

# ON THE VALIDITY OF THE RIEMANN HYPOTHESIS

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## Abstract

In this paper, we have used the partial Euler product to examine the validity of the Riemann Hypothesis. The Dirichlet series over the Mobius function  $M(s) = \sum_{n=1}^{\infty} 1/n^s$  has been modified and represented in terms of the partial Euler product by progressively eliminating the numbers that first have a prime factor 2, then 3, then 5, ..up to the prime number  $p_r$  to obtain the series  $M(s, p_r)$ . It is shown that the series  $M(s)$  and the new series  $M(s, p_r)$  have the same region of convergence for every  $p_r$ . Unlike the partial sum of  $M(s)$  that has irregular behavior, the partial sum of the new series exhibits regular behavior as  $p_r$  approaches infinity. This has allowed the use of integration methods to compute the partial sum of the new series to determine its region of convergence and to provide an answer for the validity of the Riemann Hypothesis.

**Keywords:** Riemann zeta function, Mobius function, Riemann hypothesis, conditional convergence, Euler product.

**Classification:** Number Theory, 11M26

## 1 Introduction

The Riemann zeta function  $\zeta(s)$  satisfies the following functional equation over the complex plain [1]

$$\zeta(1-s) = 2(2\pi)^2 \cos(0.5s\pi)\Gamma(s)\zeta(s), \quad (1)$$

where,  $s = \sigma + it$  is a complex variable and  $s \neq 0$ .

For  $\sigma > 1$  (or  $\Re(s) > 1$ ),  $\zeta(s)$  can be expressed by the following series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad (2)$$

or by the following product over the primes  $p_i$ 's

$$\frac{1}{\zeta(s)} = \prod_{i=1}^{\infty} \left(1 - \frac{1}{p_i^s}\right). \quad (3)$$

where,  $p_1 = 2$ ,  $\prod_{i=1}^{\infty} (1 - 1/p_i^s)$  is the Euler product and  $\prod_{i=1}^r (1 - 1/p_i^s)$  is the partial Euler product. The above series and product representations of  $\zeta(s)$  are absolutely convergent for  $\sigma > 1$ .

The region of the convergence for the sum in Equation (2) can be extended to  $\Re(s) > 0$  by using the alternating series  $\eta(s)$  where

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}, \quad (4)$$

and

$$\zeta(s) = \frac{1}{1 - 2^{1-s}} \eta(s). \quad (5)$$

One may notice that the term  $1 - 2^{1-s}$  is zero at  $s = 1$ . This zero cancels the simple pole that  $\zeta(s)$  has at  $s = 1$  enabling the extension (or analog continuation) of the zeta function series representation over the critical strip  $0 < \Re(s) < 1$ .

It is well known that all of the non-trivial zeros of  $\zeta(s)$  are located in the critical strip  $0 < \Re(s) < 1$ . Riemann stated that all non-trivial zeros were very probably located on the critical line  $\Re(s) = 0.5$  [2]. There are many equivalent statements for the Riemann Hypothesis (RH) and one of them involves the Dirichlet series with the Mobius function.

The Mobius function  $\mu(n)$  is defined as follows

$$\begin{aligned} \mu(n) &= 1, \text{ if } n = 1. \\ \mu(n) &= (-1)^k, \text{ if } n = \prod_{i=1}^k p_i, p_i\text{'s are distinct primes.} \\ \mu(n) &= 0, \text{ if } p^2 | n \text{ for some } p. \end{aligned}$$

The Dirichlet series  $M(s)$  with the Mobius function is defined as

$$M(s) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}. \quad (6)$$

This series is absolutely convergent to  $1/\zeta(s)$  for  $\Re(s) > 1$  and conditionally convergent to  $1/\zeta(s)$  for  $\Re(s) = 1$ . The Riemann hypothesis is equivalent to the statement that  $M(s)$  is conditionally convergent to  $1/\zeta(s)$  for  $\Re(s) > 0.5$ .

Gonek, Hughes and Keating [3] have done an extensive research into establishing a relationship between  $\zeta(s)$  and its partial Euler product for  $\Re(s) < 1$ . Gonek stated "Analytic number theorists believe that an eventual proof of the Riemann Hypothesis must use both the Euler product and functional equation of the zeta-function. For there are functions with similar functional equations but no Euler product, and functions with an Euler product but no functional equation." In section 4, we will present a functional equation for  $\zeta(s)$  using its partial Euler product. The method is based on writing the Euler product formula as follows

$$1/\zeta(s) = \prod_{i=1}^{\infty} \left(1 - \frac{1}{p_i^s}\right) = \prod_{i=1}^r \left(1 - \frac{1}{p_i^s}\right) \prod_{r+1}^{\infty} \left(1 - \frac{1}{p_i^s}\right).$$

The above equation is valid for  $\sigma > 1$ . To be able to represent  $\zeta(s)$  in term of its partial Euler product for  $\sigma \leq 1$ , we have to replace the term  $\prod_{r+1}^{\infty} (1 - 1/p_i^s)$  with an equivalent one that allows the analytic continuation for the representation of  $\zeta(s)$  for  $\sigma \leq 1$ . Thus, the new term, that we need to introduce to replace  $\prod_{r+1}^{\infty} (1 - 1/p_i^s)$ , must have a zero that cancels the pole that  $\zeta(s)$  has at  $s = 1$ . In the section 4, we will use the complex analysis to compute this new term. We then use the new representation to compute the sum  $\sum_{i=1}^r p_i^\sigma$  for  $\sigma < 1$ . This sum

is then used in our method, described in section 5, to examine the validity of the Riemann Hypothesis.

In this paper, we claim the the Riemann Hypothesis is invalid. We support our claim by proving that the series  $M(\sigma)$  is divergent for  $\sigma < 1$ . We have achieved this result by introducing a method to represent the Dirichlet series  $M(s)$  (defined by Equation (6)) in terms of the partial Euler product. This task is achieved (sections 2) by first eliminating the numbers that have the prime factor 2 to generate the series  $M(s, 2)$ . For the series  $M(s, 2)$ , we then eliminate the numbers with the prime factor 3 to generate the series  $M(s, 3)$ , and so on, up to the prime number  $p_r$ . In essence, we have applied the sieving technique to modify the series  $M(s)$  to include only the numbers with prime factors greater than  $p_r$ . In section 3, we have shown that the series  $M(s)$  and the new series  $M(s, p_r)$  have the same region of convergence.

