Saint-Venant's Principle: Experimental and Analytical

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Abstract

Mathematical provability , then classification, of Saint-Venant's Principle are discussed. Beginning with the simplest case of Saint-Venant's Principle, four problems of elasticity are discussed mathematically. It is concluded that there exist two categories of elastic problems concerning Saint-Venant's Principle: Experimental Problems, whose Saint-Venant's Principle is established in virtue of supporting experiment, and Analytical Problems, whose Saint-Venant's decay is proved or disproved mathematically, based on fundamental equations of linear elasticity. The boundary-value problems whose stress boundary condition consists of Dirac measure, a " singular distribution ", can not be dealt with by the mathematics of elasticity for " proof " or " disproof " of their Saint-Venant's decay, in terms of mathematical coverage.

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1 Introduction

In 1855 Saint-Venant published his famous "principle" [1, 2]. Boussinesq (1885) and Love (1927) announce statements of Saint-Venant's Principe respectively [3, 4]. Trusdell (1959) asserts, from the perspective of Rational Mechanics, that if Saint-Venant's Principle of equipollent loads is true, it " must be a mathematical consequence of the general equations" of linear elasticity [5]. It is obvious that Saint-Venant's Principle has become an academic attraction for contributors of Rational Mechanics [6, 7, 8, 9]. Authors focus their attention on establishment or proof of the principle in wide areas of research.

In this paper, we pay our attention to the problem of mathematical provability of the principle. The simplest case of Saint-Venant's Principle , the suggested simplest case of the principle in two dimensional elasticity, the 2D problem generalized from the simplest case and Saint-Venant's Principle of 2D problem are discussed mathematically. Then we realized that there exist two categories of elastic problems concerning Saint-Venant's Principle: Experimental and Analytical. The boundary-value problems whose stress boundary condition consists of Dirac measure, a " singular distribution ", can not be dealt with by the mathematics of elasticity for " proof " or " disproof " of their Saint-Venant's decay, in terms of mathematical coverage.

2 The Simplest Case of Saint-Venant's Principle

The simplest case concerning Saint-Venant's Principle should be for stress distribution of the cylinder with square cross-section under axial concentrated loading (Fig.1) [10, 11, 12]. We take the case of axial pressure for discussion.

Fig. 1. The simplest case of Saint-Venant's Principle.

Practically, except in the immediate vicinity of the points of loading, the stress is independent of the actual mode of application of the loads and uniformly distributed as

$$\sigma_x = -\frac{P}{A} \tag{1}$$

where P is the magnitude of the concentrated pressure and A is the area of the square cross-section of the cylinder.

3 Discussing the Equivalent Problem of the Simplest Case Mathematically

Equation (1) implies that removing the concentrated force P from and applying an uniformly distributed pressure $\frac{P}{A}$ to each of the ends would not change the stress distribution except in the immediate vicinity of the points of loading, and so the boundary conditions of the equivalent problem of the simplest case (Fig.2), as Saint-Venant's Principle is discussed, should be:

$$x = 0 : \quad \sigma_x = P(\delta - \frac{1}{A}), \tag{2}$$

$$\tau_{xy} = \tau_{xz} = 0; \tag{3}$$

$$x = L \quad (L \to \infty) : \; \sigma_x \to 0, \tag{4}$$

$$\tau_{xy} \to 0, \tag{5}$$

$$\tau_{xz} \to 0; \tag{6}$$

$$y = \pm a : \quad \sigma_y = \tau_{yx} = \tau_{yz} = 0; \tag{7}$$

$$z = \pm a : \quad \sigma_z = \tau_{zx} = \tau_{zy} = 0, \tag{8}$$

where δ is Dirac measure , of unit load at the origin of $O - yz$ coordinates , [13, 14, 15]; A is the area of the end, $x = 0$, of the cylinder (Fig.2).

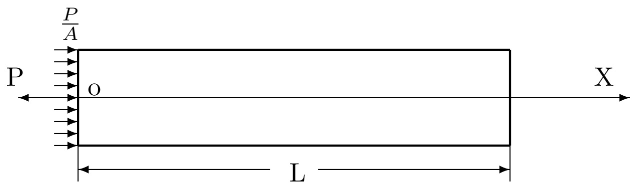

Fig. 2. The equivalent problem of the simplest case.

The fundamental equations, if applicable, of the problem defined in the domain $(0 \leq x \leq L, -a \leq y \leq a, -a \leq z \leq a)$, should be :

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0, \tag{9}$$

$$\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = 0, \tag{10}$$

$$\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0, \tag{11}$$

$$\nabla^2 \sigma_x + \frac{1}{1+\nu} \frac{\partial^2 \Theta}{\partial x^2} = 0, \tag{12}$$

$$\nabla^2 \sigma_y + \frac{1}{1+\nu} \frac{\partial^2 \Theta}{\partial y^2} = 0, \tag{13}$$

$$\nabla^2 \sigma_z + \frac{1}{1+\nu} \frac{\partial^2 \Theta}{\partial z^2} = 0, \tag{14}$$

$$\nabla^2 \tau_{xy} + \frac{1}{1+\nu} \frac{\partial^2 \Theta}{\partial x \partial y} = 0, \tag{15}$$

$$\nabla^2 \tau_{yz} + \frac{1}{1+\nu} \frac{\partial^2 \Theta}{\partial y \partial z} = 0, \tag{16}$$

$$\nabla^2 \tau_{zx} + \frac{1}{1+\nu}\frac{\partial^2 \Theta}{\partial z \partial x} = 0, \tag{17}$$

where

$$\Theta = \sigma_x + \sigma_y + \sigma_z, \tag{18}$$

ν is Poisson ratio.

