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Nested Radicals and Formal Languages:

Consider the following equation:

$$x^2 + 2x - 1 = 0$$

$$\rightarrow x^2 = 1 - 2x$$

$$x = \sqrt{1-2x} \rightarrow x = \sqrt{1 - 2\sqrt{1 - 2\sqrt{1 - 2\sqrt{1 - 2\sqrt{1 - 2\ldots}}}}}$$

The infinitely long nested radical that is generated here is an example of a nested recursive structure. This particular example can be called a trivial nested recursive structure because to determine its value one can move through the inverse process:

$$s = \sqrt{1 - 2\sqrt{1 - 2\sqrt{1 - 2\sqrt{1 - 2\sqrt{1 - 2\ldots}}}}} \rightarrow \frac{1}{2} - \frac{1}{2}s^2 = s \rightarrow s^2 + 2s - 1 = 0$$

And then use the quadratic formula to determine its solution. But we are free to generate more complex expressions. Consider the following:

$$(x+n)^2 = x^2 + 2xn + n^2 = x^2 + n(2x+n) = x^2 + n\big(x + (x+n)\big)$$

We can establish a dummy value k and note that:

$$(x+n)^2 = x^2 + n\Big(x - k + \big(x + (n+k)\big)\Big)$$

From here we can write:

$$x + n = \sqrt{x^2 + n\Big(x - k + \big(x + (n+k)\big)\Big)}$$

And then use our definition back into itself (but with n +k as opposed to n) to find that:

$$x + n$$
$$= \sqrt{x^2 + n(x - k + \sqrt{x^2 + (n+k)(x - k + \sqrt{x^2 + (n+2k)(x - k + \sqrt{x^2 + (n+3k)(x - k + \sqrt{\ldots}}}}}$$

Now we are free to set various values for example, x = 1, n = 3, k = 1 and find:

$$4 = \sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + 5\sqrt{1 + 6\sqrt{\dots}}}}}$$

The radical presented here has a very simple form and yet is extremely difficult to solve, given that any attempts to algebraically manipulate it without attempting to guess the underlying structure are futile. We are also free to scale the x value by noting an even more general formula. Notice that due to the undefined nature of k we can easily write stranger formula such as:

$$2 = \sqrt{1 + (1 - i) + \sqrt{1 + (2 + i)(1 - i + \sqrt{1 + (3 + i)(1 - i + \sqrt{1 + (4 + i)(1 - i + \sqrt{\dots}}}}}$$

which is an infinitely nested radical of complex numbers that somehow "spirals" around the complex plane to a value of 2. We can also take a look at more general formula that scale the x and the n such as:

$$x + n = \sqrt{x^2 + n(x - k_1 - k_2 + \left((x + k_1) + (n + k_2)\right)}$$

$$x = 1, n = 1, k_1 = 1, k_2 = 0 \rightarrow$$

$$2 = \sqrt{1 + \sqrt{5 + \sqrt{11 + \sqrt{19 + \sqrt{29 \dots}}}}}$$

Leading to complex patterns not only in terms of the coefficients in front of the radicals but also in the numbers underneath the radicals themselves. An entirely different approach is to look at:

$$x + n = \sqrt{x^2 + n\left(x - (k - 1)x + (kx + n)\right)} \rightarrow$$

$$x + n = \sqrt{x^2 + n(x - (k - 1)x + \sqrt{k^2x^2 + n(kx - (k - 1)kx + \sqrt{k^4x^2 + n(k^2x - (k - 1)k^2x + \sqrt{\dots}}}}$$

$$x = 1, n = 1, k = 3 \rightarrow$$

$$2 = \sqrt{\sqrt{6 + \sqrt{72 + \sqrt{702 + \sqrt{6480 + \sqrt{\dots}}}}}}$$

At this point it’s easy to argue that any kind of pattern can be formed by appropriately adding, multiplying, taking polynomials etc… of the coefficients (and or adding additional variables besides x and n) to create a variety of complex nested radicals. Clearly this area is ripe with possibilities but we will turn our attention towards a few other important concepts and unify them all into a framework.

