MANDELBROT ISO-SETS: ISO-UNIT IMPACT ASSESSMENT

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Abstract

In this introductory work, we use Santilli's iso-topic lifting as a cutting-edge platform to explore Mandelbrot's set. The objective is to upgrade Mandelbrot's complex quadratic polynomial with iso-multiplication and then computationally probe the effects on this revolutionary fractal. For this, we define the "iso-complex quadratic polynomial" and engage it to generate a locally iso-morphic array of "Mandelbrot iso-sets" by varying the iso-unit, where the connectedness property is topologically preserved in each case. The iso-unit broadens and strengthens the chaotic analysis, and authorizes an enhanced classification and demystification such complex systems because it equips us with an additional degree of freedom: the new Mandelbrot iso-set array is an improvement over the traditional Mandelbrot set because it is significantly more general. In total, the experimental results exemplify dynamic iso-spaces and indicate two modes of topological effects: scale-deformation and boundary-deformation. Ultimately, these new and preliminary developments spark further insight into the emerging realm of iso-fractals.

Keywords: Geometry and topology; Chaos theory; Complex systems; Santilli iso-number; Fractal; Iso-fractal; Mandelbrot set; Mandelbrot iso-set.

1 Introduction

The Mandelbrot set is often considered to be the most famous fractal. It is a mathematical set of points in a Euclidean complex space $\mathbb{C}$, with a distinctive boundary that characterizes a fractal structure with self-similarity [1, 2]. The set is closely related to Julia sets [3] and is named after the French mathematician Benoit Mandelbrot, the pioneer who analyzed and popularized it [1, 2]. Images of Mandelbrot's set are created by iteratively sampling complex numbers and determining, for each one, if the result tends towards infinity when a particular mathematical operation is iterated on it [1, 2]. For each complex number, the real and imaginary components serve as 2D image coordinates in $\mathbb{C}$ [4], where the pixels are colored to encode the sequence divergence rate [1, 2]. In particular, the Mandelbrot set is the set of values of $c \in \mathbb{C}$ for which the orbit of 0 under iteration of Mandelbrot's complex quadratic polynomial [1, 2]

$$z_{n+1} = z_n^2 + c, \tag{1}$$

remains bounded, where $z_n, z_{n+1}, c \in \mathbb{C}$ are complex numbers. That is, c is part of the Mandelbrot set if, when starting with $z_0 = 0$ and applying the iteration repeatedly, the absolute value of z_n remains bounded however large n gets [1, 2]. Mandelbrot initially hypothesized that the connectedness property is not preserved for his set, but after further experimentation he later revised his conjecture and claimed that the set is connected [5]. Subsequently, Douady and Hubbard proved that indeed Mandelbrot's set is connected [5, 6]. Beyond the discipline of mathematics, the Mandelbrot set has become prominent in various art forms due to its aesthetic appeal [7, 8, 9] and, moreover, because it is an emergent complex structure that arises from the application of *simple* rules [1, 2].

So why are the Mandelbrot set and other such fractals important subjects to study in science and mathematics? Well, it turns out that *fractal geometry* is the language of chaos theory [4, 10], and fractal/chaotic patterns are *abundant* in the physical, chemical, and biological expressions of nature [8, 11, 12]. Moreover, fractal geometry and chaos theory are a relatively new discipline [4]. Chaos theory examines the behavior of dynamical systems that are *highly sensitive* to initial conditions [11, 13]. In a chaotic dynamical system, *miniscule* differences in initial conditions yield *widely* *diverging* outcomes, thereby generally rendering long-term predictions impossible [11, 13]. For this, additional examples of chaos and fractals are also observed in lightning discharges [14, 15, 16, 17], weather patterns [18, 19, 20], aquatic ecosystems [21, 22], population biology [23], the biological allometric scaling laws [24, 25, 26, 27, 28], cancers and genetics [29, 30], viruses and pathogens [31, 32], the human brain [33, 34, 35], earthquakes [36, 37, 38], volcanoes [39, 40, 41], the global stock market [42, 43], and more. Certainly, fractals such as the Mandelbrot set must play a fundamental role in classifying and demystifying such *complex* systems—but how?