In reference to Eq.(2)-Eq.(8), stresses σ_x, τ_{xy} and τ_{xz} should favorably be formulated as

$$\sigma_x = C(x)P(\delta - \frac{1}{A})e^{-\lambda x} \tag{19}$$

where

$$C(0) = 1, \tag{20}$$

$$\tau_{xy} = f(x)g(y,z)e^{-\lambda x} \tag{21}$$

and

$$\tau_{xz} = f(x)s(y,z)e^{-\lambda x}, \tag{22}$$

where

$$f(0) = 0, \tag{23}$$

$$g(\pm a, z) = 0, \tag{24}$$

$$s(y, \pm a) = 0. \tag{25}$$

When Saint-Venant's Principle is discussed, stresses σ_y, σ_z and τ_{yz} should be suggested as

$$\sigma_y = u(x)h(y,z)e^{-\lambda x} \tag{26}$$

where

$$h(\pm a, z) = 0, \tag{27}$$

$$\sigma_z = u(x)j(y,z)e^{-\lambda x} \tag{28}$$

where

$$j(y, \pm a) = 0, \tag{29}$$

$$\tau_{yz} = v(x)m(y,z)e^{-\lambda x} \tag{30}$$

where

$$m(\pm a, z) = 0, \quad m(y, \pm a) = 0. \tag{31}$$

Putting Eq.(19), Eq.(21) and Eq.(22) into Eq.(9), we have

$$[C'(x) - \lambda C(x)]P(\delta - \frac{1}{A})e^{-\lambda x} + f(x)\frac{\partial g}{\partial y}e^{-\lambda x} + f(x)\frac{\partial s}{\partial z}e^{-\lambda x} = 0. \tag{32}$$

Putting Eq.(21), Eq.(26) and Eq.(30) into Eq.(10) results in

$$[f'(x) - \lambda f(x)]g(y,z)e^{-\lambda x} + u(x)\frac{\partial h}{\partial y}e^{-\lambda x} + v(x)\frac{\partial m}{\partial z}e^{-\lambda x} = 0. \tag{33}$$

Putting Eq.(22), Eq.(30) and Eq.(28) into Eq.(11) leads to

$$[f'(x) - \lambda f(x)]s(y,z)e^{-\lambda x} + v(x)\frac{\partial m}{\partial y}e^{-\lambda x} + u(x)\frac{\partial j}{\partial z}e^{-\lambda x} = 0. \tag{34}$$

From Eq.(32) we have, by separating variables, that

$$[C'(x) - \lambda C(x)]/f(x) = -(\frac{\partial g}{\partial y} + \frac{\partial s}{\partial z})/P(\delta - \frac{1}{A}) = \alpha. \tag{35}$$

Variable separation is possible for Eq.(33) and Eq.(34) under the conditions

$$g(y,z) = s(y,z), \tag{36}$$

$$u(x) = v(x) \equiv q(x), \tag{37}$$

and

$$h(y,z) = j(y,z) = m(y,z), \tag{38}$$

which results in

$$\frac{f'(x) - \lambda f(x)}{q(x)} = -(\frac{\partial h}{\partial y} + \frac{\partial h}{\partial z})/g = \beta. \tag{39}$$

Equation(35) is transformed, by means of Eq.(36), into

$$C'(x) - \lambda C(x) = \alpha f(x), \tag{40}$$

$$\frac{\partial g}{\partial y} + \frac{\partial g}{\partial z} = -\alpha P(\delta - \frac{1}{A}). \tag{41}$$

Then $C(x)$ is deduced from Eq.(40) as

$$C(x) = Ne^{\lambda x} + W(x), \tag{42}$$

where $W(x)$ is a special solution of Eq.(40), that is,

$$W'(x) - \lambda W(x) = \alpha f(x). \tag{43}$$

The function $f(x)$ is obtained from Eq.(39) as

$$f(x) = Ke^{\lambda x} + R(x), \tag{44}$$

where $R(x)$ is a special solution of Eq.(39), that is,

$$R'(x) - \lambda R(x) = \beta q(x). \tag{45}$$

From Eq.(19), Eq.(21), Eq.(22), Eq.(42) and Eq.(44) we obtain

$$\sigma_x = NP(\delta - \frac{1}{A}) + W(x)P(\delta - \frac{1}{A})e^{-\lambda x}, \tag{46}$$

$$\tau_{xy} = Kg(y,z) + R(x)g(y,z)e^{-\lambda x}, \tag{47}$$

$$\tau_{xz} = Kg(y,z) + R(x)g(y,z)e^{-\lambda x}. \tag{48}$$

Considering boundary conditions Eq.(4), Eq.(5) and Eq.(6), we have, in terms of Eq.(46)-Eq.(48), that

$$\sigma_x = W(x)P(\delta - \frac{1}{A})e^{-\lambda x}, \tag{49}$$

$$\tau_{xy} = R(x)g(y,z)e^{-\lambda x}, \tag{50}$$

$$\tau_{xz} = R(x)g(y,z)e^{-\lambda x}, \tag{51}$$

by taking

$$N = 0, \quad K = 0. \tag{52}$$

According to Eq.(26), Eq.(28), Eq.(30), Eq.(37) and Eq.(38),

$$\sigma_y = q(x)h(y,z)e^{-\lambda x}, \tag{53}$$

$$\sigma_z = q(x)h(y,z)e^{-\lambda x}, \tag{54}$$

$$\tau_{yz} = q(x)h(y,z)e^{-\lambda x}. \tag{55}$$

Now, the question of the simplest case of Saint-Venant's Principle is : Does any function of $q(x)$ satisfying equations Eq.(12)-Eq.(17) exist so that

$$W(x) \neq 0, \tag{56}$$

$$W(0) = 1 \tag{57}$$

and

$$R(0) = 0 \tag{58}$$

could be established for σ_x , τ_{xy} and τ_{xz} in terms of Eq.(49), Eq.(50) and Eq.(51)?

Unfortunately, the answer is : NO.