$$x = x - 1 + 1 = x - 1 + \frac{x+k}{x+k}$$

$$\rightarrow x = x - 1 + \cfrac{x+k}{x + k - 1 + \cfrac{x+2k}{x + 2k - 1 + \cfrac{x+3k}{x + 3k - 1 + \cfrac{x+4k}{x + 4k - 1 + \cfrac{x+5k}{\dots}}}}}$$

For example:

$$1 = \cfrac{2}{1 + \cfrac{3}{2 + \cfrac{4}{3 + \cfrac{5}{4 + \cfrac{6}{5 + \cdots}}}}} = \cfrac{1+i}{i + \cfrac{1+2i}{2i + \cfrac{1+3i}{3i + \cfrac{1+4i}{4i + \cfrac{1+5i}{5i + \cfrac{1+6i}{6i + \cdots}}}}}}$$

We can also go in the opposite direction and utilize different formula by realizing:

$$x = x - 1 + \frac{kx}{kx} \rightarrow x = x - 1 + \cfrac{kx - 1 + \cfrac{k^2x - 1 + \cfrac{k^3x - 1 + \cfrac{k^4x - 1 + \cfrac{\dots}{k^5x}}{k^4x}}{k^3x}}{k^2x}}{kx}$$

If we let $x = 1, k = 10$ $the\ formula\ produces$

$$1 = \cfrac{9 + \cfrac{99 + \cfrac{999 + \cfrac{9999 + \cfrac{99999 + \cfrac{999999 + \cfrac{9999999 + \cdots}{10000000}}{1000000}}{100000}}{10000}}{1000}}{100}}{10}$$

Another option (of stranger behavior) is to consider taking an oscillating path as we use recursion. Consider the first nested division formula and view the following:

$$x = x - 1 + \cfrac{x+k}{\cfrac{x+2k-1+\cfrac{x+3k}{x+4k-1+\cfrac{x+5k}{x+5k-1+\cfrac{x+6k-1+\cdots}{x+6k}}}}{x+k-1+\cfrac{x+3k-1+\cfrac{x+5k-1+\cfrac{x+6k-1+\cdots}{x+6k}}{x+4k}}{x+2k}}}$$

Once again substituting x,k = (1,1) we get following (rather odd structure):

$$1 = \cfrac{2}{\cfrac{2+\cfrac{4}{4+\cfrac{6}{6+\cfrac{8}{7+\cfrac{8}{9+\cdots}}}}}{1+\cfrac{3+\cfrac{5+\cfrac{7+\cfrac{8}{9+\cdots}}{7}}{5}}{3}}}$$

It is of value to mention the fact that for every appearance of a number in these aforementioned formulas one is free to nest another infinite radical or infinite division structure creating ever more intriguing algebraic formulae, none of which may is directly solvable through manipulation unless one understands the underlying structure.

But why limit ourselves to divisions, and inverses of polynomials?

We can note weirder formulae still such as the following:

$$x = x - 1 + e^{\ln\left(\frac{x+k}{x+k}\right)} \rightarrow$$

$$x = x - 1 + e^{\ln\left(\frac{x+k-1+e^{\ln\left(\frac{x+2k-1+e^{\ln(\ldots)}}{x+2k}\right.}}{x+k}\right.}$$

We can let x, k = 1 to show that:

$$1 = e^{\ln\left(\frac{1}{2}+\frac{1}{2}e^{\ln\left(\frac{2}{3}+\frac{1}{3}e^{\ln\left(\frac{3}{4}+\frac{1}{4}e^{\ln\left(\frac{4}{5}+\frac{4}{5}e^{\cdots}\right.}\right.}\right.}\right.}$$

Creating an infinitely nested power tower of changing summands. And yet even then we are restricting ourselves for one can consider the operations of repeated exponentiation (tetration) as well as its inverses, and repeated tetration (pentation) and just in general all the possible hyper operators and their inverses. Thus a question of importance arises:

How does one create a single framework that can be used to define every infinitely nested structure of recursively defined arithmetic functions?