So far, the efforts to use the series  $M(\sigma)$  to examine the validity of the Riemann Hypothesis have failed due to the irregular behavior of the partial sum of the series  $M(\sigma)$ . In section 5, we have shown that the partial sum of the new series  $M(\sigma, p_r)$  exhibits regular behavior as  $p_r$  approaches infinity. This has allowed the use of integration methods to compute the partial sum of the new series and consequently determine its region of convergence. With this analysis, we have shown that the series  $M(\sigma, p_r)$  and  $M(\sigma)$  are divergent for  $\sigma < 1$ . Thus, non-trivial zeros can be found arbitrary close to the line  $s = 1$ .

## 2 Applying the Sieving Method to the Dirichlet Series $M(s)$ .

The Dirichlet series  $M(s)$  with the Mobius function is defined as

$$M(s) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s},$$

where  $\mu(n)$  is the Mobius function. Thus,

$$M(s) = 1 - \frac{1}{2^s} - \frac{1}{3^s} + \frac{0}{4^s} - \frac{1}{5^s} + \frac{1}{6^s} \dots$$

It should be pointed out that our definition of  $M(s)$  is different from Mertins function  $M(x)$  that is commonly found in the literature and defined as  $M(x) = \sum_{1 \leq n \leq x} \mu(n)$ .

Next, we introduce the series  $M(s, 2)$  by eliminating all the numbers that have a prime factor 2. Thus,  $M(s, 2)$  can be written as

$$M(s, 2) = 1 - \frac{1}{3^s} - \frac{1}{5^s} - \frac{1}{7^s} + \frac{0}{9^s} - \frac{1}{11^s} - \frac{1}{13^s} + \frac{1}{15^s} \dots$$

Since our analysis to test the conditional convergence of these series ( $M(s)$  and  $M(s, 2)$  for  $\sigma \leq 1$ ) is based on comparing correspondent terms of these two series. Therefore, rearrangement and permutation of the terms may have a significant impact on the region of convergence of both series. Therefore, it essential to have the same index for both series  $M(s)$  and  $M(s, 2)$  refer to the same term. Hence, we will represent  $M(s, 2)$  as follows

$$M(s, 2) = 1 + \frac{0}{2^s} - \frac{1}{3^s} + \frac{0}{4^s} - \frac{1}{5^s} + \frac{0}{6^s} - \frac{1}{7^s} - \frac{0}{8^s} \dots,$$

or

$$M(s, 2) = \sum_{n=1}^{\infty} \frac{\mu(n, 2)}{n^s}, \quad (7)$$

where

$$\begin{aligned} \mu(n, 2) &= \mu(n), \text{ if } n \text{ is an odd number,} \\ \mu(n, 2) &= 0, \text{ if } n \text{ is an even number.} \end{aligned}$$

The above series  $M(s, 2)$  can be further modified by eliminating all the numbers that have a prime factor 3 to get the series  $M(s, 3)$  where

$$M(s, 3) = 1 - \frac{1}{5^s} - \frac{1}{7^s} - \frac{1}{11^s} - \frac{1}{13^s} - \frac{1}{17^s} - \frac{1}{19^s} - \frac{1}{23^s} + \frac{0}{25^s} \dots,$$

or more conveniently

$$M(s, 3) = 1 + \frac{0}{2^s} - \frac{0}{3^s} + \frac{0}{4^s} - \frac{1}{5^s} + \frac{0}{6^s} - \frac{1}{7^s} - \frac{0}{8^s} \dots,$$

and so on.

Let  $I(p_r)$  represent, in ascending order, the integers with distinct prime factors that belong to the set  $\{p_i : p_i > p_r\}$ . Let  $\{1, I(p_r)\}$  be the set of 1 and  $I(p_r)$  (for example,  $\{1, I(2)\}$  is the set of square-free odd numbers), then we define the series  $M(s, p_r)$  as

$$M(s, p_r) = \sum_{n=1}^{\infty} \frac{\mu(n, p_r)}{n^s}, \quad (8)$$

where

$$\begin{aligned} \mu(n, p_r) &= \mu(n), \text{ if } n \in \{1, I(p_r)\}, \\ \mu(n, p_r) &= 0, \text{ otherwise.} \end{aligned}$$

It can be easily shown that, for every prime number  $p_r$ , the series  $M(s, p_r)$  converges absolutely for  $\Re(s) > 1$ . Furthermore, it can be shown that, for  $\Re(s) > 1$ ,  $M(s, p_r)$  satisfies the following equation

$$M(s) = M(s, p_r) \prod_{i=1}^r \left(1 - \frac{1}{p_i^s}\right). \quad (9)$$

Since

$$M(s) = \frac{1}{\zeta(s)} = \prod_{i=1}^{\infty} \left(1 - \frac{1}{p_i^s}\right),$$

then we conclude that, for  $\Re(s) > 1$ ,  $M(s, p_r)$  approaches 1 as  $p_r$  approaches infinity.

### 3 The region of convergence for the series $M(s)$ and $M(s, p_r)$ .

In this section, we will deal with the question of relationship between the conditional convergence of the two series  $M(s, p_r)$  and  $M(s)$  over the strip  $0.5 < \Re(s) \leq 1$ . This task can be achieved by examining the convergence of the series  $M(s, p_r)$  and  $M(s)$  along the real axis (or along the line  $0.5 < \sigma \leq 1$ ). Theorems 1 and 2 establish the relationship between the conditional convergence of the two series  $M(s)$  and  $M(s, p_r)$  for  $0.5 < \sigma \leq 1$ .

**Theorem 1** For  $s = \sigma + i0$ , where  $0.5 < \sigma \leq 1$  and for every prime number  $p_r$ , the series  $M(\sigma)$  converges conditionally if and only if the series  $M(\sigma, p_r)$  converges conditionally. Furthermore,  $M(\sigma)$  and  $M(\sigma, p_r)$  are related as follows

$$M(\sigma) = M(\sigma, p_r) \prod_{i=1}^r \left(1 - \frac{1}{p_i^\sigma}\right). \quad (10)$$

The proof of Theorem 1 is outlined in Appendix 1.

**Theorem 2** For  $s = \sigma + it$ , where  $0.5 < \sigma \leq 1$  and for every prime number  $p_r$ , the series  $M(s)$  converges conditionally if and only if the series  $M(s, p_r)$  converges conditionally. Furthermore,  $M(s)$  and  $M(s, p_r)$  are related as follows

$$M(s) = M(s, p_r) \prod_{i=1}^r \left(1 - \frac{1}{p_i^s}\right). \quad (11)$$

The proof of Theorem 2 follows from the fact that  $M(s)$  and  $M(s, p_r)$  are Dirichlet series. Consequently, the series  $M(s)$  is conditionally convergent if and only if the series  $M(\sigma)$  is conditionally convergent. Also, the series  $M(s, p_r)$  is conditionally convergent if and only if the series  $M(\sigma, p_r)$  is conditionally convergent. Using Theorem 1, we then conclude that the series  $M(s)$  is conditionally convergent if and only if the series  $M(s, p_r)$  is conditionally convergent.

