_1B_1 \cap C_2A_2 = \{R_3\}$$

Then:

$$\frac{P_1B_1 \cdot P_2C_1 \cdot P_3A_1}{P_1C_1 \cdot P_2A_1 \cdot P_3B_1} \cdot \frac{Q_1B_1 \cdot Q_2C_1 \cdot Q_3A_1}{Q_1C_1 \cdot Q_2A_1 \cdot Q_3B_1} \cdot \frac{R_1B_1 \cdot R_2C_1 \cdot R_3A_1}{R_1C_1 \cdot R_2A_1 \cdot R_3B_1} = 1 \qquad (1)$$

**Proof**

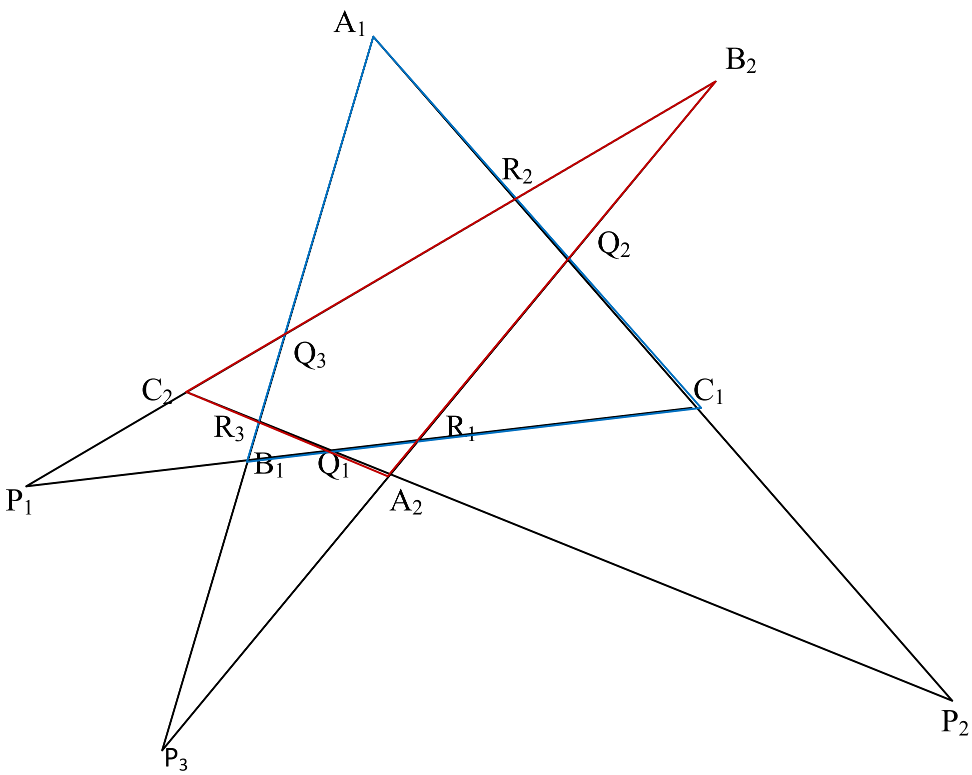



Fig. 4

We'll apply the Menelaus' theorem in the triangle $A_1B_1C_1$ for the transversals $P_1Q_3R_2$, $P_2Q_1R_3$, $P_3Q_2R_1$, (see figure 4).

We obtain

$$\frac{P_1B_1 \cdot R_2C_1 \cdot Q_3A_1}{P_1C_1 \cdot R_2A_1 \cdot Q_3B_1} = 1$$

$$\frac{P_2C_1 \cdot Q_1B_1 \cdot R_3A_1}{P_2A_1 \cdot Q_1C_1 \cdot R_3B_1} = 1$$

$$\frac{P_3A_1 \cdot R_1B_1 \cdot Q_2C_1}{P_3B_1 \cdot R_1C_1 \cdot Q_2A_1} = 1$$

Multiplying these relations side by side and re-arranging the factors, we obtain relation (1).

