A Hypothesis about Infinite Series, of the minimally centered variety

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Taylor's Theorem and Maclaurin's Special Case: A Brief Introduction

One of the first lessons in approximation and modeling that students of calculus learn is a special rule known as Taylor's Theorem. Taylor's theorem is a method of approximating a function over a local domain through the use of the value of the function as well as its derivatives. The theorem can be stated as follows:

$$f(x) \approx \sum_{n=0}^{Q} [\frac{d^n f}{dx^n}[a] \frac{(x-a)^n}{n!}$$

Around a domain (r,k) which includes the point x = a on it for x ∈ R. In the Case of x ∈ C. The domain of the Taylor series approximation is the disk defined as $|a| \leq |x| \leq |b|$. This is a powerful tool for approximation as long as a function is infinitely differentiable around a particular area (and that its derivatives are not all equal to zero or undefined). Maclaurin specifically studied a special case of Taylor's theorem where a is assumed to be zero. The series he created from this therefore were power series with only coefficients and without any subtraction/addition that needed to be complete under the power. Some examples of well known Maclaurin series include the following:

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} \dots \frac{x^\infty}{\infty!} \; for \; x \in (-\infty, \infty)$$

$$\frac{1}{1-x} = 1 + x + \; x^2 + \; x^3 \dots x^\infty \; for \; x \; \in (-1, 1)$$

$$\ln(1+x) = \; x - \frac{x^2}{2} + \; \frac{x^3}{3} \dots (-1)^{\infty+1} \frac{x^\infty}{\infty} \; for \; x \; \in [-1\,, 1]$$

The power of these tools is undoubtedly high but unfortunately their ability to conduct approximations is quite often limited to a local range. In order to use the series for points outside of their original radius of convergence, a new series all together must be generated and that too, series that are not centered at x = 0 require the use of a summation on each individual sum (or in the case of a computer a single new stored variable). Then there are some functions all together who do not have any well defined Maclaurin series and require Taylor's theorem to be applied in order to be approximated regardless if the function has a well defined limit at x= 0.

$$e^{-\frac{1}{x^2}}(centered\ at\ zero) = \ 0 + 0 + 0 \ldots for\ x = 0$$

The purpose of this piece is to explore and develop an alternative to the Taylor series for approximating functions and especially, to develop an alternative to the Maclaurin series (not only for computational efficiency but including aesthetics). These alternatives should preferably work in the regions of convergence where Taylor's theorem doesn't operate.

The Curious Case of the Infinite Power Series:

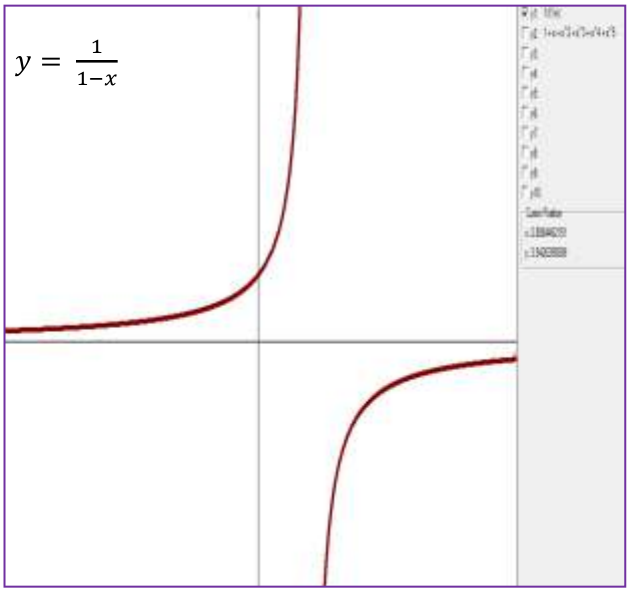
As stated earlier in the introduction the following identity is of common knowledge.

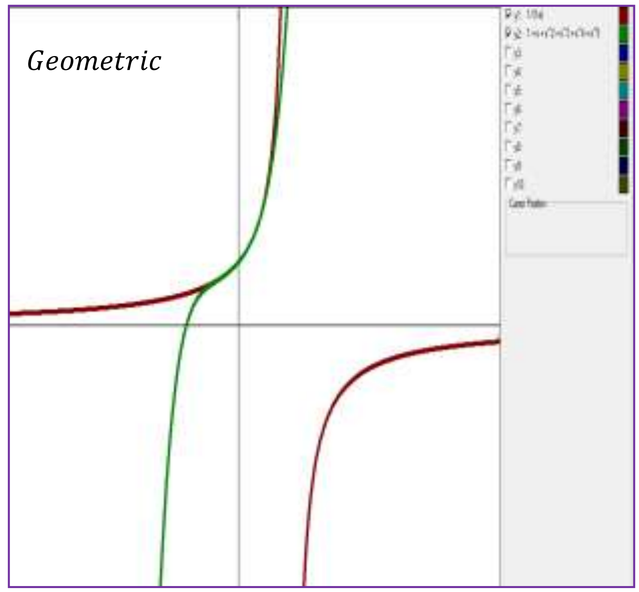
$$\sum_{i=0}^{\infty} x^n = 1 + x + \ x^2 + \ x^3 \ldots = \frac{1}{1-x}\ for\ |x| < 1$$

I pick this as a starting point for this paper because of a question that many readers might ask. This identity seems so simple, is there a way that it can be produced without the use of Calculus? The answer to that is yes, there is an algebraic approach to the problem of creating infinite series as long as the sums can be expressed in a fraction.