In fact, putting Eq.(49), Eq.(53) and Eq.(54) into Eq.(12) leads to

$$\begin{aligned} &\nabla^2[W(x)P(\delta - \frac{1}{A})e^{-\lambda x}] + \frac{1}{1+\nu} \\ &\frac{\partial^2}{\partial x^2}[W(x)P(\delta - \frac{1}{A})e^{-\lambda x} + 2q(x)h(y,z)e^{-\lambda x}] = 0. \end{aligned} \tag{59}$$

It is deduced from Eq.(59) by differential calculus that

$$\begin{aligned} &\frac{2+\nu}{1+\nu}\frac{\partial}{\partial x}\{[W'(x) - \lambda W(x)]P(\delta - \frac{1}{A})e^{-\lambda x}\} \\ &+W(x)P(\frac{\partial^2\delta}{\partial y^2} + \frac{\partial^2\delta}{\partial z^2})e^{-\lambda x} + \frac{2h(y,z)}{1+\nu}[q''(x) - 2\lambda q'(x) \\ &+\lambda^2 q(x)]e^{-\lambda x} = 0. \end{aligned} \tag{60}$$

Equation (60) is changed , by virtue of Eq.(43), to

$$\frac{2+\nu}{1+\nu}\frac{\partial}{\partial x}[\alpha f(x)P(\delta-\frac{1}{A})e^{-\lambda x}] \\ +W(x)P(\frac{\partial^2\delta}{\partial y^2}+\frac{\partial^2\delta}{\partial z^2})e^{-\lambda x}+\frac{2h(y,z)}{1+\nu}[q''(x) \\ -2\lambda q'(x)+\lambda^2 q(x)]e^{-\lambda x}=0. \tag{61}$$

It is inferred from Eq.(61) by differential calculus again that

$$\frac{2+\nu}{1+\nu}[\alpha f'(x)-\lambda\alpha f(x)]P(\delta-\frac{1}{A})e^{-\lambda x} \\ +W(x)P(\frac{\partial^2\delta}{\partial y^2}+\frac{\partial^2\delta}{\partial z^2})e^{-\lambda x}+\frac{2h(y,z)}{1+\nu}[q''(x) \\ -2\lambda q'(x)+\lambda^2 q(x)]e^{-\lambda x}=0. \tag{62}$$

Equation (62) is transformed, by virtue of Eq.(39), into

$$\frac{2+\nu}{1+\nu}\alpha\beta q(x)P(\delta-\frac{1}{A}) \\ +W(x)P(\frac{\partial^2\delta}{\partial y^2}+\frac{\partial^2\delta}{\partial z^2})+\frac{2h(y,z)}{1+\nu}[q''(x) \\ -2\lambda q'(x)+\lambda^2 q(x)]=0. \tag{63}$$

It is required, for $q(x)$ having a solution, that

$$\alpha=0 \ \ or \ \ \beta=0, \tag{64}$$

$$h(y,z)=\frac{\partial^2\delta}{\partial y^2}+\frac{\partial^2\delta}{\partial z^2}, \tag{65}$$

because of Eq.(56).

However , as

$$\alpha=0, \tag{66}$$

from Eq.(40) or Eq.(43),

$$W(x)=0 \tag{67}$$

which contradicts Eq.(56) and Eq.(57).

When the second equation of Eq.(64),

$$\beta=0, \tag{68}$$

is considered, from Eq.(39) ,

$$f(x)=Ke^{\lambda x}. \tag{69}$$

Substituting Eq.(69) into Eq.(40) and Eq.(43), we have

$$C'(x) - \lambda C(x) = \alpha K e^{\lambda x} \tag{70}$$

and

$$W'(x) - \lambda W(x) = \alpha K e^{\lambda x}. \tag{71}$$

The solution of Eq.(70) or Eq.(71) is

$$W(x) = \alpha K x e^{\lambda x} \tag{72}$$

which contradicts Eq.(57).

By the way, inconsistent with Eq.(4), Eq.(72) brings about

$$\sigma_x = \alpha K x P(\delta - \frac{1}{A}) \tag{73}$$

by means of Eq.(49).

The same result is given by discussion of Eqs (13)-(17) (omitted), and so the function of $q(x)$ expected does not exist, then the simplest case of Saint-Venant's Principle is not " mathematically proved".

4 Discussing the Simplest Case in Two Dimensional Elasticity

It is logical to suggest the simplest case of Saint-Venant's Principle in two dimensional elasticity in reference to the discussion in the last section and its boundary-value problem (Fig.3) should be suggested as

$$\nabla^2\nabla^2\varphi(x,y) = 0 \quad onD, D = \{(x,y)|0 \leq x \leq L, -c \leq y \leq +c\}; \tag{74}$$

$$x = 0: \quad \sigma_x = P(\delta - \frac{1}{2c}), \tag{75}$$

$$\tau_{xy} = 0; \tag{76}$$

$$x = L \ \ (L \to \infty): \ \sigma_x \to 0, \quad \tau_{xy} \to 0; \tag{77}$$

$$y = \pm c: \quad \sigma_y = \tau_{yx} = 0, \tag{78}$$

where $\varphi(x,y)$ is Airy stress function; δ is Dirac measure , of unit load at the origin of O-y coordinate [13, 14, 15].

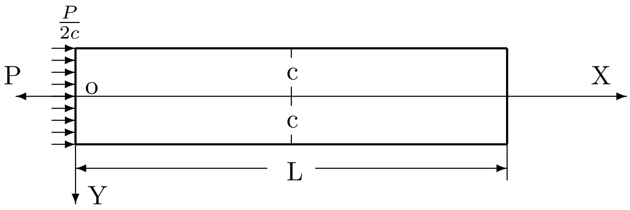

Fig. 3. The simplest case in two dimensional elasticity.