**Theorem 6**

The triangles $A_1B_1C_1$ and $A_2B_2C_2$ are homological (the lines $A_1A_2$, $B_1B_2$, $C_1C_2$ are concurrent) if and only if:

$$\frac{Q_1B_1 \cdot Q_2C_1 \cdot Q_3A_1}{Q_1C_1 \cdot Q_2A_1 \cdot Q_3B_1} = \frac{R_1C_1 \cdot R_2A_1 \cdot R_3B_1}{R_1B_1 \cdot R_2C_1 \cdot R_3A_1} \quad (2)$$

**Proof**

Indeed, if $A_1A_2$, $B_1B_2$, $C_1C_2$ are concurrent then the points $P_1, P_2, P_3$ are collinear and the Menelaus' theorem for the transversal $P_1P_2P_3$ in the triangle $A_1B_1C_1$ gives:

$$\frac{P_1B_1 \cdot P_2C_1 \cdot P_3A_1}{P_1C_1 \cdot P_2A_1 \cdot P_3B_1} = 1 \quad (3)$$

This relation substituted in (1) leads to (2)

**Reciprocal**

If the relation (2) takes place then substituting it in the relation (1) we obtain (3) which shows that $P_1, P_2, P_3$ is the homology axis of the triangles $A_1B_1C_1$ and $A_2B_2C_2$.

**Remark 2**

If in relation (1) two fractions are equal to 1, then the third fraction will be equal to 1, and this leads to the following:

**Theorem 7**

If the triangles $A_1B_1C_1$ and $A_2B_2C_2$ are homological in two modes (are double-homological) then these are homological in three modes (are tri-homological).

**Remark 3**

The precedent theorem can be formulated in a different mod that will allow us to construct tri-homological triangles with a given triangle and of some tri-homological triangles.

Here is the theorem that will do this:

**Theorem 8**

(i) Let $ABC$ a given triangle and $P, Q$ two points in its plane such that $BP$ intersects $CQ$ in $A_1$, $CP$ intersects $AQ$ in $B_1$ and $AP$ intersects $BQ$ in $C_1$.

Then $AA_1, BB_1, CC_1$ intersect in a point $R$.

(ii) If $\cap\, CP = \{A_2\}, CQ \cap AP = \{B_2\}$, BP$\cap\, AQ = \{C_2\}$then the triangles $ABC$, $A_1B_1C_1$, $A_2B_2C_2$ are two by two homological and their homological centers are collinear.

**Proof**

(i). From the way how we constructed the triangle $A_1B_1C_1$, we observe that $ABC$ and $A_1B_1C_1$ are double homological, their homology centers being two given points $P, Q$ (see figure 5). Using theorem 7 it results that the triangles $ABC$, $A_1B_1C_1$ are tri-homological, therefore $AA_1, BB_1, CC_1$ are concurrent in point noted R.

