The Concept of the Effective Mass Tensor in GR

The Equation of Motion

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Abstract: *In the papers [1, 2] we presented the concept of the effective mass tensor (EMT) in General Relativity (GR). According to this concept under the influence of the gravitational field the bare mass tensor* $m_{\mu\nu}^{bare}$ *becomes the EMT* $m_{\mu\nu}$. *The concept of the EMT is a new physical interpretation of GR, where the curvature of space-time has been replaced by the EMT. In this paper we consider again the concept of the EMT in the GR but in the aspect of the equation of motion.*

keywords: *the effective mass tensor, the equation of motion, Mach's principle*

I. Introduction

In the papers [1, 2] we presented the concept of *the effective mass tensor* (EMT) in General Relativity (GR). According to this concept under the influence of the gravitational field *the bare mass tensor* $m_{\mu\nu}^{bare}$ becomes the EMT $m_{\mu\nu}$ (see Table I). The concept of the EMT is a new physical interpretation of GR, where the curvature of space-time has been replaced by the EMT.

In this paper we again consider the concept of the EMT in the but in the aspect of the equation of motion. In the Table I we compared a few physical features concerning of the space-time curvature with the physical features of the EMT which were discussed in this paper.

As we know from the papers [1, 2] the metric tensor $g_{\mu\nu}$ we can express by the EMT $m_{\mu\nu}$

$$g_{\mu\nu} = \frac{m_{\mu\nu}}{m} \tag{1}$$

where: m is *the bare mass* of the body, the space-time components μ, ν = 0, 1, 2, 3.

Therefore the metric

$$ds^2\left(g_{\mu\nu}\right) = ds^2\left(m_{\mu\nu}\right) \tag{2}$$

where: $ds^2(g_{\mu\nu}) = g_{\mu\nu}dx^\mu dx^\nu$ and $ds^2(m_{\mu\nu}) = \frac{m_{\mu\nu}}{m}dx^\mu dx^\nu$.

II. The equation of motion

Let us consider the Lagrangian function for the free body in the curved space, which is moving with the small velocity $\frac{dx^\mu}{d\tau}$ ($\frac{dx^\mu}{d\tau}$ << c), where c is the speed of the light, τ is the proper time.

$$L = \frac{1}{2}m \cdot g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \tag{3}$$

If we replace in the eq. (3) the metric tensor $g_{\mu\nu}$ with the EMT $m_{\mu\nu}$ (see eq. (1)) then we have

$$L = \frac{1}{2}m_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \tag{4}$$

The equation of motion for the Lagrangian function (4) have the form (see to Appendix)

$$\frac{d^2x^\beta}{d\tau^2} + \Gamma^{*\beta}_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0 \tag{5}$$

where the term $\Gamma^{*\beta}_{\mu\nu}$ we will call *the modified Christoffel symbols of the second kind* and

$$\Gamma^{*\beta}_{\mu\nu} = \frac{1}{2}m^{\beta\alpha}\left(\frac{\partial m_{\alpha\mu}}{\partial x^\nu} + \frac{\partial m_{\alpha\nu}}{\partial x^\mu} - \frac{\partial m_{\mu\nu}}{\partial x^\alpha}\right) \tag{6}$$

In the weak gravitational field we can decompose of the EMT of the body to the simple form: $m_{\mu\nu} = m^{bare}_{\mu\nu} + m^*_{\mu\nu}$, where: $m^{bare}_{\mu\nu} = m \cdot \eta_{\mu\nu} = \mathrm{diag}(-m,+m,+m,+m)$ we will call *the bare mass tensor*, $\eta_{\mu\nu}$ is the Minkowski tensor, $m^*_{\mu\nu} = m \cdot |h_{\mu\nu}| << 1$ is a small EMT "perturbation" [1] (see also to Table I). Note that in the absence of the gravitational field the EMT becomes the bare mass tensor $m_{\mu\nu} \to m^{bare}_{\mu\nu}$. The $\Gamma^{*\beta}_{\mu\nu}$ describes the anisotropy of the EMT. For the bare mass tensor $m^{bare}_{\mu\nu}$ the $\Gamma^{*\beta}_{\mu\nu} = 0$.