To satisfy Eq.(74) , Eq.(77) and Eq.(78), the stress function is suggested as

$$\varphi_1(x,y) = Ce^{-\gamma\frac{x}{c}}(\kappa\cos\frac{\gamma y}{c} + \frac{\gamma y}{c}\sin\frac{\gamma y}{c}), \tag{79}$$

whose eigenvalue problem is discussed by Timoshenko and Goodier [16].

The shearing stress τ_{xy} is deduced from Eq.(79), which is

$$\begin{aligned} \tau_{xy} &= -\frac{\partial^2\varphi_1}{\partial x\partial y} \\ &= C(\frac{\gamma}{c})^2[(1-\kappa)\sin\gamma\frac{y}{c} + \gamma\frac{y}{c}\cos\gamma\frac{y}{c}]e^{-\gamma\frac{x}{c}}. \end{aligned} \tag{80}$$

If Eq.(76) is to be satisfied, it is required that

$$C = 0 \tag{81}$$

that implies

$$\varphi_1(x,y) = 0, \quad \sigma_x = 0, \tag{82}$$

which is impossible to satisfy Eq.(75).

Considering, for the stress function, the odd function of y

$$\varphi_2(x,y) = Ce^{-\gamma\frac{x}{c}}(\kappa'\sin\frac{\gamma y}{c} + \frac{\gamma y}{c}\cos\frac{\gamma y}{c}) \tag{83}$$

gives the same result of discussion: satisfaction of Eq.(76) ruins satisfaction of Eq.(75).

Therefore, the Saint-Venant's decay in terms of Eq.(79) or Eq.(83) for the two dimensional problem in terms of Eq.(74)-Eq.(78) is not proved.

5 Two-Dimensional Problems Mathematically Discussed

5.1 Generalized 2D Problem

We generalize the 2D simplest case of Saint-Venant's Principle discussed in the last section by replacing $P(\delta-\frac{1}{2c})$ in Eq.(75) with an arbitrary " self-equilibrated " function $p(y)$ and suggest the boundary-value problem as

$$\nabla^2\nabla^2\varphi(x,y) = 0 \qquad onD, D = \{(x,y)|0 \le x \le L, -c \le y \le +c\}, \tag{84}$$

where $\varphi(x,y)$ is Airy stress function;

$$x = 0: \quad \sigma_x = p(y) \neq 0, \tag{85}$$

$$\tau_{xy} = 0, \tag{86}$$

where

$$\begin{aligned} \int_{-c}^{+c} p(y)dy &= 0, \\ \int_{-c}^{+c} p(y)ydy &= 0; \end{aligned} \tag{87}$$

$$x = L \quad (L \to \infty): \ \sigma_x \to 0, \quad \tau_{xy} \to 0; \tag{88}$$

$$y = \pm c: \quad \sigma_y = \tau_{yx} = 0. \tag{89}$$

It is easy to argue, by means of the logic used in the last section, that it is impossible to establish Saint-Venant's decay in terms of

$$\varphi_1(x,y) = Ce^{-\gamma\frac{x}{c}}(\kappa \cos\frac{\gamma y}{c} + \frac{\gamma y}{c}\sin\frac{\gamma y}{c}) \tag{90}$$

or

$$\varphi_2(x,y) = Ce^{-\gamma\frac{x}{c}}(\kappa' \sin\frac{\gamma y}{c} + \frac{\gamma y}{c}\cos\frac{\gamma y}{c}), \tag{91}$$

for the problem Eq.(84)-Eq.(89). In other words, Saint-Venant's decay in terms of Eq.(90) and Eq.(91) for the problem Eq.(84)-Eq.(89) is disproved.

5.2 Saint-Venant's Principle of 2D Problem

The boundary-value problem is defined as

$$\nabla^2\nabla^2\varphi(x,y) = 0 \qquad on D, D = \{(x,y)|0 \leq x \leq L, -c \leq y \leq +c\}, \tag{92}$$

where $\varphi(x,y)$ is Airy stress function;

$$x = 0: \quad \sigma_x = f(y), \tag{93}$$

$$\tau_{xy} = g(y), \tag{94}$$

where $f(y)$ and $g(y)$ are functions to be determined;

$$x = L \quad (L \to \infty): \ \sigma_x \to 0, \quad \tau_{xy} \to 0; \tag{95}$$

$$y = \pm c: \quad \sigma_y = \tau_{yx} = 0. \tag{96}$$

The condition Eq.(96) is satisfied when

$$\varphi(x,y)=0, \quad \frac{\partial\varphi(x,y)}{\partial y}=0 \quad (on\ y=\pm c) \tag{97}$$

because it implies

$$\frac{\partial^2\varphi(x,y)}{\partial x^2}=0, \quad \frac{\partial^2\varphi(x,y)}{\partial x\partial y}=0 \quad (on\ y=\pm c). \tag{98}$$

[16].

To satisfy the biharmonic equation Eq.(92) , the conditions Eq.(95) and Eq.(97), the stress function is suggested as

$$\varphi_1(x,y)=Ce^{-\gamma\frac{x}{c}}(\kappa\cos\frac{\gamma y}{c}+\frac{\gamma y}{c}\sin\frac{\gamma y}{c}). \tag{99}$$

The eigenvalue problem is discussed by Timoshenko and Goodier [16]. The eigenvalue equation of γ is

$$\sin 2\gamma+2\gamma=0, \tag{100}$$

which results from

$$\kappa\cos\gamma+\gamma\sin\gamma=0 \tag{101}$$

and

$$\gamma\cos\gamma+(1-\kappa)\sin\gamma=0 \tag{102}$$

by eliminating κ.