In this paper, we resume the iso-fractal developments of [4] and attack this complex problem by utilizing the power of Santilli's new iso-topic lifting [44, 45, 46, 47, 48, 49] to probe Mandelbrot's set [1, 2]. For this, we launch with Section 2, where we deploy Santilli's iso-numbers [44, 45, 46, 47, 48, 49] to upgrade Mandelbrot's complex quadratic polynomial—eq. (1)—with iso-multiplication to construct the *iso-complex quadratic polynomial*, which is used to construct a *Mandelbrot iso-set*. For this, we identify the procedure and results for the *main* computational experiment that assesses the impact of Santilli's iso-unit [44, 45, 46, 47, 48, 49] for various Mandelbrot iso-sets, where the connectedness property is preserved. Afterwards, in the *extended* experiment of Section 3, we expand the examination and outcomes to include a Mandelbrot iso-complex polynomial of degree k along with a version based on the tangent trigonometric function. Finally, we conclude with Section 4, where we briefly recapitulate this mode of research and suggest future actions to take.

2 Main Experiment

Here, motivated by the iso-fractal initiation of [4], we engage Santilli's iso-numbers [44, 45, 46, 47, 48, 49] to explore Mandelbrot's set [1, 2] in Euclidean complex space. In the procedure of Section 2.1, we attack our objective by upgrading Mandelbrot's complex quadratic polynomial—eq. (1)—with Santilli's iso-multiplication [44, 45, 46, 47, 48, 49] to construct the iso-complex quadratic polynomial, which is used to construct a Mandelbrot iso-set. Afterwards, in Section 2.2, we examine the computational results for an array of Mandelbrot iso-sets with distinct iso-topic liftings to assess the impact of varying the iso-units.

2.1 Main Procedure

In this section, the iso-complex quadratic polynomial for the experiment is assembled as follows:

1. First, in accordance to Santilli's iso-number methodology [44, 45, 46, 47, 48, 49], we select the positive-definite iso-unit $\hat{r} > 0$ with the corresponding inverse $\hat{\kappa} = \frac{1}{\hat{r}} > 0$.

2. Second, given that $\mathbb{C}$ is the set of all complex numbers, then we demonstrate that $\mathbb{C}$ is iso-topically lifted via $\mathbb{C} \to \mathbb{C}_{\hat{r}}$ to establish $\mathbb{C}_{\hat{r}}$, which is the set of all iso-complex numbers [4, 44, 45, 46, 47, 48, 49]. Thus, if $z_1, z_2 \in \mathbb{C}$ are complex numbers, then the corresponding iso-complex numbers $\hat{z}_1, \hat{z}_2 \in \mathbb{C}_{\hat{r}}$ are directly related via [4, 44, 45, 46, 47, 48, 49]

$$\begin{array}{rcl} \hat{z}_1 & = & z_1 \times \hat{r} \\ \hat{z}_2 & = & z_2 \times \hat{r} \end{array}, \ \forall z_1, z_2 \in \mathbb{C} \to \forall \hat{z}_1, \hat{z}_2 \in \mathbb{C}_{\hat{r}}, \tag{2}$$

   where the conventional complex multiplication $\hat{z}_1 \times \hat{z}_2$ is upgraded with the iso-multiplication [4, 44, 45, 46, 47, 48, 49]

$$\hat{z}_1 \ \hat{\times} \ \hat{z}_2 = \hat{z}_1 \times \hat{\kappa} \times \hat{z}_2 = \hat{z}_1 \times \frac{1}{\hat{r}} \times \hat{z}_2. \tag{3}$$

3. Third, given the iso-multiplication of eq. (3), we deduce the iso-square via the expansion

$$\begin{array}{rcl} \hat{z}_n^2 & = & \hat{z}_n \ \hat{\times} \ \hat{z}_n \\ & = & (z_n \times \hat{r}) \times \hat{\kappa} \times (z_n \times \hat{r}) \\ & = & (z_n \times \hat{r}) \times \frac{1}{\hat{r}} \times (z_n \times \hat{r}) \\ & = & z_n \times z_n \times \hat{r}. \end{array} \tag{4}$$

4. Fourth, we prove that the axiom of the multiplicative units of eqs. (2–4) is confirmed by the expressions [4, 44, 45, 46, 47, 48, 49]

$$1 \ \hat{\times} \ \hat{z}_n = 1 \times \hat{\kappa} \times \hat{z}_n = \hat{z}_n \times \frac{1}{\hat{r}} \times 1 = \hat{z}_n \ \hat{\times} \ 1, \ \forall \hat{z}_n \in \mathbb{C}_{\hat{r}}. \tag{5}$$

5. Fifth, we establish that eqs. (2–5) are characterized by the iso-topic lifting and its inverse [4, 44, 45, 46, 47, 48, 49]

$$\begin{array}{rccc} f(\hat{r}): & \mathbb{C} & \rightarrow & \mathbb{C}_{\hat{r}} \\ \\ f^{-1}(\hat{r}): & \mathbb{C}_{\hat{r}} & \rightarrow & \mathbb{C}, \end{array} \tag{6}$$

respectively, where indeed $\mathbb{C}$ and $\mathbb{C}_{\hat{r}}$ are *locally iso-morphic*.