The Algebraic Method:

In this case our goal is not to prove that the infinite sum of all powers of a number converges but that the original function is equal to the infinite sum of all powers of a number. We thus proceed onward with our original statement $\frac{1}{1-x}$. By simply carrying out long division (divide 1 by 1-x) we end up with a never ending division sum that produces the terms of the geometric series. So basically one simply had "solve" the fraction to be able to produce the series. Now either by graphing the infinite sum or by actually using logic must one deduct that this series cannot possible converge for x with an absolute value greater than (or equal to for that matter) 1. Here I place the graph of 1/(1-x) as well as the power series to 10 terms (in green) to compare the two series.





For many people this might now appear to be a closed point; however, we haven't fully concluded it yet.

Generating Infinite Series is not always Commutative:

A rather unobvious question that one may ask is whether performing the same long division problem of $\frac{1}{1-x}$ but using the term (-x + 1) in division will change the answer. One would naturally say no (or not likely) to that answer because that would appear to be as counterintuitive as $\frac{1}{1-2} \neq \frac{1}{-2+1}$ (which obviously is not true) but that seems to be the case here. Upon dividing by (-x + 1) in that order we are left with a very different series than our original geometric series.

$$\frac{1}{1-x} = -\frac{1}{x} - \frac{1}{x^2} - \frac{1}{x^3} \dots \frac{1}{x^\infty}$$

Curiously enough, this series has some very unique properties. It converges to the function $\frac{1}{1-x}$ (which I now believe is important enough function that it should be given its own name) for all x whose absolute value exceeds 1. Basically, this series makes up for where the original geometric series fails to converge. A graph of all three series graphed together reveals this relationship in more detail.

At this point it is necessary to say that this series behaves very similarly to the Maclaurin series that we are more accustomed to except for a couple (major) differences. One, the series always converges outside a particular interval of convergence where as values of x being approximated are located further away, fewer terms are required to accurately model a function. The rest will be discussed in the next chapter (if that's the appropriate word) of this paper.

The Inverse Series:

The first type of alternative-Taylor series to be considered here will be the inverse or reciprocal Series. Their currently does not exist a formal way of defining this series using only f and a value of f (+ derivatives and integral values) at a particular point but the following information can be discerned:

The Inverse series when fully complete takes on a form that can be modeled by the following expression:

$$f(x) = \sum_{i=-\infty}^{\infty} a_i \frac{d(\frac{1}{x})^i}{dx^i} + \sum_{q=0}^{n} a_q x^q$$

Because all the terms in the sequence (for i>-1) approach zero as x approaches both positive and negative infinity, it is necessary to determine the values of the endpoints of the function being evaluated.

Thus we know that a constant term must be added that satisfies:

$$c(x) = \lim_{x \to \pm\infty} f(x)\,; c'(x) = 0.$$

Except in the situation where f(x) can be expressed as:

$$q(\ln(x))$$

where q does not remove ln(x) all together from the most algebraically simple expression of the function. An example of such a function would be ln(1-x). Through the use of integration of the derivative series (-1/1-x) we find that:

$$\ln(1-x) = -\ln(x) + \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{3x^3} \cdots \frac{1}{\infty x^\infty}$$

Clearly this series will have a value of infinity at the end. It is therefore very necessary to resort back to a more general way of determining the constant term:

$$a_q = \lim_{x \to \pm\infty} f^{[q]}(x)\,;$$

For those who are worried that evaluating at both ends of infinity will result in two different expressions a simple tool to counter that is the use of the function sign(x), sgn(x), or $\frac{|x|}{x}$. This function is capable of creating a positive and negative constant function of two different values (which itself can be multiplied and added by constants) to create the individual terms in the positive power series portion of the reciprocal series. The golden question remains however; how does one derive the coefficients of the reciprocal terms and iterated-logarithmic-integral-terms? That latter is really a mystery to me but the former appears to have the same coefficients as the corresponding power series of f(x) except for one very important point. The sign of these terms changes back and forth as one integrates or derives the series repeatedly.