Substituting the complex eigenvalues of γ into Eq.(99), we transform the stress function $\varphi_1(x,y)$ into

$$\varphi_1(x,y)=\varphi_{1r}(x,y)+i\varphi_{1i}(x,y), \tag{103}$$

where

$$\begin{aligned}
\varphi_{1r}(x,y) &= \frac{1}{2}\sum C_n e^{-\gamma_{nr}\frac{x}{c}}\{\cos\frac{\gamma_{ni}}{c}x[\kappa_{nr}(e^{-\frac{\gamma_{ni}}{c}y} \\
&+ e^{\frac{\gamma_{ni}}{c}y})\cos\frac{\gamma_{nr}}{c}y+\kappa_{ni}(-e^{-\frac{\gamma_{ni}}{c}y}+e^{\frac{\gamma_{ni}}{c}y})\sin\frac{\gamma_{nr}}{c}y \\
&+ \frac{\gamma_{ni}}{c}y(e^{-\frac{\gamma_{ni}}{c}y}-e^{\frac{\gamma_{ni}}{c}y})\cos\frac{\gamma_{nr}}{c}y \\
&+ \frac{\gamma_{nr}}{c}y(e^{-\frac{\gamma_{ni}}{c}y}+e^{\frac{\gamma_{ni}}{c}y})\sin\frac{\gamma_{nr}}{c}y] \\
&+ \sin\frac{\gamma_{ni}}{c}x[\kappa_{nr}(e^{-\frac{\gamma_{ni}}{c}y}-e^{\frac{\gamma_{ni}}{c}y})\sin\frac{\gamma_{nr}}{c}y \\
&+ \kappa_{ni}(e^{-\frac{\gamma_{ni}}{c}y}+e^{\frac{\gamma_{ni}}{c}y})\cos\frac{\gamma_{nr}}{c}y \\
&+ \frac{\gamma_{ni}}{c}y(e^{-\frac{\gamma_{ni}}{c}y}+e^{\frac{\gamma_{ni}}{c}y})\sin\frac{\gamma_{nr}}{c}y \\
&+ \frac{\gamma_{nr}}{c}y(-e^{-\frac{\gamma_{ni}}{c}y}+e^{\frac{\gamma_{ni}}{c}y})\cos\frac{\gamma_{nr}}{c}y]\},
\end{aligned} \tag{104}$$

$$
\begin{aligned}
\varphi_{1i}(x,y) &= \frac{1}{2}\Sigma C_n e^{-\gamma_{nr}\frac{x}{c}}\{\cos\frac{\gamma_{ni}}{c}x[\kappa_{nr}(e^{-\frac{\gamma_{ni}}{c}y} \qquad (105)\\
&- e^{\frac{\gamma_{ni}}{c}y})\sin\frac{\gamma_{nr}}{c}y\\
&+ \kappa_{ni}(e^{-\frac{\gamma_{ni}}{c}y}+e^{\frac{\gamma_{ni}}{c}y})\cos\frac{\gamma_{nr}}{c}y+\frac{\gamma_{ni}}{c}y(e^{-\frac{\gamma_{ni}}{c}y}\\
&+ e^{\frac{\gamma_{ni}}{c}y})\sin\frac{\gamma_{nr}}{c}y\\
&+ \frac{\gamma_{nr}}{c}y(-e^{-\frac{\gamma_{ni}}{c}y}+e^{\frac{\gamma_{ni}}{c}y})\cos\frac{\gamma_{nr}}{c}y]\\
&+ \sin\frac{\gamma_{ni}}{c}x[-\kappa_{nr}(e^{-\frac{\gamma_{ni}}{c}y}+e^{\frac{\gamma_{ni}}{c}y})\cos\frac{\gamma_{nr}}{c}y\\
&+ \kappa_{ni}(e^{-\frac{\gamma_{ni}}{c}y}-e^{\frac{\gamma_{ni}}{c}y})\sin\frac{\gamma_{nr}}{c}y\\
&+ \frac{\gamma_{ni}}{c}y(-e^{-\frac{\gamma_{ni}}{c}y}+e^{\frac{\gamma_{ni}}{c}y})\cos\frac{\gamma_{nr}}{c}y\\
&- \frac{\gamma_{nr}}{c}y(e^{-\frac{\gamma_{ni}}{c}y}+e^{\frac{\gamma_{ni}}{c}y})\sin\frac{\gamma_{nr}}{c}y]\},
\end{aligned}
$$

where γ_{nr} and γ_{ni} are the real and imaginary part of n^{th} non-zero eigenvalue of γ respectively, in virtue of (100); κ_{nr} and κ_{ni} are the real and imaginary part of the related n^{th} eigenvalue of κ respectively, in virtue of (101) or (102).

Since $\varphi_1(x,y)$ in (99) or (103) satisfies the biharmonic equation (92), its real part $\varphi_{1r}(x,y)$ in (104) and imaginary part $\varphi_{1i}(x,y)$ in (105) individually satisfy this biharmonic equation and can be used as the stress functions of the problem respectively.

