IMPORTANCE OF THE GAUSS' 2^{nd} DIFFERENTIAL FORM IN RELATIVITY

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Abstract: Because undervalued in classical Relativity, following Ferrarese's approach, I simply expose the necessity of dealing with this topic of surface geometry as basis for extension of Einstein's theory into Tailherer's Vortex theory of gravity by the introduction of the metric deformation speed tensor.

Since General Relativity treats the universe's geometry in terms of hyper-surface's metric of Space-Time (the V4 chronotope), and since a surface is *differentially* characterized by the two Gauss' quadratic forms (whose the components of the 1^{st} one are the coefficients of the metric), it follows that the single Einstein's equation would be insufficient to determine the right evolution of the cosmos. Indeed, two tensorial equations are requisite in two unknown tensorial variables as in like manner as many there are needed to univocally feature a curve in the space by its curvature and torsion through Frenet's formulas. Thus, in my opinion, as for compelling math, the lacking one is the Tailherer's equation which ties the two fundamental tensors.
Certainly, as far back as pioneering studies on the evolution of surfaces, one might wonder it took so long to understand such a simple thing, but physics up to now could not imagine the vorticity $\omega_{\mu\nu}$ related to the tensor of curvature, ground of Tailherer's theory, because the quantity S that links them , as shall we see, lacked among the fundamental constants.
By following Ferrarese's books [1] we want to show how the coefficients[1] K_{ij} of the 2^{nd} differential form of a surface σ are directly related to the derivative of metric with respect to a parameter (the proper time in relativity). First of all, once fixed a coordinate reference system upon the the surface, the K_{ij}'s are related to the basis vectors on the surface relative to the point P: $\underline{e}_i = \partial OP/\partial x^i$ (i =1,2) - whose metric tensor $g_{ij} = \underline{e}_i \cdot \underline{e}_j$ - through the differential link : $\partial_i \underline{e}_j = \Gamma^m_{ij}\, \underline{e}_m + K_{ij}\boldsymbol{\nu}$
with $\boldsymbol{\nu} = (\underline{e}_1 \times \underline{e}_2)/\|\underline{e}_1 \times \underline{e}_2\|$ normal versor to the surface and so to the basis vectors and introducing the Christoffel symbols of 2^{nd} kind Γ^m_{ij}.
This definition assures consistency with well-known consequences, above all the symmetry of K_{ij} by the Schwartz theorem $\partial_i \underline{e}_j = \partial_j \underline{e}_i$ and the equivalence of the first two invariants of $K_{in}\, g^{jn}$ with the mean and Gauss' curvature as plainly inferred in [1]. Multiplying by the direct product the former will yield:
$K_{ij} = \partial_i \underline{e}_j \cdot \boldsymbol{\nu} = \partial_i (\underline{e}_j \cdot \boldsymbol{\nu}) - \underline{e}_j \cdot \partial_i \boldsymbol{\nu} = - \underline{e}_j \cdot \partial_i \boldsymbol{\nu}$.
Now, taking the generic surface σ' parallel to σ defined hence by $OP' = OP + \tau\boldsymbol{\nu}$, on applying the definition $\underline{e}'_i = \partial_i OP' = \underline{e}_i + \tau\, \partial_i \boldsymbol{\nu}$ we have for the metric of the parallel surface $g'_{ij} = \underline{e}'_i \cdot \underline{e}'_j = g_{ij} - 2\tau K_{ij} + \tau^2 K_{im} K^m{}_j$ and to the limit as $\tau \to 0$, i.e. at less of τ^2 order terms : $K_{ij} = -1/2\, \partial_\tau g_{ij}$.