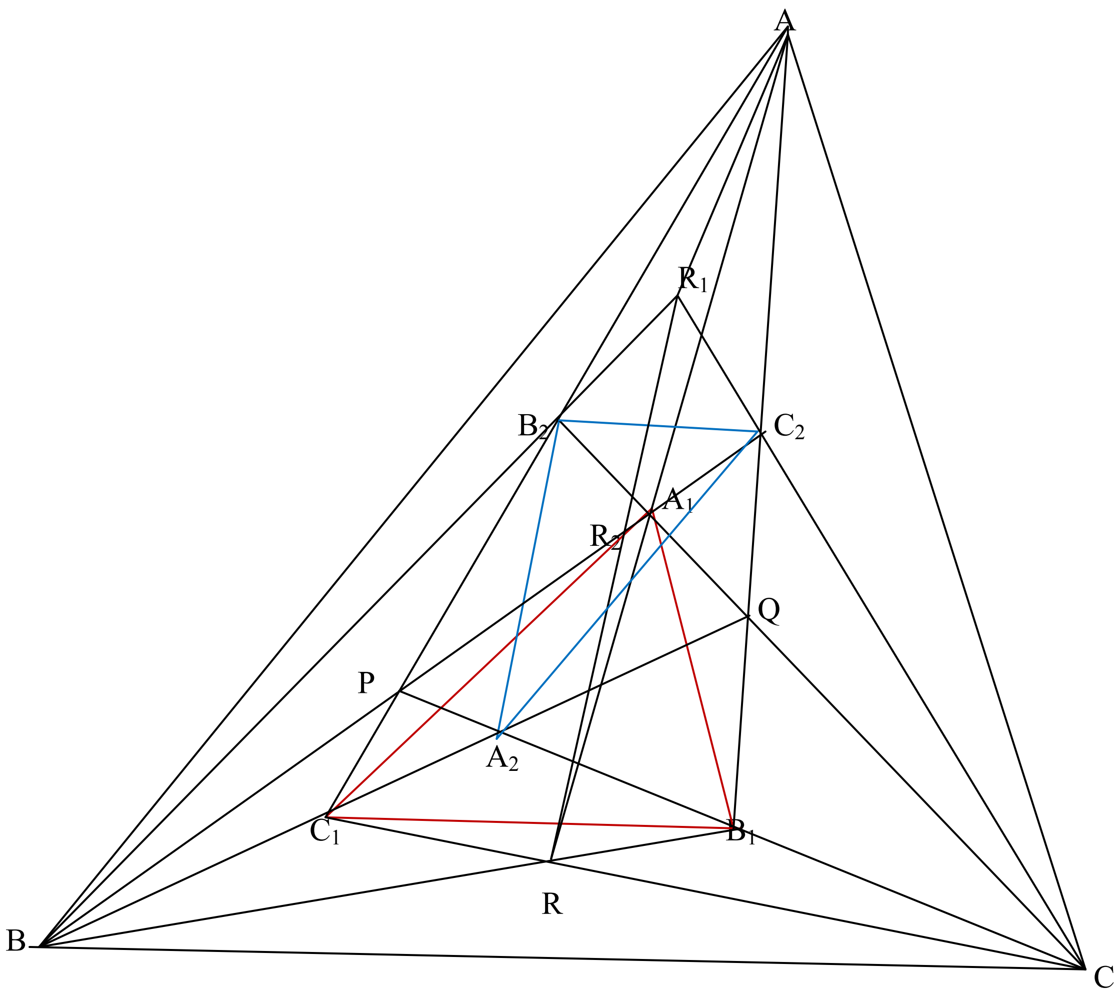



Fig. 5

(ii) The conclusion results by applying the Veronese theorem for the homological triangles $ABC$, $A_1B_1C_1$ that have as homological center the point $R$.

**Remark 4**

We observe that the triangles $ABC$ and $A_2B_2C_2$ are bi-homological, their homological centers being the given points $P, Q$. It results that these are tri-homological and therefore $AA_2, BB_2, CC_2$ are concurrent in the third homological center of these triangle, which we'll note $R_1$.

Similarly we observe that the triangles $A_1B_1C_1,\ A_2B_2C_2$ are double homological with the homological centers $P,Q$; it results that these are tri-homological, therefore $A_1A_2, B_2B_2, C_2C_2$ are concurrent, their concurrence point being notated with $R_2$. In accordance to the Veronese's theorem, applied to any pair of triangles from the triplet $\left(ABC,\ A_1B_1C_1,\ A_2B_2C_2\right)$ we find that the points $R,\ R_1,\ R_2$ are collinear.

**Remark 5**

Considering the points $P,R$ and making the same constructions as in theorem 8 we obtain the triangle $A_3B_3C_3$ which along with the triangles $ABC,\ A_1B_1C_1$ will form another triplet of triangles tri-homological two by two.

**Remark 6**

The theorem 8 provides us a process of getting a triplet of tri-homological triangles two by two beginning with a given triangle and from two given points in its plane. Therefore if we consider the triangle $ABC$ and as given points the two points of Brocard $\mathbf{\Omega}\Omega$ and $\mathbf{\Omega}'$, the triangle $A_1B_1C_1$ constructed as in theorem 8 will be the first Brocard's triangle and we'll find that this is a theorem of J. Neuberg: the triangle $ABC$ and the first Brocard triangle are tri-homological. The third homological center of these triangles is noted $\mathbf{\Omega}''$and it is called the Borcard's third point and $\mathbf{\Omega}''$ is the isometric conjugate of the simedian center of the triangle $ABC$

**Open problems**

1) If $T_1,T_2,T_3$ are triangles in a plane, such that $\left(T_1,T_2\right)$ are tri-homological, $\left(T_2,T_3\right)$ are tri-homological, then are the $\left(T_1,T_3\right)$ tri-homological?