Differentiating (104), we find the stresses as

$$
\begin{aligned}
\sigma_x &= \frac{\partial^2\varphi_{1r}(x,y)}{\partial y^2} \qquad (106)\\
&= \frac{1}{2}\sum_{n=1}^{N} C_n e^{-\gamma_{nr}\frac{x}{c}}\{\cos\frac{\gamma_{ni}}{c}x[(\kappa_{nr}\frac{\gamma_{ni}^2}{c^2}-\kappa_{nr}\frac{\gamma_{nr}^2}{c^2}+2\frac{\gamma_{nr}^2}{c^2}-2\frac{\gamma_{ni}^2}{c^2}\\
&+ 2\kappa_{ni}\frac{\gamma_{nr}}{c}\frac{\gamma_{ni}}{c})(e^{-\frac{\gamma_{ni}}{c}y}+e^{\frac{\gamma_{ni}}{c}y})\cos\frac{\gamma_{nr}}{c}y+(\kappa_{ni}\frac{\gamma_{nr}^2}{c^2}-\kappa_{ni}\frac{\gamma_{ni}^2}{c^2}\\
&+ 2\kappa_{nr}\frac{\gamma_{nr}}{c}\frac{\gamma_{ni}}{c}-4\frac{\gamma_{nr}}{c}\frac{\gamma_{ni}}{c})(e^{-\frac{\gamma_{ni}}{c}y}-e^{\frac{\gamma_{ni}}{c}y})\sin\frac{\gamma_{nr}}{c}y\\
&+ (\frac{\gamma_{ni}^3}{c^3}-3\frac{\gamma_{nr}^2\gamma_{ni}}{c^3})y(e^{-\frac{\gamma_{ni}}{c}y}-e^{\frac{\gamma_{ni}}{c}y})\cos\frac{\gamma_{nr}}{c}y\\
&- (\frac{\gamma_{nr}^3}{c^3}-3\frac{\gamma_{nr}\gamma_{ni}^2}{c^3})y(e^{-\frac{\gamma_{ni}}{c}y}+e^{\frac{\gamma_{ni}}{c}y})\sin\frac{\gamma_{nr}}{c}y]\\
&+ \sin\frac{\gamma_{ni}}{c}x[(\kappa_{nr}\frac{\gamma_{ni}^2}{c^2}-\kappa_{nr}\frac{\gamma_{nr}^2}{c^2}+2\frac{\gamma_{nr}^2}{c^2}-2\frac{\gamma_{ni}^2}{c^2}+2\kappa_{ni}\frac{\gamma_{nr}}{c}\frac{\gamma_{ni}}{c})\\
&\quad (e^{-\frac{\gamma_{ni}}{c}y}-e^{\frac{\gamma_{ni}}{c}y})\sin\frac{\gamma_{nr}}{c}y+(\kappa_{ni}\frac{\gamma_{ni}^2}{c^2}-\kappa_{ni}\frac{\gamma_{nr}^2}{c^2}-2\kappa_{nr}\frac{\gamma_{nr}}{c}\frac{\gamma_{ni}}{c}
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad 4\frac{\gamma_{nr}}{c}\frac{\gamma_{ni}}{c})(e^{-\frac{\gamma_{ni}}{c}y} + e^{\frac{\gamma_{ni}}{c}y})\cos\frac{\gamma_{nr}}{c}y \\
&+ \quad (\frac{\gamma_{ni}^3}{c^3} - 3\frac{\gamma_{nr}^2\gamma_{ni}}{c^3})y(e^{-\frac{\gamma_{ni}}{c}y} + e^{\frac{\gamma_{ni}}{c}y})\sin\frac{\gamma_{nr}}{c}y \\
&+ \quad (\frac{\gamma_{nr}^3}{c^3} - 3\frac{\gamma_{nr}\gamma_{ni}^2}{c^3})y(e^{-\frac{\gamma_{ni}}{c}y} - e^{\frac{\gamma_{ni}}{c}y})\cos\frac{\gamma_{nr}}{c}y]\},
\end{aligned}
$$

$$
\begin{aligned}
\sigma_y &= \frac{\partial^2\varphi_{1r}(x,y)}{\partial x^2} \qquad (107) \\
&= \frac{1}{2}\sum_{n=1}^{N} C_n e^{-\gamma_{nr}\frac{x}{c}}\{[(\frac{\gamma_{nr}^2}{c^2} - \frac{\gamma_{ni}^2}{c^2})\cos\frac{\gamma_{ni}}{c}x + 2\frac{\gamma_{nr}}{c}\frac{\gamma_{ni}}{c}\sin\frac{\gamma_{ni}}{c}x] \\
&\quad [\kappa_{nr}(e^{-\frac{\gamma_{ni}}{c}y} + e^{\frac{\gamma_{ni}}{c}y})\cos\frac{\gamma_{nr}}{c}y + \kappa_{ni}(-e^{-\frac{\gamma_{ni}}{c}y} + e^{\frac{\gamma_{ni}}{c}y})\sin\frac{\gamma_{nr}}{c}y \\
&+ \frac{\gamma_{ni}}{c}y(e^{-\frac{\gamma_{ni}}{c}y} - e^{\frac{\gamma_{ni}}{c}y})\cos\frac{\gamma_{nr}}{c}y + \frac{\gamma_{nr}}{c}y(e^{-\frac{\gamma_{ni}}{c}y} + e^{\frac{\gamma_{ni}}{c}y})\sin\frac{\gamma_{nr}}{c}y] \\
&+ [(\frac{\gamma_{nr}^2}{c^2} - \frac{\gamma_{ni}^2}{c^2})\sin\frac{\gamma_{ni}}{c}x - 2\frac{\gamma_{nr}}{c}\frac{\gamma_{ni}}{c}\cos\frac{\gamma_{ni}}{c}x] \\
&\quad [\kappa_{nr}(e^{-\frac{\gamma_{ni}}{c}y} - e^{\frac{\gamma_{ni}}{c}y})\sin\frac{\gamma_{nr}}{c}y + \kappa_{ni}(e^{-\frac{\gamma_{ni}}{c}y} + e^{\frac{\gamma_{ni}}{c}y})\cos\frac{\gamma_{nr}}{c}y \\
&+ \frac{\gamma_{ni}}{c}y(e^{-\frac{\gamma_{ni}}{c}y} + e^{\frac{\gamma_{ni}}{c}y})\sin\frac{\gamma_{nr}}{c}y \\
&+ \frac{\gamma_{nr}}{c}y(-e^{-\frac{\gamma_{ni}}{c}y} + e^{\frac{\gamma_{ni}}{c}y})\cos\frac{\gamma_{nr}}{c}y]\},
\end{aligned}
$$