The second part of the theorem can be also proved by first defining  $M(s, p_r; N_1, N_2)$  as the partial sum

$$M(s, p_r; N_1, N_2) = \sum_{n=N_1}^{N_2} \frac{\mu(n, p_r)}{n^s}, \quad (12)$$

where  $N_2 \geq p_r$ . Then, we have

$$M(s, p_{r-1}; 1, Np_r) = M(s, p_r; 1, Np_r) - \frac{1}{p_r^s} M(s, p_r; 1, N). \quad (13)$$

Since the series  $M(s, p_r)$  is conditionally convergent, then the partial sums  $M(s, p_r; 1, Np_r)$  and  $M(s, p_r; 1, N)$  are both convergent to  $M(s, p_r)$  as  $N$  approaches infinity. Hence, as  $N$  approaches infinity, we obtain

$$M(s, p_{r-1}) = \lim_{N \rightarrow \infty} M(s, p_{r-1}; 1, Np_r) = M(s, p_r) \left(1 - \frac{1}{p_r^s}\right).$$

By repeating this process  $r - 1$  times, we then obtain

$$M(s) = M(s, p_r) \prod_{i=1}^r \left(1 - \frac{1}{p_i^s}\right).$$

Note that if we multiply both sides of the above equation by  $\prod_{i=1}^r (1 + p_i^{-s})$

$$M(s, p_r) = \frac{1}{\zeta(s) \prod_{i=1}^r (1 - p_i^{-2s})} \prod_{i=1}^r \left(1 + \frac{1}{p_i^s}\right).$$

As  $p_r$  approaches infinity, we then have

$$M(s, p_r) = \frac{\zeta(2s)}{\zeta(s)} \prod_{i=1}^r \left(1 + \frac{1}{p_i^s}\right).$$

#### 4 Functional representation of $\zeta(s)$ using its partial Euler product.

Theorem 1 of the previous section provides a relationship between  $\zeta(s) = 1/M(s)$  and the partial Euler product  $\prod_{i=1}^r (1 - 1/p_i^s)$ . In this section, we will use the prime counting function to derive a functional representation for  $\zeta(s)$  using its partial Euler product. This functional representation is then used to provide a second relationship between  $\zeta(s) = 1/M(s)$  and the partial Euler product  $\prod_{i=1}^r (1 - 1/p_i^s)$ . These two relationships are then analyzed in sections (5) and (6) and then used to examine the validity of the Riemann Hypothesis.

We will start this task by first writing  $\zeta(s)$  for  $\sigma > 1$  as follows

$$1/\zeta(s) = \prod_{i=1}^{\infty} \left(1 - \frac{1}{p_i^s}\right) = \prod_{i=1}^r \left(1 - \frac{1}{p_i^s}\right) \prod_{i=r+1}^{\infty} \left(1 - \frac{1}{p_i^s}\right). \quad (14)$$

For  $\sigma > 0.5$ , we have

$$\log \prod_{i=r+1}^{r^2} \left(1 - \frac{1}{p_i^s}\right) = \sum_{i=r+1}^{r^2} \log \left(1 - \frac{1}{p_i^s}\right),$$

or

$$\log \prod_{i=r+1}^{r^2} \left(1 - \frac{1}{p_i^s}\right) = \sum_{i=r+1}^{r^2} \left(-\frac{1}{p_i^s} - \frac{1}{2p_i^{2s}} - \frac{1}{3p_i^{3s}} - \dots\right).$$

Let  $\delta$  be defined as the sum

$$\delta = \sum_{i=r+1}^{r^2} \left(-\frac{1}{2p_i^{2s}} - \frac{1}{3p_i^{3s}} - \frac{1}{4p_i^{4s}} \dots\right). \quad (15)$$

Thus,

$$\log \prod_{i=r+1}^{r^2} \left(1 - \frac{1}{p_i^s}\right) = -\sum_{i=r+1}^{r^2} \frac{1}{p_i^s} + \delta. \quad (16)$$

Since  $|\delta| < \sum_{n=p_{r+1}}^{\infty} \left(\frac{1}{2n^{2\sigma}} + \frac{1}{3n^{3\sigma}} + \frac{1}{4n^{4\sigma}} \dots\right)$ , thus  $\delta = O(p_{r+1}^{1-2\sigma}/(2\sigma - 1))$ . Furthermore, if  $2\sigma - 1$  is a fixed positive number, then  $\delta = O(p_{r+1}^{1-2\sigma})$ .

Using the Prime Number Theorem (PNT) with a suitable constant  $a > 0$ , the number of primes less than  $x$  is given by [4, page 43]

$$\pi(x) = \text{Li}(x) + O\left(xe^{-a\sqrt{\log x}}\right), \quad (17)$$

or

$$\pi(x) = \text{Li}(x) + O\left(x/(\log x)^k\right), \quad (18)$$

where  $\text{Li}(x)$  is the Logarithmic Integral of  $x$  and  $k$  is a number greater than zero.

Using Stieltjes integral [5], we may write the sum  $\sum_{i=r_1}^{r_2} \frac{1}{p_i^\sigma}$  for  $\sigma > 1$  as follows

$$\sum_{i=r_1}^{r_2} \frac{1}{p_i^\sigma} = \int_{x=p_{r_1}}^{p_{r_2}} \frac{d\pi(x)}{x^\sigma}. \quad (19)$$

Using Equation (18) for the representation of  $\pi(x)$ , we may then write the integral in Equation (19) as [5, Theorem 2, page 57]

$$\sum_{i=r_1}^{r_2} \frac{1}{p_i^\sigma} = \int_{p_{r_1}}^{p_{r_2}} \frac{1}{x^\sigma} \frac{1}{\log x} dx + O\left(\frac{1}{(\log p_{r_1})^k}\right), \quad (20)$$

where  $k$  is a number greater than zero. Therefore,

$$\sum_{i=r_1}^{r_2} \frac{1}{p_i^\sigma} = \int_{p_{r_1}}^{\infty} \frac{1}{x^\sigma} \frac{1}{\log x} dx - \int_{p_{r_2}}^{\infty} \frac{1}{x^\sigma} \frac{1}{\log x} dx + O\left(\frac{1}{(\log p_{r_1})^k}\right). \quad (21)$$