Thus, the second quadratic form tells us how the metric varies on passing to a parallel surface infinitely nearby, the sign in front being unessential in account of the arbitrariness of the choice of the normal versor (as implied by the local pseudo-invariance of the 2^{nd} differential form).
As elicited also in [4,5], this can be associated with a the deformation speed tensor in continuum mechanics.
By his ansatz [2] Tailherer has been able to get a second gravitational equation (∇_μ is the covariant derivative):

$$(*) \qquad \nabla_\sigma C_{\mu\nu} = S\,(\nabla_\mu K_{\nu\sigma} - \nabla_\nu K_{\mu\sigma}) \qquad (\mu,\nu,\sigma = 0,1,2,3)$$

$$(**) \qquad C_{\mu\nu} = S\omega_{\mu\nu}\,, \quad C_{\mu\nu} = R_{\alpha\beta\mu\nu}\in^{\alpha\beta} \approx {}^{(1)}R_{\alpha\beta\mu\nu}\in^{\alpha\beta} = -k(g_{\alpha\nu}/4)(T_{\beta\mu} - 1/2\, g_{\beta\mu} T)\in^{\alpha\beta}$$

with ${}^{(1)}R_{\alpha\beta\mu\nu} = g_{\alpha\nu} R_{\beta\mu}/4$ curvature tensor in the first approximation estimate as starting position in a

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[1] In the Gauss' original notation these are denoted by the letters [3] $L = K_{11}$; $M = K_{12}$; $N = K_{22}$.

successive approximations process since $R_{\beta\mu} = g^{\alpha\nu} R_{\alpha\beta\mu\nu}$; $\in^{\alpha\beta}= 1$ for α, β consecutive indexes and[2] $\in^{\alpha\beta} = - \in^{\beta\alpha}$; $T_{\beta\mu}$ energy momentum tensor, $k = 8\pi G/c^4$; $S|_1 = 1.6E14\ m^{-1}$ (more than else an efficacious value as pointed out in [4]).

The reckoning of $S|_2$ to the second order of approximation of the theory about the PSR 1913+16 binary system can also be found in [4] but a more direct method than first two approximations and with a reliable result, with the crucial interpretation of the inverse of graviton wavelength . Of course each constant's estimate can only be used to the corresponding order of approximation in successive calculations, e.g. $S|_2$ cannot be employed adopting the first approximation in the procedure of finding the solution of (*) in getting the gravitational radiation energy loss in that it would output meaningless results.

(*) is an equation relating the variation of curvature expressed by the contracted Riemann tensor (that we remind to be the only one containing linearly second order derivatives of the metric) to the curl (rotor) of the deformation tensor of metric [2]. As said by Tailherer, this would play the role of Faraday's law in Relativity. By a double curl the equation of waves comes out.

If the energy momentum tensor in matter, expressed through the Riemann tensor by Einstein's equations, does not depend on the metric coordinates - isotropic case -, we have a null LHS in (*) , thus retrieving the Mainardi-Codazzi condition for the coefficients of the 2^{nd} quadratic form of a surface [5] expressing its propriety of symmetry with respect to covariant derivative.

Someone could object that the ansatz (**) is not feasible in that a null vorticity does not imply necessarily a null gravitational field ($R_{\mu\nu\rho\sigma} = 0$) as the simple case of a force acting in straight line between two masses. However this is not true because even though the space part of vorticity $\omega_{hk} = 1/2(\partial_h v_k - \partial_k v_h)$ is null, not so it is the time-radial component ω_{01} in polar coordinates as promptly checked.

To exemplify, by applying the theory, the calculus of energy losses to the first approximation with $S|_1$ of a solar mass m_2 falling radially into a black hole of $m_1=10M_\odot$ yields $-\Delta E|_1 = c\int \Phi_\rho{}^{\sigma\alpha}\Phi^\rho{}_{\sigma\alpha}\, n^\rho\, r^2 d\Omega d\tau = 1E100\ J$ (with n^ρ normal versor on a 2-sphere enclosing the system and $\Phi_{\nu\sigma\mu} = K_{\nu\sigma/\mu} - K_{\mu\sigma/\nu}$ the strength tensor), evidently meaningless with respect to $-\Delta E=0.0104 m_2 c^2 (m_2/m_1)= 2e44\ J$ for Einstein's theory [6]. At this stage the energy loss is found sensibly dependent on the distance from observer, what should not. Thus a calculus to the 2^{nd} approximation gets mandatory, yet still to be performed at the moment, in account of exceeding RAM performances required > 20 Gb.

References

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[2]The choice of $\in^{\alpha\beta}$ gauges the constant S, see [4] p.271. We note also that a cosmological term $\Lambda c^4 g_{\mu\nu}/(8\pi G)$ to be added to the energy momentum tensor do not affect the equation because of the null derivative of the metric.