The modified Christoffel symbols (6) (with accuracy to first order) have the form

$$\Gamma^{*i}_{00} = \frac{1}{2m}\delta^{ij}\partial_j m^*_{00} \tag{7}$$

(components i and j are the Roman indices to denote spatial components: i, j = 1, 2, 3) and similarly

$$\Gamma^{*i}_{0j} = -\frac{1}{2m}\delta^{ik}\left(\partial_j m^*_{0k} - \partial_k m^*_{0j}\right) \tag{8}$$

Now the equation of the motion (5) have the form

$$\frac{d^2x^i}{dt^2} = -\frac{1}{2m}c^2\delta^{ij}\partial_j m^*_{00} + \frac{1}{m}\delta^{ik}c\left(\partial_j m^*_{0k} - \partial_k m^*_{0j}\right)\frac{dx^j}{dt} \tag{9}$$

where we omitted the term $\partial_0 m^*_{\mu\nu}$. Now we can interpret this equation in the Newtonian language. The second right term in the eq. (9) is velocity-dependent and is associated with the rotation and *the Coriolis acceleration*. Let's assume that the body is not rotate and $\partial_j m^*_{0k} - \partial_k m^*_{0j} = 0$. Motion of the body in a weak gravitational field is described by the equation

$$\frac{d^2x^i}{dt^2} = -\frac{1}{2m}c^2\delta^{ij}\partial_j m^*_{00} \tag{10}$$

According to the eq. (10) we can say that the gradient of *the time component* of the EMT – $\partial_j m^*_{00}$ determines the acceleration $\frac{d^2x^i}{dt^2}$ of the tested body in the gravitational field. If the $\partial_j m^*_{00} \neq 0$ (the EMT $m_{\mu\nu}$ is an anisotropic[1] and $m_{\mu\nu} \neq m^{bare}_{\mu\nu}$) then the tested body is moving with an acceleration $\frac{d^2x^i}{dt^2} \neq 0$. But if the $\partial_j m^*_{00} = 0$ (the EMT $m_{\mu\nu}$ is an isotropic and $m_{\mu\nu} = m^{bare}_{\mu\nu}$) then the accelera-tion of the body $\frac{d^2x^i}{dt^2} = 0$ and the test body move takes place at a constant velocity. Eq. (10) deter-mines the first Newton's law of motion in the EMT framework.

Of course we must remember that our consideration we used for a weak gravitational field in the New-tonian approximation. The more generally condition $\Gamma^{*\beta}_{\mu\nu} = 0$ in eq. (5) determines the first Newton's law of motion in EMT framework and is more precise.

In the classical mechanics the Newtonian equation of motion have the form

$$\frac{d^2x^i}{dt^2} = -\nabla V \tag{11}$$

where: *V* is the Newtonian gravitational potential of the source. Both eq. (10) and (11) are equivalent if and only if the time component of the EMT takes the form

$$\frac{m^*_{00}}{m} = \frac{2V}{c^2} = \frac{2GM}{c^2 r} \tag{12}$$

We can see that the concept of the EMT correctly describes gravitational phenomena in the Newton's law of gravity.

Let us consider the Lagrangian function (eq. (4)) with the Schwarzschild EMT (the EMT in the Schwarzschild metric)

[1] The EMT is isotropic if and only if his properties do not depend on the directions in the space-time [2]. So far, the concept of the mass isotropy was associated only with the 3-dimensional space but not with the 4-dimensional space-time. It is a new paradigm.

$$m_{\mu\nu}^{Schwarzschild} = m\begin{bmatrix} -\left(1-\frac{2GM}{c^2 r}\right) & 0 & 0 & 0 \\ 0 & \left(1-\frac{2GM}{c^2 r}\right)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{bmatrix} \tag{13}$$

Well known calculations for the Schwarzschild EMT (see eq.(4)) satisfy the classical tests of GR – for example: *the perihelion shift*, *the deflection of light by the Sun* and *the gravitational redshift*, but their physical interpretation is different.

The concept of the EMT correctly describes gravitational phenomena known from GR but these physical phenomena are not generated by the curvature of the space-time, but by the effective mass tensor of the body.

Let's compare a few physical features concerning of the space-time curvature with the physical features of the EMT discussed in this paper. Results of comparison are presented in Table I below.

Table I. The space-time curvature vs. the EMT.