Recalling that the Exponential Integral  $E_1(r)$  is given by

$$E_1(r) = \int_r^{\infty} \frac{e^{-u}}{u} du,$$

and using the substitutions  $u = (\sigma - 1) \log p_r$ ,  $du = (\sigma - 1) dx/x$  and  $x^\sigma/x = e^u$ , then for  $\sigma > 1$ , we may write Equation (21) as

$$\sum_{i=r_1}^{r_2} \frac{1}{p_i^\sigma} = E_1((\sigma - 1) \log p_{r_1}) - E_1((\sigma - 1) \log p_{r_2}) + O\left(\frac{1}{(\log p_{r_1})^k}\right). \quad (22)$$

Combining Equations (16) and ((22)) and noting that, for  $\sigma > 1$ ,  $E_1((\sigma - 1) \log p_{r_2})$  approaches zero as  $p_{r_2}$  approaches infinity, we may write Equation (14) for  $\sigma > 1$  as

$$-\log \zeta(\sigma) = \sum_{i=1}^r \log\left(1 - \frac{1}{p_i^\sigma}\right) - \sum_{i=r+1}^{\infty} \frac{1}{p_i^\sigma} + \delta,$$

or

$$\log \zeta(\sigma) + \sum_{i=1}^r \log\left(1 - \frac{1}{p_i^\sigma}\right) - E_1((\sigma - 1) \log p_{r+1}) = \epsilon,$$

where  $\epsilon = O(1/(\log p_{r+1})^k)$  is an arbitrarily small number attained by setting  $p_r$  sufficiently large. Therefore,

$$\zeta(\sigma) \prod_{i=1}^r \left(1 - \frac{1}{p_i^\sigma}\right) \exp(-E_1((\sigma - 1) \log p_{r+1})) = 1 + \epsilon. \quad (23)$$

As  $p_r$  approaches infinity,  $\epsilon$  approaches zero. Hence, the right side of the above equation approaches 1 as  $p_r$  approaches infinity.

Similarly, for  $\Re(s) > 1$ , we can use the following expression for  $E_1(s)$

$$E_1(s) = \int_1^{\infty} \frac{e^{-xs}}{x} dx,$$

to show that

$$\lim_{r \rightarrow \infty} \left\{ \zeta(s) \prod_{i=1}^r \left( 1 - \frac{1}{p_i^s} \right) \exp(-E_1((s-1) \log p_{r+1})) \right\} = 1. \quad (24)$$

Let the function  $G(s, p_r)$  be defined as

$$G(s, p_r) = \zeta(s) \prod_{i=1}^r \left( 1 - \frac{1}{p_i^s} \right) \exp(-E_1((s-1) \log p_{r+1})) \quad (25)$$

where,  $G(s, p_r)$  is a regular function for  $\Re(s) > 1$ . Referring to Equation (24), the function  $G(s, p_r)$  approaches 1 as  $p_r$  approaches infinity. It should be noted that, for every  $p_r$ , the function  $\exp(-E_1((s-1) \log p_{r+1}))$  is an entire function, the function  $\zeta(s)$  is analytic everywhere except at  $s = 1$  and the function  $\prod_{i=1}^r (1 - 1/p_i^s)$  is analytic for  $\Re(s) > 0$ . Thus, for any  $\sigma > 1$ , the function  $G(s, p_r)$  can be considered as a sequence of analytic functions. Furthermore, as  $p_r$  (or  $r$ ) approaches infinity, this sequence is uniformly convergent over the half plane with  $\sigma > 1 + \epsilon$  (where,  $\epsilon$  is an arbitrary small number). Therefore, by the virtue of the Weiestrass theorem, the limit is also analytic function [6] (Weiestrass theorem states that if the function sequence  $f_n$  is analytic over the region  $\Omega$  and  $f_n$  is uniformly convergent to a function  $f$ , then  $f$  is also analytic on  $\Omega$  and  $f_n'$  converges uniformly to  $f'$  on  $\Omega$ ). If we define this limit as  $G(s)$ , where

$$G(s) = \lim_{r \rightarrow \infty} G(s, p_r) \quad (26)$$

then,  $G(s)$  is analytic over the half plane  $\Re(s) > 1$  and it is equal to 1 by the virtue of Equation (24).

The Prime Number Theorem (PNT) allows us to extend the above results to the line  $s = 1 + it$ . Moreover, we will show that if RH is valid, then for the strip  $s = \sigma + it$  where,  $0.5 < \sigma < 1$ , the above results will also be valid with the limit of  $G(s, p_r)$  is 1 as  $p_r$  approaches infinity.

We will start this task by showing that although both  $\zeta(s)$  and  $E_1((s-1) \log p_{r+1})$  have a singularity at  $s = 1$ , the product  $G(s, p_r)$  has a removable singularity at  $s = 1$  for every  $p_r$ . This can be shown by first expanding  $\zeta(s)$  as a Laurent series about its singularity at  $s = 1$

$$\zeta(s) = \frac{1}{s-1} + \gamma - \gamma_1(s-1) + \gamma_2 \frac{(s-1)^2}{2!} - \gamma_3 \frac{(s-1)^3}{3!} + \dots, \quad (27)$$

where  $\gamma$  is the Euler-Mascheroni constant and  $\gamma_i$ 's are the Stieltjes constants. For  $s = 1 + \epsilon$ , where  $\epsilon = \epsilon_1 + i\epsilon_2$ ,  $\epsilon_1$  and  $\epsilon_2$  are arbitrary small numbers, the above equation can be written as

$$\zeta(s) = \frac{1}{\epsilon} + \gamma - \gamma_1 \epsilon + \gamma_2 \frac{\epsilon^2}{2!} - \gamma_3 \frac{\epsilon^3}{3!} + \dots \quad (28)$$

Furthermore, for  $\sigma > 1$ , using the definition of the Exponential Integral, we may write  $E_1(s)$  as

$$E_1(s) = -\gamma - \log s + s - \frac{s^2}{2!} + \frac{s^3}{3!} - \frac{s^4}{4!} + \dots \quad (29)$$

Thus, for  $s = 1 + \epsilon$ , we have

$$\exp(-E_1((s-1) \log p_r)) = e^\gamma \epsilon \log p_r \exp \left( -\epsilon \log p_r + \frac{(\epsilon \log p_r)^2}{2!} - \frac{(\epsilon \log p_r)^3}{3!} + \dots \right). \quad (30)$$

By taking the product  $\zeta(s) \exp(-E_1((s-1) \log p_r))$  and allowing  $\epsilon$  to approach zero, we then obtain

$$\lim_{s \rightarrow 1} \{\zeta(s) \exp(-E_1((s-1) \log p_r))\} = e^\gamma \log p_r. \quad (31)$$

However, it is well known that the partial Euler product at  $s = 1$  can be written as [8]

$$\prod_{i=1}^r \left(1 - \frac{1}{p_i}\right) = \frac{e^{-\gamma}}{\log p_r} + O\left(\frac{1}{(\log p_r)^2}\right). \quad (32)$$