2) If $T_1,T_2,T_3$ are triangles in a plane such that $\left(T_1,T_2\right)$ are tri-homological, $\left(T_2,T_3\right)$ are tri-homological, $\left(T_1,T_3\right)$ are tri-homological and these pairs of triangles have in common two homological centers, then are the three remaining non-common homological centers collinear?

**References**

[1] Roger A Johnson – Advanced Euclidean Geometry, Dover Publications, Inc. Mineola, New-York, 2007

[2] Florentin Smarandache – Multispace & Multistructure, NeutrosophicTrandisciplinarity (100 Collected Papers of Sciences) Vol IV, North-European Scientific Publishers, Hanko- Finland 2010

# A Class of Orthohomological Triangles

Prof. Claudiu Coandă, National College “Carol I”, Craiova, Romania.
Prof. Florentin Smarandache, ENSIETA, Brest, France.
Prof. Ion Pătraşcu, National College “Fraţii Buzeşti”, Craiova, Romania.

**Abstract.**

In this article we propose to determine the triangles’ class $A_iB_iC_i$ orthohomological with a given triangle $ABC$, inscribed în the triangle $ABC$ ($A_i \in BC,\ B_i \in AC,\ C_i \in AB$).

We’ll remind, here, the fact that if the triangle $A_iB_iC_i$ inscribed in $ABC$ is orthohomologic with it, then the perpendiculars in $A_i$, $B_i$, respectively in $C_i$ on $BC$, $CA$, respectively $AB$ are concurrent in a point $P_i$ (the orthological center of the given triangles), and the lines $AA_i$, $BB_i$, $CC_i$ are concurrent in point (the homological center of the given triangles).

To find the triangles $A_iB_iC_i$, it will be sufficient to solve the following problem.

**Problem.**

Let’s consider a point $P_i$ in the plane of the triangle $ABC$ and $A_iB_iC_i$ its pedal triangle. Determine the locus of point $P_i$ such that the triangles $ABC$ and $A_iB_iC_i$ to be homological.

**Solution.**

Let’s consider the triangle $ABC$, $A(1,0,0),\ B(0,1,0),\ C(0,0,1)$, and the point $P_i(\alpha,\beta,\gamma),\ \alpha+\beta+\gamma=0$.

The perpendicular vectors on the sides are:

$$U_{BC}^{\perp}\left(2a^2,\ -a^2-b^2+c^2,\ -a^2+b^2-c^2\right)$$

$$U_{CA}^{\perp}\left(-a^2-b^2+c^2,\ 2b^2,\ a^2-b^2-c^2\right)$$

$$U_{AB}^{\perp}\left(-a^2+b^2-c^2,\ a^2-b^2-c^2,\ 2c^2\right)$$

The coordinates of the vector $\overrightarrow{BC}$ are $(0,-1,1)$, and the line $BC$ has the equation $x=0$.

The equation of the perpendicular raised from point $P_i$ on $BC$ is:

$$\begin{vmatrix} x & y & z \\ \alpha & \beta & \gamma \\ 2a^2 & -a^2-b^2+c^2 & -a^2+b^2-c^2 \end{vmatrix} = 0$$

We note $A_i(x,y,z)$, because $A_i \in BC$ we have:

$$x=0 \text{ and } y+z=1.$$

The coordinates $y$ and $z$ of $A_i$ can be found by solving the system of equations

$$\begin{cases} \begin{vmatrix} x & y & z \\ \alpha & \beta & \gamma \\ 2a^2 & -a^2-b^2+c^2 & -a^2+b^2-c^2 \end{vmatrix} = 0 \\ y+z=0 \end{cases}$$

We have:

$$y \cdot \begin{vmatrix} \alpha & \gamma \\ 2a^2 & -a^2+b^2-c^2 \end{vmatrix} = z \cdot \begin{vmatrix} \alpha & \beta \\ 2a^2 & -a^2-b^2+c^2 \end{vmatrix},$$

$$y\left[\alpha\left(-a^2+b^2-c^2\right)-2\gamma a^2\right] = z\left[\alpha\left(-a^2-b^2+c^2\right)-2\beta a^2\right],$$

$$y + y \cdot \frac{\alpha\left(-a^2+b^2-c^2\right)-2\gamma a^2}{\alpha\left(-a^2-b^2+c^2\right)-2\beta a^2} = 1,$$