$$
\begin{aligned}
\tau_{xy} &= -\frac{\partial^2\varphi_{1r}(x,y)}{\partial x\partial y} \qquad (108) \\
&= \frac{1}{2}\sum_{n=1}^{N} C_n e^{-\gamma_{nr}\frac{x}{c}}\{(\frac{\gamma_{nr}}{c}\cos\frac{\gamma_{ni}}{c}x + \frac{\gamma_{ni}}{c}\sin\frac{\gamma_{ni}}{c}x) \\
&\quad [(-\kappa_{nr}\frac{\gamma_{ni}}{c} - \kappa_{ni}\frac{\gamma_{nr}}{c} + \frac{\gamma_{ni}}{c})(e^{-\frac{\gamma_{ni}}{c}y} - e^{\frac{\gamma_{ni}}{c}y})\cos\frac{\gamma_{nr}}{c}y \\
&+ (-\kappa_{nr}\frac{\gamma_{nr}}{c} + \kappa_{ni}\frac{\gamma_{ni}}{c} + \frac{\gamma_{nr}}{c})(e^{-\frac{\gamma_{ni}}{c}y} + e^{\frac{\gamma_{ni}}{c}y})\sin\frac{\gamma_{nr}}{c}y \\
&+ (\frac{\gamma_{nr}^2}{c^2} - \frac{\gamma_{ni}^2}{c^2})y(e^{-\frac{\gamma_{ni}}{c}y} + e^{\frac{\gamma_{ni}}{c}y})\cos\frac{\gamma_{nr}}{c}y - 2\frac{\gamma_{nr}}{c}\frac{\gamma_{ni}}{c}y \\
&\quad (e^{-\frac{\gamma_{ni}}{c}y} - e^{\frac{\gamma_{ni}}{c}y})\sin\frac{\gamma_{nr}}{c}y] + (\frac{\gamma_{nr}}{c}\sin\frac{\gamma_{ni}}{c}x - \frac{\gamma_{ni}}{c}\cos\frac{\gamma_{ni}}{c}x) \\
&\quad [(-\kappa_{nr}\frac{\gamma_{ni}}{c} - \kappa_{ni}\frac{\gamma_{nr}}{c} + \frac{\gamma_{ni}}{c})(e^{-\frac{\gamma_{ni}}{c}y} + e^{\frac{\gamma_{ni}}{c}y})\sin\frac{\gamma_{nr}}{c}y \\
&+ (\kappa_{nr}\frac{\gamma_{nr}}{c} - \kappa_{ni}\frac{\gamma_{ni}}{c} - \frac{\gamma_{nr}}{c})(e^{-\frac{\gamma_{ni}}{c}y} - e^{\frac{\gamma_{ni}}{c}y})\cos\frac{\gamma_{nr}}{c}y \\
&+ (\frac{\gamma_{nr}^2}{c^2} - \frac{\gamma_{ni}^2}{c^2})y(e^{-\frac{\gamma_{ni}}{c}y} - e^{\frac{\gamma_{ni}}{c}y})\sin\frac{\gamma_{nr}}{c}y \\
&+ 2\frac{\gamma_{nr}}{c}\frac{\gamma_{ni}}{c}y(e^{-\frac{\gamma_{ni}}{c}y} + e^{\frac{\gamma_{ni}}{c}y})\cos\frac{\gamma_{nr}}{c}y]\},
\end{aligned}
$$

where

$$N < \infty \tag{109}$$

for satisfaction of (95). Then the functions $f(y)$ in (93) and $g(y)$ in (94) are determined from (106) and (108) as

$$\begin{aligned} f(y) &= \frac{1}{2}\sum_{n=1}^{N} C_n[(\kappa_{nr}\frac{\gamma_{ni}^2}{c^2} - \kappa_{nr}\frac{\gamma_{nr}^2}{c^2} + 2\frac{\gamma_{nr}^2}{c^2} - 2\frac{\gamma_{ni}^2}{c^2} \\ &+ 2\kappa_{ni}\frac{\gamma_{nr}}{c}\frac{\gamma_{ni}}{c})(e^{-\frac{\gamma_{ni}}{c}y} + e^{\frac{\gamma_{ni}}{c}y})\cos\frac{\gamma_{nr}}{c}y \\ &+ (\kappa_{ni}\frac{\gamma_{nr}^2}{c^2} - \kappa_{ni}\frac{\gamma_{ni}^2}{c^2} + 2\kappa_{nr}\frac{\gamma_{nr}}{c}\frac{\gamma_{ni}}{c} \\ &- 4\frac{\gamma_{nr}}{c}\frac{\gamma_{ni}}{c})(e^{-\frac{\gamma_{ni}}{c}y} - e^{\frac{\gamma_{ni}}{c}y})\sin\frac{\gamma_{nr}}{c}y \\ &+ (\frac{\gamma_{ni}^3}{c^3} - 3\frac{\gamma_{nr}^2\gamma_{ni}}{c^3})y(e^{-\frac{\gamma_{ni}}{c}y} - e^{\frac{\gamma_{ni}}{c}y})\cos\frac{\gamma_{nr}}{c}y \\ &- (\frac{\gamma_{nr}^3}{c^3} - 3\frac{\gamma_{nr}\gamma_{ni}^2}{c^3})y(e^{-\frac{\gamma_{ni}}{c}y} + e^{\frac{\gamma_{ni}}{c}y})\sin\frac{\gamma_{nr}}{c}y], \end{aligned} \tag{110}$$