Multiplying Equations (31) and (32), we may conclude that at  $s = 1$ ,  $G(s, p_r)$  approaches 1 as  $p_r$  approaches infinity. Furthermore, for  $s = 1 + it$  and  $t \neq 1$ , the value of  $\exp(-E_1(it \log p_r))$  approaches 1 as  $p_r$  approaches infinity and since

$$\lim_{r \rightarrow \infty} \left\{ \zeta(s) \prod_{i=1}^r \left(1 - \frac{1}{p_i^s}\right) \right\} = 1,$$

therefore, for  $s = 1 + it$ , we have the following

$$\lim_{r \rightarrow \infty} G(s, p_r) = \lim_{r \rightarrow \infty} \left\{ \zeta(s) \prod_{i=1}^r \left(1 - \frac{1}{p_i^s}\right) \exp(-E_1((s-1) \log p_{r+1})) \right\} = 1.$$

So far, we have shown that the function  $G(s, p_r)$  is uniformly convergent to 1 when  $\Re(s) > 1$  and using PNT,  $G(s, p_r)$  is convergent to 1 for  $\Re(s) = 1$ . In the following, we will show that, assuming the validity of the Riemann Hypothesis, the function  $G(s, p_r)$  is uniformly convergent to 1 for every value of  $s$  with  $\Re(s) > 0.5 + \epsilon$ , where  $\epsilon$  is an arbitrary small number. Toward this goal, we will first show that the function  $G(s, p_r)$  is convergent for any value of  $s$  on the real axis with  $\sigma > 0.5$ . This can be achieved by first writing the expressions for  $G(\sigma, p_{r1})$  and  $G(\sigma, p_{r2})$  (where  $r2$  is an arbitrary large number greater than  $r1$ )

$$G(\sigma, p_{r1}) = \zeta(\sigma) \exp(-E_1((\sigma-1) \log p_{r1+1})) \prod_{i=1}^{r1} \left(1 - \frac{1}{p_i^\sigma}\right), \quad (33)$$

$$G(\sigma, p_{r2}) = \zeta(\sigma) \exp(-E_1((\sigma-1) \log p_{r2+1})) \prod_{i=1}^{r2} \left(1 - \frac{1}{p_i^\sigma}\right). \quad (34)$$

Since the function  $G(s, p_r)$  is analytic that is not equal to 0 for  $\sigma > 0.5$ , hence we can divide Equation (34) by Equation (33) and then take the logarithm to obtain

$$\log \left( \frac{G(\sigma, p_{r2})}{G(\sigma, p_{r1})} \right) = E_1((\sigma-1) \log p_{r1+1}) - E_1((\sigma-1) \log p_{r2+1}) + \log \left( \prod_{i=r1+1}^{r2} \left(1 - \frac{1}{p_i^\sigma}\right) \right). \quad (35)$$

To compute the logarithm of the partial Euler product in Equation (35), we recall Equation (16)

$$\log \prod_{r1+1}^{r2} \left(1 - \frac{1}{p_i^\sigma}\right) = - \sum_{i=r1+1}^{r2} \frac{1}{p_i^\sigma} + \delta,$$

where  $\delta = O(p_{r1}^{1-2\sigma}/(2\sigma-1))$ . Furthermore, on RH, we have

$$\pi(x) = \text{Li}(x) + O(\sqrt{x} \log x), \quad (36)$$

where  $\text{Li}(x)$  is the Logarithmic Integral of  $x$ . Using Equation (36) for the representation of the prime counting function, we may then obtain (Appendix 2)

$$\sum_{i=r_1+1}^{r_2} \frac{1}{p_i^\sigma} = E_1((\sigma - 1) \log p_{r_1+1}) - E_1((\sigma - 1) \log p_{r_2}) + \varepsilon,$$

where  $\varepsilon = O(p_{r_1}^{0.5-\sigma} \log p_{r_1} / (\sigma - 0.5)^2)$ . Hence, Equation (35) can be written as

$$\log \left( \frac{G(\sigma, p_{r_2})}{G(\sigma, p_{r_1})} \right) = \varepsilon + \delta + E_1((\sigma - 1) \log p_{r_2}) - E_1((\sigma - 1) \log p_{r_2+1}).$$

Since  $E_1((\sigma - 1) \log p_{r_2}) - E_1((\sigma - 1) \log p_{r_2+1})$  approaches zero as  $p_{r_2}$  approaches zero, thus

$$\lim_{p_{r_2} \rightarrow \infty} \log \left( \frac{G(\sigma, p_{r_2})}{G(\sigma, p_{r_1})} \right) = \varepsilon + \delta.$$

Furthermore, for  $\sigma > 0.5 + \epsilon$ ,  $\varepsilon + \delta$  can be made arbitrary small by choosing  $p_{r_1}$  arbitrary large, thus the limit of  $G(\sigma, p_r)$  exists as  $p_r$  approaches infinity and it is given by

$$G(\sigma) = \lim_{r \rightarrow \infty} G(\sigma, p_r) \quad (37)$$

This proves that, on RH,  $G(\sigma, p_r)$  is convergent as  $p_r$  approaches infinity and thus  $G(\sigma)$  exists for  $\sigma > 0.5$ . In Appendix 3, we have shown that, on RH and for  $\Re(s) > 0.5$ , we have

$$\sum_{i=r_1}^{r_2} \frac{1}{p_i^s} = E_1((s - 1) \log p_{r_1}) - E_1((s - 1) \log p_{r_2}) + \varepsilon, \quad (38)$$

where  $\varepsilon = O\left(\frac{|s|+1}{(\sigma-0.5)^2} p_{r_1}^{0.5-\sigma} \log p_{r_1}\right)$ . Thus, we can follow the same steps and show that  $G(s, p_r)$  is convergent as  $p_r$  approaches infinity and thus  $G(s)$  exists for  $\Re(s) > 0.5$  (it should be pointed out, that the term  $\varepsilon$  in Equation (38) can be determined in terms of the non-trivial zero if the von Mangoldt function is used in deriving Equation (38) instead of using the prime counting function).