$$y \cdot \frac{\alpha\left(-a^2-b^2+c^2\right)-2\beta a^2+\alpha\left(-a^2+b^2-c^2\right)-2\gamma a^2}{\alpha\left(-a^2-b^2+c^2\right)-2\beta a^2} = 1,$$

$$y \cdot \frac{-2a^2\left(\alpha+\beta+\gamma\right)}{\alpha\left(-a^2-b^2+c^2\right)-2\beta a^2} = 1,$$

it results

$$y = \frac{\alpha}{2a^2}\left(a^2+b^2-c^2\right)+\beta$$

$$z = 1-y = 1-\beta-\frac{\alpha}{2a^2}\left(a^2+b^2-c^2\right) = \alpha+\gamma-\frac{\alpha}{2a^2}\left(a^2+b^2-c^2\right).$$

Therefore,

$$A_i\left(0,\ \ \frac{\alpha}{2a^2}\left(a^2+b^2-c^2\right)+\beta,\ \ \frac{\alpha}{2a^2}\left(a^2-b^2+c^2\right)+\gamma\right).$$

Similarly we find:

$$B_i\left(\frac{\beta}{2b^2}\left(a^2+b^2-c^2\right)+\alpha,\ 0,\ \frac{\beta}{2b^2}\left(-a^2+b^2+c^2\right)+\gamma\right),$$

$$C_i\left(\frac{\gamma}{2c^2}\left(a^2-b^2+c^2\right)+\alpha,\ \frac{\gamma}{2c^2}\left(-a^2+b^2+c^2\right)+\beta,\ \ 0\right).$$

We have:

$$\frac{\overrightarrow{A_iB}}{\overrightarrow{A_iC}}=-\frac{\frac{\alpha}{2a^2}\left(a^2-b^2+c^2\right)+\gamma}{\frac{\alpha}{2a^2}\left(a^2+b^2-c^2\right)+\beta}=-\frac{\alpha c\cos B+\gamma a}{\alpha b\cos C+\beta a}$$

$$\frac{\overrightarrow{B_iC}}{\overrightarrow{B_iA}}=-\frac{\frac{\beta}{2b^2}\left(a^2+b^2-c^2\right)+\alpha}{\frac{\alpha}{2a^2}\left(-a^2+b^2+c^2\right)+\gamma}=-\frac{\beta a\cos C+\alpha b}{\beta c\cos A+\gamma b}.$$

$$\frac{\overrightarrow{C_iA}}{\overrightarrow{C_iB}}=-\frac{\frac{\gamma}{2c^2}\left(-a^2+b^2+c^2\right)+\beta}{\frac{\gamma}{2c^2}\left(a^2-b^2+c^2\right)+\alpha}=-\frac{\gamma b\cos A+\beta c}{\gamma a\cos B+\alpha c}$$

(We took into consideration the cosine's theorem: $a^2=b^2+c^2-2bc\cos A$).

In conformity with Ceva's theorem, we have:

$$\frac{\overrightarrow{A_iB}}{\overrightarrow{A_iC}}\cdot\frac{\overrightarrow{B_iC}}{\overrightarrow{B_iA}}\cdot\frac{\overrightarrow{C_iA}}{\overrightarrow{C_iB}}=-1.$$

$$\left(a\gamma+\alpha c\cos B\right)\left(b\alpha+\beta a\cos C\right)\left(c\beta+\gamma b\cos A\right)=$$
$$=\left(a\beta+\alpha b\cos C\right)\left(b\gamma+\beta c\cos A\right)\left(c\alpha+\gamma a\cos B\right)$$
$$a\alpha\left(b^2\gamma^2-c^2\beta^2\right)\left(\cos A-\cos B\cos C\right)+b\beta\left(c^2\alpha^2-a^2\gamma^2\right)\left(\cos B-\cos A\cos C\right)+$$
$$+c\gamma\left(a^2\beta^2-b^2\alpha^2\right)\left(\cos C-\cos A\cos B\right)=0.$$