$$\begin{aligned} g(y) &= \frac{1}{2}\sum_{n=1}^{N} C_n\{\frac{\gamma_{nr}}{c}[(-\kappa_{nr}\frac{\gamma_{ni}}{c} - \kappa_{ni}\frac{\gamma_{nr}}{c} \\ &+ \frac{\gamma_{ni}}{c})(e^{-\frac{\gamma_{ni}}{c}y} - e^{\frac{\gamma_{ni}}{c}y})\cos\frac{\gamma_{nr}}{c}y \\ &+ (-\kappa_{nr}\frac{\gamma_{nr}}{c} + \kappa_{ni}\frac{\gamma_{ni}}{c} + \frac{\gamma_{nr}}{c})(e^{-\frac{\gamma_{ni}}{c}y} + e^{\frac{\gamma_{ni}}{c}y})\sin\frac{\gamma_{nr}}{c}y \\ &+ (\frac{\gamma_{nr}^2}{c^2} - \frac{\gamma_{ni}^2}{c^2})y(e^{-\frac{\gamma_{ni}}{c}y} + e^{\frac{\gamma_{ni}}{c}y})\cos\frac{\gamma_{nr}}{c}y \\ &- 2\frac{\gamma_{nr}}{c}\frac{\gamma_{ni}}{c}y(e^{-\frac{\gamma_{ni}}{c}y} - e^{\frac{\gamma_{ni}}{c}y})\sin\frac{\gamma_{nr}}{c}y] \\ &- \frac{\gamma_{ni}}{c}[(-\kappa_{nr}\frac{\gamma_{ni}}{c} - \kappa_{ni}\frac{\gamma_{nr}}{c} + \frac{\gamma_{ni}}{c})(e^{-\frac{\gamma_{ni}}{c}y} + e^{\frac{\gamma_{ni}}{c}y})\sin\frac{\gamma_{nr}}{c}y \\ &+ (\kappa_{nr}\frac{\gamma_{nr}}{c} - \kappa_{ni}\frac{\gamma_{ni}}{c} - \frac{\gamma_{nr}}{c})(e^{-\frac{\gamma_{ni}}{c}y} - e^{\frac{\gamma_{ni}}{c}y})\cos\frac{\gamma_{nr}}{c}y \\ &+ (\frac{\gamma_{nr}^2}{c^2} - \frac{\gamma_{ni}^2}{c^2})y(e^{-\frac{\gamma_{ni}}{c}y} - e^{\frac{\gamma_{ni}}{c}y})\sin\frac{\gamma_{nr}}{c}y \\ &+ 2\frac{\gamma_{nr}}{c}\frac{\gamma_{ni}}{c}y(e^{-\frac{\gamma_{ni}}{c}y} + e^{\frac{\gamma_{ni}}{c}y})\cos\frac{\gamma_{nr}}{c}y]\}. \end{aligned} \tag{111}$$

Now we have established Saint-Venant's decay (106)-(108) for the boundary-value problem (92)- (96) with explicit $f(y)$ (in (110)) and $g(y)$ (in (111)) loading on $x = 0$.

It is reasonable to expect that discussion of $\varphi_{1i}(x, y)$ should give another solution of Saint-Venant's decay for the boundary-value problem (92)- (96) with another explicit couple of stresses of σ_x (or $f(y)$) and τ_{xy} (or $g(y)$) loading on $x = 0$.

Considering, for the stress function, the odd function

$$\varphi_2(x,y) = \sum C_n e^{-\gamma_n \frac{x}{c}} (\kappa_n' \sin \frac{\gamma_n y}{c} + \frac{\gamma_n y}{c} \cos \frac{\gamma_n y}{c}), \tag{112}$$

should result in other solutions of Saint-Venant's decay for the boundary-value problem (92)- (96). (We omit the detail of discussion here.)

Here we reiterate that the discussion in every section of this paper is based on the argument that Saint-Venant's decay is decay of stress ,or strain, or strain energy density, as is discussed by Zhao [9].

6 Classification of Saint-Venant's Principle

6.1 Experimental Saint-Venant's Principle

The first category is Experimental Saint-Venant's Principle, of which the simplest case of Saint-Venant's Principle discussed in Sec.2 and Sec.3 is the typical example. Saint-Venant's Principle is not " mathematically proved " for the case, but stands true by experimental testing. In principle and in fact, the fundamental equations of elasticity, where stresses are defined to be " functions " , can not deal with the boundary-value problems of this category, where the stress boundary condition consists of Dirac measure, a " singular distribution ". Therefore, Saint-Venant's Principle of this category is not " a mathematical consequence of the general equations " of elasticity. In other words, the mathematics of elasticity does not cover this category of problems of Saint-Venant's Principle.

More complicated cases of Saint-Venant's Principle are mentioned by Budynus [17]. The simplest case of Saint-Venant's Principle is impossible to be mathematically proved , let alone the more complicated cases.

As is discussed in Sec.4, the simplest case of Saint-Venant's Principle in two dimensional elasticity is not " mathematically proved " . Furthermore, Saint-Venant's Principle is not mathematically disproved for the problem either because, as mentioned before, the mathematics of the problem is beyond the coverage of the fundamental equations of elasticity. The only way to test the validity of Saint-Venant's Principle for the problem should be approached by experiment.

Counterexamples to " traditional verbal statements " of Saint-Venant's Principle are mentioned in the articles of Toupin [18, 19] . The arguments against broad validity of Saint-Venant's Principle are based on " qualitative, intuitive observations " rather than mathematical analysis.

6.2 Analytical Saint-Venant's Principle

The second category is Analytical Saint-Venant's Principle. Saint-Venant's decay of this category is proved or disproved mathematically for boundary-value problems of elasticity. The discussion of the problem in Sec.5.2 is an example for " proof " of Saint-Venant's decay that is " a mathematical consequence of the general equations ", or, a theorem, of linear elasticity. The discussion of the problem in Sec.5.1 is an example for " disproof " of Saint-Venant's decay.

7 Conclusion

1. There exist two categories of elastic problems concerning Saint-Venant's Principle. The first category is of Experimental Saint-Venant's Principle. Saint-Venant's Principle is tested, confirmed and then established in virtue of supporting experimental evidence and data . The second category is of Analytical Saint-Venant's Principle. Saint-Venant's decay is proved or disproved mathematically, based on fundamental equations of linear elasticity.

2. The boundary-value problems whose stress boundary condition consists of Dirac measure, a " singular distribution ", are beyond the coverage of the fundamental equations of elasticity, and can not be dealt with by the mathematics of elasticity for " proof " or " disproof " of their Saint-Venant's decay.

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