It should be noted that, while the function sequence  $G(s, p_r)$  is not uniformly convergent when the region of convergence is extended all the way to the line  $\sigma = 0.5$ , it is however uniformly convergence for any rectangle extending from  $-iT$  to  $iT$  (for any arbitrary large  $T$ ) and with  $\sigma > 0.5 + \epsilon$ , where  $\epsilon$  is an arbitrary small number. This follows from the fact that, on RH,  $\varepsilon$  (or, the  $O$  term) is bounded for any  $\sigma > 0.5 + \epsilon$ . Since  $G(s, p_r)$  is analytic for  $\Re(s) > 0$  and it is uniformly convergent for  $\Re(s) > 0.5 + \epsilon$ , thus  $G(s)$  is analytic for the half right complex plain with  $\Re(s) > 0.5 + \epsilon$  (Weiestrass theorem [6]). Since we have shown that  $G(s) = 1$  for  $\Re(s) \geq 1$ , thus on RH,  $G(s) = 1$  for  $\Re(s) > 0.5 + \epsilon$ . Hence, we have the following theorem

**Theorem 3** For  $s = \sigma + it$  and  $\sigma > 0.5$ , the following holds if RH is valid

$$\lim_{r \rightarrow \infty} \left\{ \zeta(s) \prod_{i=1}^r \left( 1 - \frac{1}{p_i^s} \right) \exp(-E_1((s - 1) \log p_{r+1})) \right\} = 1. \quad (39)$$

$$\lim_{r \rightarrow \infty} \{M(s, p_r) \exp(E_1((s - 1) \log p_{r+1}))\} = 1. \quad (40)$$

It should be also pointed out that Theorem 3 can be generalized to the case where there are no non-trivial zeros for values of  $s$  with  $\Re(s) > a$  (where  $a > 0.5$ ). For this case, Equation (39) is valid for every  $s$  with  $\Re(s) > a$  and  $\varepsilon$  in Appendix 3 is given by  $O\left(\frac{|s+1|}{(\sigma-a)^2} p_r^{1-a-\sigma} \log p_r\right)$ .

In the next section, we will use Theorem 3 and Equation (38) in conjunction with Theorems 1 and 2 to show that the series  $M(\sigma, p_r)$  and  $M(\sigma)$  diverge for  $\sigma < 1$

## 5 The series $M(\sigma, p_r)$ at $\sigma = 1$ .

In this section, we will provide an estimate for the partial sum  $M(1, p_r; 1, p_r^a)$  as  $a$  approaches infinity. This estimate will be computed by using Equation (38) and noting that  $M(1, p_r)$  equals zero for every  $p_r$ . Therefore, for every  $p_r$ ,  $M(1, p_r; 1, p_r^a)$  approaches zero as  $a$  approaches infinity. We have also shown in Appendix 4 that, for every  $p_r$  and  $N$ , we have

$$|M(1, p_r; 1, N)| = \left| \sum_{n=1}^N \frac{\mu(n, p_r)}{n} \right| \leq 2.$$

Before we present the details of our method, it is important to note that the use of  $y$ -smooth numbers can be also used to generate and estimate the partial sum  $M(1, p_r; 1, p_r^a)$ . The  $y$ -smooth numbers are the numbers that have only prime factors less than or equal to  $y$ . These numbers have been extensively analyzed in the literature [10]. In [10], Granville presented a clever method to generate a series that is identical to the partial sum  $M(1, p_r; 1, p_r^a)$ . With his method and using the inclusion-exclusion principle [10, page 248], one can then provide an estimate for the partial sum  $M(1, p_r; 1, p_r^a)$ . In this section, we will provide a more general approach to compute  $M(1, p_r; 1, p_r^a)$ . The main advantage of our approach is the ability to extend it to compute the partial sum for values of  $s$  other than 1. In the following, we will present our method in the following two steps.

- In the first step of our approach, we will show that, for every  $a$  and as  $p_r$  approaches infinity, the partial sum  $M(1, p_r; 1, p_r^a)$  is a function of only  $a$  (independent of  $p_r$ ).

Toward this end, we define the function  $f(a, p_r)$  as

$$f(a, p_r) = M(1, p_r; 1, p_r^a) = \sum_{n=1}^{p_r^a} \frac{\mu(n, p_r)}{n}.$$

We will then show that, for every  $a$  and as  $p_r$  approaches infinity, the function  $f(a, p_r)$  approaches a deterministic function  $\rho(a)$ . In other words; if we plot  $M(1, p_r; 1, N)$  (where  $N = p_r^a$ ) as a function of  $a = \log N / \log p_r$ , then for each value of  $a$  and as  $p_r$  approaches infinity,  $f(a, p_r)$  approaches a unique value  $\rho(a)$ . This is equivalent to the statement

$$\rho(a) = \lim_{p_r \rightarrow \infty} f(a, p_r) = \lim_{p_r \rightarrow \infty} M(1, p_r; 1, p_r^a).$$

This result can be achieved by first noting that the partial sum  $M(1, p_r; 1, p_r^a)$  for  $1 < a < 2$  is given by

$$M(1, p_r; 1, p_r^a) = 1 - \sum_{p_r \leq p_i < p_r^a} \frac{1}{p_i}.$$

If we define  $M_1(1, p_r; 1, p_r^a)$  as

$$M_1(1, p_r; p_r, p_r^a) = \sum_{p_r \leq p_i < p_r^a} \frac{1}{p_i},$$

then, using Stieltjes integral, we obtain

$$M(1, p_r; 1, p_r^a) = 1 - M_1(1, p_r; p_r, p_r^a) = 1 - \int_{p_r}^{p_r^a} \frac{d\pi(x)}{x} = 1 - \int_1^a \frac{d\pi(p_r^y)}{p_r^y}.$$

On RH, we have

$$d\pi(p_r^y) = d\text{Li}(p_r^y) + dO(\sqrt{p_r^y} \log(p_r^y)),$$

or

$$d\pi(p_r^y) = \frac{1}{\log(p_r^y)} dp_r^y + dO(\sqrt{p_r^y} \log(p_r^y)) = \frac{p_r^y}{y} dy + dO(\sqrt{p_r^y} \log(p_r^y)).$$

Hence, for  $1 < a < 2$ , we have

$$M(1, p_r; 1, p_r^a) = 1 - \int_1^a \frac{dy}{y} + \int_1^a \frac{dO(\sqrt{p_r^y} \log(p_r^y))}{p_r^y} = 1 - \log(a) + O(g_1(p_r, a)),$$

where

$$O(g_1(p_r, a)) = \int_1^a \frac{dO(\sqrt{p_r^y} \log(p_r^y))}{p_r^y}.$$

As  $p_r$  approaches infinity,  $O(g_1(p_r, a))$  approaches zero. Consequently,

$$\lim_{p_r \rightarrow \infty} M(1, p_r; 1, p_r^a) = 1 - \log a.$$

The terms of the partial sum  $M(1, p_r; 1, p_r^a)$  in the range  $p_r \leq x < p_r^3$  are either a reciprocal of a prime or a reciprocal of the product of two primes. Therefore, for  $1 < a < 3$ , we have

$$M(1, p_r; 1, p_r^a) = 1 - \sum_{p_r \leq p_i < p_r^a} \frac{1}{p_i} + \sum_{p_r \leq p_{i1} p_{i2} < p_r^a} \frac{1}{p_{i1} p_{i2}},$$