Dividing it by $a^2b^2c^2$, we obtain that the equation in barycentric coordinates of the locus $\mathfrak{L}$ of the point $P_i$ is:

$$\frac{\alpha}{a}\left(\frac{\gamma^2}{c^2}-\frac{\beta^2}{b^2}\right)\left(\cos A-\cos B\cos C\right)+\frac{\beta}{b}\left(\frac{\alpha^2}{a^2}-\frac{\gamma^2}{c^2}\right)\left(\cos B-\cos A\cos C\right)+$$
$$+\frac{\gamma}{c}\left(\frac{\beta^2}{b^2}-\frac{\alpha^2}{a^2}\right)\left(\cos C-\cos A\cos B\right)=0.$$

We note $\overline{d}_A,\ \overline{d}_B,\ \overline{d}_C$ the distances oriented from the point $P_i$ to the sides $BC$, $CA$ respectively $AB$, and we have:

$$\frac{\alpha}{a}=\frac{\overline{d}_A}{2s},\ \frac{\beta}{b}=\frac{\overline{d}_B}{2s},\ \frac{\gamma}{c}=\frac{\overline{d}_C}{2s}.$$

The locus' $\mathfrak{L}$ equation can be written as follows:

$$\overline{d}_A\left(\overline{d}_C^2-\overline{d}_B^2\right)\left(\cos A-\cos B\cos C\right)+\overline{d}_B\left(\overline{d}_A^2-\overline{d}_C^2\right)\left(\cos B-\cos A\cos C\right)+$$
$$+\overline{d}_C\left(\overline{d}_B^2-\overline{d}_A^2\right)\left(\cos C-\cos A\cos B\right)=0$$

**Remarks.**

1. It is obvious that the triangle's $ABC$ orthocenter belongs to locus $\mathcal{L}$. The orthic triangle and the triangle $ABC$ are orthohomologic; a orthological center is the orthocenter $H$, which is the center of homology.
2. The center of the inscribed circle in the triangle $ABC$ belongs to the locus $\mathcal{L}$, because $\overline{d}_A = \overline{d}_B = \overline{d}_C = r$ and thus locus' equation is quickly verified.

**Theorem (Smarandache-Pătraşcu).**

If a point $P$ belongs to locus $\mathcal{L}$, then also its isogonal $P'$ belongs to locus $\mathcal{L}$.

**Proof.**

Let $P(\alpha,\beta,\gamma)$ a point that verifies the locus' $\mathcal{L}$ equation, and $P'(\alpha',\beta',\gamma')$ its isogonal in the triangle $ABC$. It is known that $\frac{\alpha\alpha'}{a^2}=\frac{\beta\beta'}{b^2}=\frac{\gamma\gamma'}{c^2}$. We'll prove that $P'\in \mathcal{L}$, i.e.

$$\sum\frac{\alpha'}{a}\left(\frac{\gamma'^2}{c^2}-\frac{\beta'^2}{b^2}\right)(\cos A-\cos B\cos C)=0$$

$$\sum\frac{\alpha'}{a}\left(\frac{\gamma'^2b^2-\beta'^2c^2}{b^2c^2}\right)(\cos A-\cos B\cos C)=0$$

$$\sum\frac{\alpha'}{ab^2c^2}\left(\gamma'^2b^2-\beta'^2c^2\right)(\cos A-\cos B\cos C)=0\Leftrightarrow$$

$$\Leftrightarrow\sum\frac{\alpha'}{ab^2c^2}\left(\frac{\gamma'\beta\beta'c^2}{\gamma}-\frac{c^2\gamma\gamma'\beta'}{\beta}\right)(\cos A-\cos B\cos C)=0\Leftrightarrow$$

$$\Leftrightarrow\sum\frac{\alpha'\beta'\gamma'}{ab^2c^2}\left(\frac{\beta c^2}{\gamma}-\frac{\gamma b^2}{\beta}\right)(\cos A-\cos B\cos C)=0\Leftrightarrow$$

$$\Leftrightarrow\sum\frac{\alpha'\beta'\gamma'}{ab^2c^2}\left(\frac{\beta^2c^2-\gamma^2b^2}{\beta\gamma}\right)(\cos A-\cos B\cos C)=0\Leftrightarrow$$

$$\Leftrightarrow\sum\frac{\alpha}{a}\left(\frac{\alpha'\beta'\gamma'}{\alpha\beta\gamma}\right)\frac{1}{b^2c^2}\cdot b^2c^2\left(\frac{\beta^2}{b^2}-\frac{\gamma^2}{c^2}\right)(\cos A-\cos B\cos C)=0.$$

We obtain that:

$$\frac{\alpha'\beta'\gamma'}{\alpha\beta\gamma}\sum\frac{\alpha}{a}\left(\frac{\gamma^2}{c^2}-\frac{\beta^2}{b^2}\right)(\cos A-\cos B\cos C)=0,$$

this is true because $P\in \mathcal{L}$.