The answer to this lies in the notion of a formal language:

Consider the following language that invokes the use of a:

$$"f", "0", "|", "(", ")", "="$$

Set of characters such that the language develops the following rules to relate it modern mathematics:

$\Delta =$ *set of all expressions in the language*

f: is a function of 3 arguments $f(u_1 \in \Delta, u_2 \in \Delta, u_3 \in \Delta)$

Whereas:

$f(u_1, u_2, u_3) = f(u_1 - 1, f(u_1 - 1, f(u_1 - 1 \ldots u_2, u_3) \ldots u_3,)\, u_3)$ $(nested\ in\ u_2\ times)$

$0 = 0, f(0, x, y) = x + 1$

Notice that we are free to define:

$$g(u) = h |\, f(0, h, 0) = 0$$

Allowing us to remove the minus signs from our original definition and from here we can readily use recursion (repeating the increment operator) to create addition, multiplication, powers, tetration, and keep moving on and on, as well happily using the such that operator to define inverses for these functions, thus create a framework for handling any and all elementary recursive expressions. Furthermore we can use that operator to create all forms of infinitely nested expressions such as the following:

$$3 = \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + \cdots}}}}$$

$$3 = z(0) |\, z(n) = n\sqrt{1 + z(n+1)}$$

$$(1+1) + 1 = z(0) \big|\, z(n) = n * h\big(1 + z(n+1)\big) \,\big| h(k) * h(k) = k$$

$$f(0, f(0, f(0,0,0), 0), 0) = z(0) \,\Big| z(n) = f\Big(2, n, h\big(1 + z(n+1)\big)\Big) \Big| f\big(2, h(k), h(k)\big) = k$$

$$\begin{aligned} &f(0, f(0, f(0,0,0), 0), 0) \\ &\quad = z(0) \,\Big| z(n) \\ &\quad = f\Big(f(0, f(0,0,0), 0), n, h\big(1 + z(n+1)\big)\Big) \Big| f\big(f(0, f(0,0,0), 0), h(k), h(k)\big) = k \end{aligned}$$

For the sake of compactness we will have it such that our notation now transforms the comma separated list: (a,b,c… n) into the vertical vector $\begin{pmatrix} a \\ b \\ c \\ \vdots \\ n \end{pmatrix}$

$$f\begin{pmatrix}0\\ f\begin{pmatrix}0\\ f\begin{pmatrix}0\\0\\0\end{pmatrix}\\0\end{pmatrix}\\0\end{pmatrix} = z(0)\ \Bigg|\ z(n) = f\begin{pmatrix} f\begin{pmatrix}0\\ f\begin{pmatrix}0\\0\\0\end{pmatrix}\\0\end{pmatrix}\\ n\\ h\left(f\begin{pmatrix}0\\0\\ z\left(f\begin{pmatrix}0\\n\\0\end{pmatrix}\right)\end{pmatrix}\right)\end{pmatrix}\ \Bigg|\ f\begin{pmatrix} f\begin{pmatrix}0\\ f\begin{pmatrix}0\\0\\0\end{pmatrix}\\0\end{pmatrix}\\ h(k)\\ h(k)\end{pmatrix} = k$$

Thus the challenge now becomes given an expression in this formal language such as:

$$z(0)\ \Bigg|\ z(n) = f\begin{pmatrix} f\begin{pmatrix}0\\ f\begin{pmatrix}0\\0\\0\end{pmatrix}\\0\end{pmatrix}\\ n\\ h\left(f\begin{pmatrix}0\\0\\ z\left(f\begin{pmatrix}0\\n\\0\end{pmatrix}\right)\end{pmatrix}\right)\end{pmatrix}\ \Bigg|\ f\begin{pmatrix} f\begin{pmatrix}0\\ f\begin{pmatrix}0\\0\\0\end{pmatrix}\\0\end{pmatrix}\\ h(k)\\ h(k)\end{pmatrix} = k$$

Can one determine if it can be converted into an expression that minimizes the appearance of the initial operator (z in this case) if not totally removing the need for such an operator?

And thus we have reached an area of algebra and for that matter general arithmetic that is not well understood but may be able to be attacked with advanced algorithmic methods.