Example:

$$\frac{1}{1-x} = \begin{cases} 1 + x + x^2 + \ \ldots \\ -\frac{1}{x} - \frac{1}{x^2} - \frac{1}{x^3} \ldots \end{cases}$$

$$-\ln(1-x) = \begin{cases} x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots \\ -\ln(x) + \frac{1}{x} + \frac{1}{x^2} \ldots \end{cases}$$

The magnitude of the coefficients between the Maclaurin and reciprocal series of functions has always been equal but the sign has been an issue. Sometimes the signs match, other times they do not. It appears as though every function as a number 1 or -1 assigned to it to define the sign orientation for its corresponding series. (Perhaps complex numbers may appear in future generalizations). The integrated log portions really have no discernible pattern at the moment. Which makes the existence of a close form expression of these series (taking on the form mention above) an open question or (for the sake of the title) a hypothesis.

Complex Series:

A Further Generalization from the reciprocal series can be committed. Consider the following functions (defined on C now as opposed to just R).

$$y = \frac{1}{1+\sqrt{x}}; \ y = \frac{3}{7 + x^{\pi+2}}; \ y = \frac{17}{x^{i/2} + 12}$$

If one goes through the divide-terms-and-see-what-happens approach a variety of new types of series come into existence. They are of similar form to the reciprocal series but now a variety of rational, irrational, transcendental, and complex terms become possible for powers of x. It appears that a series can be constructed using any power of x. Just try it out for oneself and results will start to appear. This leaves a lot of possibilities for how to generalize the Maclaurin (and I think we've sidetracked the goal of generalizing Taylor's Theorem) series. I believe that the use of fractional calculus might be a way to go about approaching the problem. A closed form expression for generating any type of power series (simply define the power of x who (along with derivative and integral terms) will be used and through calculus determine coefficients) should exist (no matter how un-elegant it will be). The question is how to determine this pattern.

An interesting note here is that when defining a series on C the direction or path that one wants to take along the x-plane to approximate the function becomes changeable and thus multiple series can be created. A very similar idea to this is the notion of the Laurent Series which defines a variety of infinite series over a path integral.

Permutation Groups:

There is more than meets the eye in terms of possible series when attempting the original division problem of f(x)/g(x). Consider the following case as an example:

$$y = \frac{1}{1 - x + x^2 - 5x^3}$$

Depending on how one orders the terms of 1, -x, $+x^2$, and $-5x^3$ a different series is produced as a solution. Even more interestingly is the fact that four all together different series (no matching terms to the point that terms containing an x to the same power have different coefficients) can be generated depending on whether 1, -x, x^2, and $-5x^3$ were the leading coefficient when conducting division. This zoo of different series (all of whom are some form or another of General Complex Power Series) can be grouped together as the Permutation group of defined series on the function. All integrals and derivatives of this function will have an integral and derivative series associated with those generated from this function. If it is possible to determine the number a permutation series a particular function has (say something abstract such as ln(x) + e^x – 1/(1-x)) then a General Complex Power Series Rule can be used quickly and effectively to only produce those series which do help approximate the function and not just any series. This comes down to the point of determining when a particular series should be used and when not.

Higher Order Generalizations:

The very notion of a series could be attached to any sort of iterated approximation. This paper primarily discusses power series but the idea can be considered with series of exponentials, series of wave functions (fourier is an example), series of tetrations, and further hyper-operators. Each has its own place and use for approximating functions. The other notion that can be considered is creating n-dimensional series for approximation. Partial derivatives make the problem more complex because the notion of a total derivative for a function is much harder to arrive at and instead a matrix of series will need to be used that allows one to decide along which axis (or path) they take to approximate a function. Perhaps path efficiency would be a field of value in that case.

Why Use It:

Besides the convenience of being able to on the spot generate an infinite series that can approximate a given function (or differential equation) this idea has additional weight to it. It creates a framework from which additional integral methods (as opposed to trapezoidal, simpson's rule etc… there exists the possibility of an reciprocal method, reciprocal quadratic, etc.. which can be used right now) can be generated. This also adds the convenience of being able to generalize the Runge-Kutta methods to allow for even more efficient methods of approximation (in the sense that these methods better fit the equation that is being modeled). The latter of which would be invaluable to the fields of ODE and PDE.

A Curious Side Note:

While I was working on this idea I realized that linear algebra could be made even more invaluable to the field. By using matrices (or higher dimensional arrays if dealing with higher dimensional equations) one is able to very accurately locally approximate a function. This can be achieved by not only solving for the individual points that the function exists in and fitting a curve but also by measuring the values of the derivatives (and even more curiously) the values of definite integrals of that function so that curve fitting can become even more accurate. This method may be much more efficient than even Taylor's Theorem or the proposed generalization (depending on the situation) as it may be much easier to compute. The main punch here is that by also evaluating a function at infinity (and perhaps using the matrices again for very accurate end point approximation) it may be possible to grasp the shape of the entire function without ever having to actually use a series approximation. This is just a thought though and needs to be explored further.