Space-time curvature	**The EMT**
The metric tensor $g_{\mu\nu}$	*The effective mass tensor* $m_{\mu\nu} = m \cdot g_{\mu\nu}$
The metric $ds^2(g_{\mu\nu}) = g_{\mu\nu} dx^\mu dx^\nu$	*The metric* $ds^2(m_{\mu\nu}) = \frac{m_{\mu\nu}}{m} dx^\mu dx^\nu$
Decomposition of the metric tensor in the weak gravitational field $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ where: $\eta_{\mu\nu}$ = diag(-1, +1, +1, +1) is the Minkowski tensor, $\lvert h_{\mu\nu} \rvert << 1$ is a small "perturbation".	*Decomposition of the EMT tensor in the weak gravitational field* $m_{\mu\nu} = m_{\mu\nu}^{bare} + m_{\mu\nu}^{*}$ where: $m_{\mu\nu}^{bare} = m \cdot \eta_{\mu\nu} =$ diag(-*m*, +*m*, +*m*, +*m*) is *the bare mass tensor*, $m_{\mu\nu}^{*} = m \cdot \lvert h_{\mu\nu} \rvert << 1$ is a small EMT "perturbation".
Lagrangian: $L = \frac{1}{2} m \cdot g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}$	*Lagrangian:* $L = \frac{1}{2} m_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}$

The equation of motion (m = 1): $$\frac{d^2x^\beta}{d\tau^2}+\Gamma^\beta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}=0$$ where the term $\Gamma^\beta_{\mu\nu}$ is *the Christoffel symbols* and $$\Gamma^\beta_{\mu\nu}=\frac{1}{2}g^{\beta\alpha}\left(\frac{\partial g_{\alpha\mu}}{\partial x^\nu}+\frac{\partial g_{\alpha\nu}}{\partial x^\mu}-\frac{\partial g_{\mu\nu}}{\partial x^\alpha}\right)$$	*The equation of motion:* $$\frac{d^2x^\beta}{d\tau^2}+\Gamma^{*\beta}_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}=0$$ where the term $\Gamma^{*\beta}_{\mu\nu}$ we will call *the modified Christoffel symbols* and $$\Gamma^{*\beta}_{\mu\nu}=\frac{1}{2}m^{\beta\alpha}\left(\frac{\partial m_{\alpha\mu}}{\partial x^\nu}+\frac{\partial m_{\alpha\nu}}{\partial x^\mu}-\frac{\partial m_{\mu\nu}}{\partial x^\alpha}\right)$$
Equation of motion in the Newtonian limit $$\frac{d^2r}{dt^2}=-\frac{1}{2}c^2\delta^{ij}\partial_j h_{00}=-\nabla V$$	*Equation of motion in the Newtonian limit* $$\frac{d^2r}{dt^2}=-\frac{1}{2m}c^2\delta^{ij}\partial_j m^*_{00}=-\nabla V$$

III. The Foucault's pendulum

There are two different ways of measuring the Earth’s rotation about its polar axis. The first one is astronomical method where we can determine Earth's rotation with respect to the background of distant stars. The second method is dynamically method where Earth's rotation we can determine by means of Foucault's pendulum or the gyroscope. Both methods have the same results. The first method describes Earth’s rotation with respect to *the fixed stars*. The second one with respect to the absolute space. Is it important coincidence or not? According to E. Mach this coincidence is not trivial and only the mass rotation $\left(\partial_j m^*_{0k}-\partial_k m^*_{0j}\right)$ in equation (8) changes the plane of Foucault's pendulum.

IV. Mach’s Principle in a new sound

Eq. (5) we can rewrite in a slightly different form

$$m_{\beta\nu}\frac{d^2x^\nu}{d\tau^2}+\Gamma^*_{\beta\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}=0 \tag{14}$$

where the term $\Gamma^*_{\beta\mu\nu}$ we will call *the modified Christoffel symbols of the first kind* expressed by the formula

$$\Gamma^*_{\beta\mu\nu}=\frac{1}{2}\left(\frac{\partial m_{\beta\nu}}{\partial x^\mu}+\frac{\partial m_{\beta\mu}}{\partial x^\nu}-\frac{\partial m_{\mu\nu}}{\partial x^\beta}\right) \tag{15}$$

Now the motion of the body in the gravitational field is described by two interesting expressions: $m_{\beta\nu}\frac{d^2x^\nu}{d\tau^2}$ and $\Gamma^*_{\beta\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$. The first term $m_{\beta\nu}\frac{d^2x^\nu}{d\tau^2}$ describes *the four–force*, where: $m_{\beta\nu}$ is the EMT, $\frac{d^2x^\nu}{d\tau^2}$ is *the four–acceleration* (see Appendix). Dimension of this term is [N = kg m/s^2]. The second term $\Gamma^{*}_{\beta\mu\nu}$ describes *"the four-gradient"* of the EMT. Dimension of this term is [kg/m]. While the term $\Gamma^{*}_{\beta\mu\nu}\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau}$ describes "the four-gradient" of the EMT and *four-velocities* of the effective masses. Dimension of this term is [N].