6. Finally, we engage eqs. (2–6) to upgrade eq. (1) to define the iso-complex quadratic polynomial as

$$\hat{z}_{n+1} \equiv \hat{z}_n^2 + \hat{c} \equiv (\hat{z}_n \hat{\times} \hat{z}_n) + \hat{c} \equiv (z_n \times z_n \times \hat{r}) + (c \times \hat{r}) \equiv z_{n+1} \times \hat{r}, \tag{7}$$

where $\hat{z}_n, \hat{z}_{n+1}, \hat{c} \in \mathbb{C}_{\hat{r}}$ are iso-complex numbers and $z_n, z_{n+1}, c \in \mathbb{C}$ are the corresponding complex numbers. Hence, we can computationally generate a Mandelbrot iso-set by systematically iterating eq. (7)! Here, we note that the restricted iso-unit case of $\hat{r} = 1$ for eq. (7) yields the conventional Mandelbrot set of eq. (1) with the standard multiplicative unit, therefore the Mandelbrot iso-set array includes the Mandelbrot set as an element so they are connected.

Thus, given that the Mandelbrot set generated by eq. (1) has been proven to have the connectedness property by Douady and Hubbard [5, 6], and given that the iso-topic lifting of eqs. (2–6) is topologically-preserving to the Mandelbrot set, then we deduce that a Mandelbrot iso-set generated by eq. (7)—which is locally iso-morphic—*must inherit the connectedness property*. Here, we *hypothesize* that the Julia set for a given point in a Mandelbrot iso-set is connected, but this notion of "Julia iso-sets" is beyond the limited scope of this paper and will thus be a subject for future research.

At this point, we've successfully upgraded Mandelbrot's complex quadratic polynomial [1, 2] of eq. (1) with Santilli's iso-multiplication [4, 44, 45, 46, 47, 48, 49] to synthesize the iso-complex quadratic polynomial of eq. (7), which is used to generate Mandelbrot iso-sets.

2.2 Main Results

In total, we computationally experimented with the 5 distinct iso-units:

$$\hat{r} \in \{\frac{1}{2}, \frac{3}{4}, 1, \frac{4}{3}, 2\}. \tag{8}$$

In eq. (8), we observe that $\frac{1}{2}$ is the inverse of 2, 1 is the inverse of 1, and $\frac{3}{4}$ is the inverse of $\frac{4}{3}$, so these iso-unit values are in fact *dual*. Our objective is to insert the various iso-units of eq. (8) into the iso-complex quadratic polynomial of eq. (7) to observe the effect of Santilli's iso-topic lifting [4, 44, 45, 46, 47, 48, 49] on Mandelbrot's set [1, 2].

For our experimental control case, we started with $\hat{r} = 1$ and generated the Mandelbrot set—see the *middle* graphic in Figure 1. Afterwards, we varied the iso-unit for $\hat{r} \neq 1$, such that $\hat{r} = \frac{1}{2}, \frac{3}{4}, \frac{4}{3}, 2$, to generate the Mandelbrot iso-sets—see the *non-middle* graphics in Figure 1. In this preliminary assessment, we observe that the iso-unit variation impact results of Figure 1 indicate that Santilli's iso-topic lifting [4, 44, 45, 46, 47, 48, 49] yields—at minimum—*two* general topological effects:

1. **scale-deformation**, where the fractal is magnified ("zoom-in") or de-magnified ("zoom-out"); and

2. **boundary-deformation**, where the relative position of the fractal boundaries and sequence divergence rates are restructured.

Effects 1 and 2 prove the iso-mathematical existence of the proposed Mandelbrot iso-sets, which are indeed *locally iso-morphic* to the Mandelbrot set. Moreover, these computational results are an experimental implementation of the discrete dynamic iso-spaces in [50], where the iso-unit is treated as a dynamic iso-unit function of a parameter that varies by taking on discrete values.