where  $p_{i1}$  and  $p_{i2}$  are two distinct primes that are greater than or equal to  $p_r$ . Let  $M_2(1, p_r; 1, p_r^a)$  be defined as

$$M_2(1, p_r; 1, p_r^a) = \sum_{p_r \leq p_{i1} p_{i2} < p_r^a} \frac{1}{p_{i1} p_{i2}} = \frac{1}{2} \sum_{p_r \leq p_i < p_r^{a-1}} \frac{1}{p_i} M_1(1, p_r; p_r, p_r^a/p_i) + R_2,$$

where the factor of half was added since each term of the form  $1/(p_{i1} p_{i2})$  is repeated twice. It should be also noted that the second sum of the above equation includes non square-free terms (notice that, there is no repetition in any of the non square-free terms). The term  $R_2$  was added to offset the contribution by these non square-free terms. We will show later that the

contribution by these terms (or  $R_2$ ) approaches zero as  $p_r$  approaches infinity. Using Stieltjes integral, we then have

$$M_2(1, p_r; 1, p_r^a) = \frac{1}{2} \int_1^{a-1} \frac{d\pi(p_r^y)}{p_r^y} (\log(a-y) + O(g_1(p_r, a-y))) + R_2.$$

Hence

$$M(1, p_r; 1, p_r^a) = 1 - \log(a) + O(g_1(p_r, a)) + \frac{1}{2} \int_1^{a-1} \frac{\log(a-y)}{y} dy + O(g_2(p_r, a-1)),$$

where

$$\begin{aligned} O(g_2(p_r, a)) &= \int_1^{a-1} \frac{O(g_1(p_r, a-y))}{y} dy + \int_1^{a-1} \log(a-y) \frac{dO(\sqrt{p_r^y} \log(p_r^y))}{p_r^y} + \\ &\quad \int_1^{a-1} O(g_1(p_r, a-y)) \frac{dO(\sqrt{p_r^y} \log(p_r^y))}{p_r^y} + R_2. \end{aligned}$$

It can be easily shown that, for any fixed value for  $a$ , the three integrals on the right side of the above equation approach zero as  $p_r$  approaches infinity. We will also show later that  $R_2$  approaches zero as  $p_r$  approaches infinity. Thus, for  $1 \leq a < 3$ , we have

$$\lim_{p_r \rightarrow \infty} M(1, p_r; 1, p_r^a) = 1 - \log a + \int_1^{a-1} \frac{\log(a-y)}{y} dy$$

Therefore, as  $p_r$  approaches infinity,  $M(1, p_r; 1, p_r^a)$  is only dependent on  $a$ .

Repeating the previous process  $[a]$  times (where  $[x]$  is the integer value of  $x$ ) and by using the induction method, we can show that, as  $p_r$  approaches infinity, the partial sum  $M(1, p_r; 1, p_r^a)$  is dependent on only  $a$ . Specifically, we first write the partial sum  $M(1, p_r; 1, p_r^a)$  as follows

$$\begin{aligned} M(1, p_r; 1, p_r^a) &= 1 - M_1(1, p_r; 1, p_r^a) + M_2(1, p_r; 1, p_r^a) - \dots + (-1)^j M_j(1, p_r; 1, p_r^a) + \dots + \\ &\quad (-1)^{[a]-1} M_{[a]-1}(1, p_r; 1, p_r^a) + (-1)^{[a]} M_{[a]}(1, p_r; 1, p_r^a), \end{aligned}$$

where

$$M_j(1, p_r; 1, p_r^a) = \sum_{p_r \leq p_{i1} p_{i2} \dots p_{ij} < p_r^a} \frac{1}{p_{i1} p_{i2} \dots p_{ij}}.$$

and  $p_{i1}, p_{i2}, \dots, p_{ij}$  are  $j$  distinct prime numbers greater than or equal to  $p_r$ . If we assume that  $M_{j-1}(1, p_r; 1, p_r^a)$  is given by

$$M_{j-1}(1, p_r; 1, p_r^a) = h_{j-1}(a) + O(g_{j-1}(p_r, a))$$

where  $h_{j-1}(a)$  is a function of  $a$  and  $O(g_{j-1}(p_r, a))$  approaches zero as  $p_r$  approaches infinity, then

$$M_j(1, p_r; 1, p_r^a) = \sum_{p_r \leq p_{i1} p_{i2} \dots p_{ij} < p_r^a} \frac{1}{p_{i1} p_{i2} \dots p_{ij}} = \frac{1}{j} \sum_{p_r \leq p_i < p_r^{a-1}} \frac{1}{p_i} M_{j-1}(1, p_r; p_r, p_r^a/p_i) + R_j,$$

where the factor of  $1/j$  was added since each term of the form  $1/(p_{i1}p_{i2}\dots p_{ij})$  is repeated  $j$  times. It should be also noted that the second sum of the above equation includes non square-free terms. The term  $R_j$  was added to offset the contribution by these non square-free terms. We will show later that the contribution by these terms (or  $R_j$ ) approaches zero as  $p_r$  approaches infinity. Using Stieltjes integral, we then have

$$M_j(1, p_r; 1, p_r^a) = \frac{1}{j} \int_1^{a-1} \frac{d\pi(p_r^y)}{p_r^y} (h_{j-1}(a-y) + O(g_{j-1}(p_r, a-y))) + R_j.$$

Hence

$$M_j(1, p_r; 1, p_r^a) = \frac{1}{j} \int_1^{a-1} \frac{h_{j-1}(a-y)}{y} dy + O(g_j(p_r, a)),$$

where the first term is a definite integral with only one variable  $a$ . Thus, the definite integral is a function of  $a$ . We define this function as  $h_j(a)$ . The second term is given by

$$\begin{aligned} O(g_j(p_r, a)) &= \int_1^{a-1} \frac{O(g_{j-1}(p_r, a-y))}{y} dy + \int_1^{a-1} h_{j-1}(a-y) \frac{dO(\sqrt{p_r^y} \log(p_r^y))}{p_r^y} + \\ &\int_1^{a-1} O(g_{j-1}(p_r, a-y)) \frac{dO(\sqrt{p_r^y} \log(p_r^y))}{p_r^y} + R_j. \end{aligned}$$