According to eq. (14) we can say that:

the effective mass of the body is not an intrinsic property but is a result of interactions between this body and the effective mass other bodies in the Universe.

This is Mach's Principle in a new sound.

V. Conclusion

In this paper we considered the concept of the EMT in the aspect of the equation of the motion. According to this concept under the influence of the gravitational field the bare mass tensor becomes the EMT. The concept of the EMT is a new physical interpretation of GR, where the curvature of space-time has been replaced by the EMT.

Analyzing the Lagrangian function (4) and the equation of motion (5) we see that: in a weak gravitational field the Lagrangian function and the equations of motion for the body with the mass *m* moving in the space-time curvature with the metric tensor $g_{\mu\nu}$ **are the same** like the Lagrangian function and the equations of motion for the body moving with the EMT $m_{\mu\nu}$ in the flat Minkowski space-time. Both descriptions are equivalent (see Table I). In the Newtonian limit the equation of motion (5) gives the equation (11) well known from classical mechanics.

Let's note that the motion of the bodies we describe relative to other bodies with the effective or bare mass, but not to the respect to the massless reference frames.

The concept of the EMT is a very attractive because the equation of the motion (5) includes full information about all fields (in this case a weak gravitational field) surrounding the body without their exact analysis. The EMT can be isotropic or anisotropic. Us we known from the solid-state physics the EMT can be positive or negative. Is a negative effective mass can exist in a Universe? It is a very important question.

The concept of the EMT offers Mach's Principle in a new sound: **the effective mass of the body is not an intrinsic property but is a result of interactions between this body and the effective mass other bodies in the Universe**.

We believe that the concept of the EMT will help better understand a fascinating gravitational phenomena.

Appendix

Let us consider the Lagrangian function with the EMT $m_{\mu\nu}(x^{\beta})$

$$L = \frac{1}{2} m_{\mu\nu}(x^{\beta}) \frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau} \tag{A1}$$

The *Euler – Lagrangian equations* has the form

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial\left(dx^{\beta}/d\tau\right)}\right)-\frac{\partial L}{\partial x^{\beta}}=0 \qquad \text{(A2)}$$

After well-known calculation we get the equation of motion (5). If EMT $m_{\mu\nu}$ does not depends on coordinate x^{β} then

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial\left(dx^{\beta}/d\tau\right)}\right)=0 \qquad \text{(A3)}$$

which implies that $\frac{\partial L}{\partial\left(dx^{\beta}/dt\right)}$ is a constant. But $\frac{\partial L}{\partial\left(dx^{\beta}/d\tau\right)}=m_{\mu\nu}\frac{dx^{\nu}}{d\tau}$, so we have that the four-momentum p_{μ} is constant. Thus, we obtained a very important relationship between four-momentum p_{μ} and EMT $m_{\mu\nu}$ and

$$p_{\mu}=m_{\mu\nu}\frac{dx^{\nu}}{d\tau} \qquad \text{(A4)}$$

Note that this relationship is something different than in GR, where mass is a scalar. Using eq. (A4) we can rewrite eq. (14) in a more general form

$$\frac{dp_{\beta}}{d\tau}+\Gamma^{*}_{\beta\mu\nu}\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau}=0 \qquad \text{(A5)}$$

where: $p_{\beta}=m_{\beta\nu}\frac{dx^{\nu}}{d\tau}$ is the four-momentum.

Differentiating the four-momentum in eq. (A5) with respect to $\frac{d}{d\tau}$ we obtain

$$m_{\beta\nu}\frac{d^{2}x^{\nu}}{d\tau^{2}}+\frac{dm_{\beta\nu}}{d\tau}\frac{dx^{\nu}}{d\tau}+\Gamma^{*}_{\beta\mu\nu}\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau}=0 \qquad \text{(A6)}$$

Although the mass does not exist explicit in eq. (5) but plays a very important role in the eq. (A6). If we assume that $\frac{dm_{\beta\nu}}{d\tau}=0$ then eq. (A6) becomes the eq. (14).

References

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