**Remark.**

We saw that the triangle 's $ABC$ orthocenter $H$ belongs to the locus, from the precedent theorem it results that also $O$, the center of the circumscribed circle to the triangle $ABC$ (isogonable to $H$), belongs to the locus.

**Open problem:**

What does it represent from the geometry's point of view the equation of locus $\mathcal{L}$?

In the particular case of an equilateral triangle we can formulate the following:

**Proposition:**

The locus of the point $P$ from the plane of the equilateral triangle $ABC$ with the property that the pedal triangle of $P$ and the triangle $ABC$ are homological, is the union of the triangle's heights.

**Proof:**

Let $P(\alpha,\beta,\gamma)$ a point that belongs to locus $\mathfrak{L}$. The equation of the locus becomes:

$$\alpha\left(\gamma^2-\beta^2\right)+\beta\left(\alpha^2-\gamma^2\right)+\gamma\left(\beta^2-\alpha^2\right)=0$$

Because:

$$\alpha\left(\gamma^2-\beta^2\right)+\beta\left(\alpha^2-\gamma^2\right)+\gamma\left(\beta^2-\alpha^2\right)=\alpha\gamma^2-\alpha\beta^2+\beta\alpha^2-\beta\gamma^2+\gamma\beta^2-\gamma\alpha^2=$$

$$=\alpha\beta\gamma+\alpha\gamma^2-\alpha\beta^2+\beta\alpha^2-\beta\gamma^2+\gamma\beta^2-\gamma\alpha^2-\alpha\beta\gamma=$$

$$=\alpha\beta(\gamma-\beta)+\alpha\gamma(\gamma-\beta)-\alpha^2(\gamma-\beta)-\beta\gamma(\gamma-\beta)=$$

$$=(\gamma-\beta)\left[\alpha(\beta-\alpha)-\gamma(\beta-\alpha)\right]=(\beta-\alpha)(\alpha-\gamma)(\gamma-\beta).$$

We obtain that $\alpha=\beta$ or $\beta=\gamma$ or $\gamma=\alpha$, that shows that $P$ belongs to the medians (heights) of the triangle $ABC$.

**References:**


[1] C. Coandă , Geometrie analitică în coordanate baricentrice, Editura Reprograph, Craiova, 2005.

[2] Multispace & Multistructure. Neutrosophic Trandisciplinarity (100 Collected Papers of Sciences), vol. IV, North European Scientific Publishers, Hanko, Finland, 2010.

**This book contains 21 papers of plane geometry.**

**It deals with various topics, such as: quasi-isogonal cevians, nedians, polar of a point with respect to a circle, anti-bisector, aalsonti-symmedian, anti-height and their isogonal.**

**A *nedian* is a line segment that has its origin in a triangle's vertex and divides the opposite side in $n$ equal segments.**

**The papers also study distances between remarkable points in the 2D-geometry, the circumscribed octagon and the inscribable octagon, the circles adjointly ex-inscribed associated to a triangle, and several classical results such as: Carnot circles, Euler's line, Desargues theorem, Sondat's theorem, Dergiades theorem, Stevanovic's theorem, Pantazi's theorem, and Newton's theorem.**

**Special attention is given in this book to *orthological triangles, bi-orthological triangles, ortho-homological triangles*, and *tri-homological triangles.***

**Each paper is independent of the others. Yet, papers on the same or similar topics are listed together one after the other.**

**The book is intended for College and University students and instructors that prepare for mathematical competitions such as National and International Mathematical Olympiads, or for the AMATYC (American Mathematical Association for Two Year Colleges) student competition, Putnam competition, Gheorghe Ţiţeica Romanian competition, and so on.**

**The book is also useful for geometrical researchers.**

ISBN 9781599732473

90000 >

9 781599 732473