3 Extended Experiment: Procedure and Results

Here, inspired by the procedure and outcomes of Section 2, we further probe the Mandelbrot iso-set array by experimenting with two alternative variations of the iso-complex polynomial of eq. (7) to open up additional iso-fractal cases for investigation.

First, for the *degree-k Mandelbrot iso-set* experimental case, we replace the $\hat{z}_n^2$ of eq. (7) with the generalization $\hat{z}_n^k$ to define the *degree-k Mandelbrot iso-complex polynomial*

$$\hat{z}_{n+1} \equiv \hat{z}_n^k + \hat{c}, \tag{9}$$

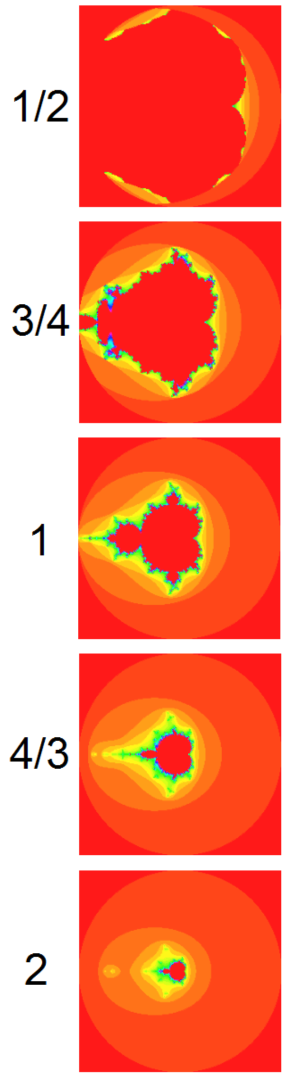

Fig. 1: A depiction of the iso-unit impact variation for the *main* computational experiment. In the *left* column, the varying iso-units of eq. (8) are listed. In the *right* column, the middle graphic is the Mandelbrot set and the non-middle graphics are the Mandelbrot iso-sets that are generated by eq. (7). Observe that Santilli's iso-topic lifting [44, 45, 46, 47, 48, 49] yields two general topological effects: scale-deformation and boundary-deformation.

such that $k > 0$ is a positive-definite natural number. Subsequently, for eq. (9), we set $k = 3$ and re-execute the computational experiment with the same 5 distinct iso-unit parameters of eq. (8): the resulting degree-3 Mandelbrot iso-sets are displayed in the *left* column of Figure 2, which are indeed *locally iso-morphic* to each other. Here, similar to the computational results of Section 2.2, we observe the two general topological effects: scale-deformation and boundary-deformation.

Second, for the *tangent Mandelbrot iso-set* experimental case, we replace the $\hat{z}_n^2$ of eq. (7) with the variation $\tan(\hat{z}_n^2)$ to define the *tangent Mandelbrot iso-complex polynomial*

$$\hat{z}_{n+1} \equiv \tan(\hat{z}_n^2) + \hat{c}. \tag{10}$$

Subsequently, for eq. (10), we re-execute the computations with the same 5 distinct iso-unit parameters of eq. (8): the resulting tangent Mandelbrot iso-sets are displayed in the *right* column of Figure 2, which are indeed *locally iso-morphic* to each other. Here, similar to the computational results of the degree-3 Mandelbrot iso-set case, we observe the two general topological effects: scale-deformation and boundary-deformation.

Thus, in total, the extended outcomes of eqs. (9–10) and Figure 2 indicate that it is possible to develop theoretical and experimental computations for generating such Mandelbrot iso-set variations.

4 Conclusion

The outcomes of this investigation reveal and assess the preliminary impact of Santilli's iso-unit [44, 45, 46, 47, 48, 49] on Mandelbrot's set [1, 2]. More precisely, we were inspired by the iso-fractal developments of [4] and deployed iso-topic liftings [44, 45, 46, 47, 48, 49] to transform Mandelbrot's complex quadratic polynomial into an iso-complex quadratic polynomial, which thereby enabled us to forge the new Mandelbrot iso-set—the Mandelbrot set and a given Mandelbrot iso-set are locally iso-morphic. Subsequently, the initial results of the computational experiments revealed that varying the iso-units causes two general topological effects: scale-deformation and boundary-deformation. For this, we noted that this experiment is an implementation of discrete dynamic iso-spaces [50].

The main results of Section 2.2 clearly indicate that the Mandelbrot set is a specific type of Mandelbrot iso-set when the iso-unit $\hat{r}$ is *not* iso-topically

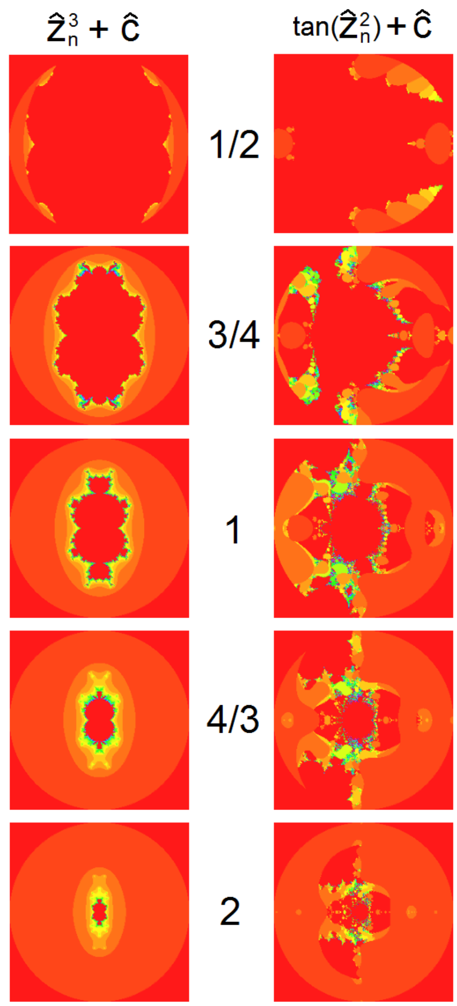

Fig. 2: A depiction of the iso-unit impact variation for the *extended* computational experiment. In the *left* column, the *degree-k Mandelbrot iso-sets* generated by $\hat{z}_n^k + \hat{c}$ in eq. (9) for $k = 3$ are listed. In the *middle* column, the varying iso-units of eq. (8) are listed. In the *right* column, the *tangent Mandelbrot iso-sets* generated by $\tan(\hat{z}_n^2) + \hat{c}$ in eq. (10) are listed. Again, in both experimental cases, observe that Santilli's iso-topic lifting [44, 45, 46, 47, 48, 49] yields two general topological effects: scale-deformation and boundary-deformation.

lifted and thus restricted to the conventional case $\hat{r} = 1$; one could classify the Mandelbrot set as the "base case" of the Mandelbrot iso-set array. Thus, the iso-topic liftings [44, 45, 46, 47, 48, 49] preserve *all* of the Mandelbrot set's characteristics and extend them. For this, the iso-topic liftings [44, 45, 46, 47, 48, 49] enable us to vary $\hat{r}$, which serves as an additional degree of freedom for iso-fractal construction and analysis, and therefore authorizes an enhanced classification and demystification methodology for such chaotic systems because we can take into further account the iso-unit impact of the scale-deformations and boundary-deformations. Hence, we recapitulate these outcomes and implications by pointing out that: *the Mandelbrot iso-set is an improvement over the Mandelbrot set* because the iso-topic liftings [44, 45, 46, 47, 48, 49] render it to be *significantly more general*.

The extended results of Section 3 further probed the Mandelbrot iso-set array by experimenting with two alternative variations of the iso-complex polynomial: the degree-k Mandelbrot iso-set and the tangent Mandelbrot iso-set. Here, the two experimental cases proved the existence of such iso-fractals, which are indeed locally iso-morphic to each other under the topological-preservations of the iso-topic liftings [44, 45, 46, 47, 48, 49], where again, we observed the two said modes of topological deformation effects.

In our opinion, the examination and results indicate an exciting and promising future for this mode of cutting-edge research: the territory of iso-fractals is a vast, uncharted frontier. Ultimately, the implications of this venture are significant because they advance the borderland of iso-mathematics to new trajectories of thought, inquiry, and experimentation. Hence, for future research, we plan to conduct additional related surveys of iso-mathematics that investigate novel patterns for advancement, with the intent to simplify and classify these emerging, developing iso-fractals. More precisely, in future work, we plan to identify the area and Hausdorff dimension [51] of the Mandelbrot iso-set and develop Julia iso-sets to test our connectedness hypothesis in a more generalized iso-fractal framework.

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