It can be easily shown that, for a fixed value of  $a$ , the three integrals on the right side of the above equation approach zero as  $p_r$  approaches infinity. We will also show later that  $R_j$  approaches zero as  $p_r$  approaches infinity. Hence, as  $p_r$  approaches infinity, we have

$$\lim_{p_r \rightarrow \infty} M_j(1, p_r; 1, p_r^a) = \frac{1}{j} \int_1^{a-1} \frac{h_{j-1}(a-y)}{y} dy = h_j(a)$$

where  $h_1(a) = \log(a)$ . Hence, for every  $a$  and as  $p_r$  approaches infinity, we have

$$\lim_{p_r \rightarrow \infty} M(1, p_r; 1, p_r^a) = 1 - h_1(a) + h_2(a) - h_3(a) + \dots + (-1)^{\lfloor a \rfloor} h_{\lfloor a \rfloor}(a) = \rho(a). \quad (41)$$

It should be pointed out that the above equation implies that the partial sums  $M(1, p_r; 1, p_r^a)$  and  $M(1, p_r^y; 1, p_r^{ay})$  (where,  $p_r^y$  is a prime number) have the same limit as  $p_r$  approaches infinity. Hence,

$$\lim_{p_r \rightarrow \infty} M(1, p_r; 1, p_r^a) = \lim_{p_r \rightarrow \infty} M(1, p_r^y; 1, p_r^{ay}) = \rho(a). \quad (42)$$

Equation (42) will be used in the next step to estimate the asymptotic behavior of the function  $\rho(a)$  as  $a$  approaches infinity.

As it mentioned earlier, the partial sum  $M(1, p_r; 1, p_r^a)$  constructed by this process included non square-free terms (i.e  $R_i$ 's). In the following, we will show that, for every  $a$  and as  $p_r$  approaches infinity, the total contribution by these non square-free terms approaches zero as well. Toward this end, let  $S_0$  be the sum of the terms with the factor  $1/p_r^2$ . Let  $S_1$  be the sum of the remaining terms with the factor  $1/(p_{r+1})^2$ ,  $S_2$  be the sum of the remaining terms with the factor  $1/(p_{r+2})^2$ , and so on. Let  $R$  be sum of all the terms associated with non square-free terms. Thus,  $R$  is given by

$$R = \frac{1}{p_r^2} S_0 + \frac{1}{p_{r+1}^2} S_1 + \dots + \frac{1}{p_{r+l}^2} S_L,$$

where  $p_{r+l}$  is the largest prime where its square is less than  $p_r^a$ . However,

$$|S_0|, |S_1|, \dots, |S_l| < 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{p_r^a}.$$

Thus,

$$|S_0|, |S_1|, \dots, |S_l| = O(a \log p_r).$$

Therefore

$$R = \left( \frac{1}{p_r^2} + \frac{1}{p_{r+1}^2} + \dots + \frac{1}{p_{r+l}^2} \right) O(a \log p_r).$$

Hence, the contribution by the non square-free terms  $R$  is given by,

$$R = O(a \log p_r / p_r).$$

Consequently, for every  $a$  and as  $p_r$  approaches infinity,  $R$  (or the contribution by the non square-free terms) approaches zero.

- In the second step, we write the partial sum  $M(1, p_r; 1, p_r^a)$  as the sum of two components. The first is the deterministic or regular component and it is given by  $\rho(a)$ . The second one is the irregular component  $R(1, p_r; 1, p_r^a)$  given by  $M(1, p_r; 1, p_r^a) - \rho(a)$ . We will then show that the function  $\rho(a)$  is the Dickman function that has been extensively used to analyze the properties of  $y$ -smooth numbers.

Toward this end, we write the partial sum  $M(1, p_r; 1, p_r^a)$  as the following sum

$$M(1, p_r; 1, p_r^a) = 1 - \sum_{p_r \leq p_i < p_r^{a/2}} \frac{1}{p_i} M(1, p_i; 1, p_r^a / p_i) - \sum_{p_r^{a/2} \leq p_i < p_r^a} \frac{1}{p_i}. \quad (43)$$

Notice that the above equation is justified by the virtue that  $M(1, p_i; 1, p_r^a / p_i)$  is comprised of 1 and the terms of the form  $1/n$  where  $p_i < n \leq p_r^a / p_i$ . Furthermore, every factor of  $n$  is greater than  $p_i$ . The second sum was added since the first sum is void of the terms  $1/p_i$ 's for  $p_i^{a/2} \leq p_i \leq p_r^a$ . Using Stieltjes integral, we can write the above equation as follows

$$M(1, p_r; 1, p_r^a) = 1 - \int_1^{a/2} \frac{d\pi(p_r^y)}{p_r^y} M(1, p_r^y; 1, p_r^a / p_r^y) - \int_{a/2}^a \frac{d\pi(p_r^y)}{p_r^y}, \quad (44)$$

where, on RH,  $d\pi(p_r^y)$  is given by  $d\text{Li}(p_r^y) + dO(\sqrt{p_r^y} \log(p_r^y))$ . It should be pointed out that while Equations (43) and (44) provide the value of the partial sum  $M(s, p_r; 1, p_r^a)$  at  $s = 1$ , they can be easily modified to compute the partial sum for any values of  $s$  to the right of the line  $\sigma = 0.5$ . This task will be achieved in the next section and it will be the key step to examine the validity of the Riemann Hypothesis

As  $p_r$  approaches infinity,  $M(1, p_r^y; 1, p_r^{a-y})$  approaches  $\rho(a/y - 1)$  (refer to Equation (42)). Therefore, as  $p_r$  approaches infinity, we have

$$\rho(a) = 1 - \int_1^{a/2} \frac{\rho\left(\frac{a}{y} - 1\right)}{y} dy - \int_{a/2}^a \frac{dy}{y}. \quad (45)$$

It is shown in Appendix 4 that  $|M(1, p_r; 1, p_r^a)| \leq 2$  for every  $p_r$  and  $a$ . Hence,  $|\rho(a)| \leq 2$ . Consequently,  $\rho(a)$  approaches zero as  $a$  approaches infinity (this follows from the fact that if  $\rho(a)$  does not converge to zero, then the first integral of the above equation diverges as  $a$  approaches infinity which then leads to the divergence of  $\rho(a)$ ). This contradicts our earlier statement that  $|\rho(a)| \leq 2$ . Thus, as  $a$  approaches infinity, we have

$$\int_1^{a/2} \frac{\rho\left(\frac{a}{y} - 1\right)}{y} dy = 1 - \log 2. \quad (46)$$

A key step in our method to examine the validity of the the Riemann Hypothesis is the computation of the rate at which  $\rho(a)$  decays to zero. This task will be achieved by using Equation (45) to compute the difference  $\rho(a + \Delta a) - \rho(a)$  (where,  $\Delta a$  is an arbitrary small number) to obtain

$$\rho(a + \Delta a) - \rho(a) = - \int_1^{(a+\Delta a)/2} \frac{\rho\left(\frac{a+\Delta a}{y} - 1\right)}{y} dy + \int_1^{a/2} \frac{\rho\left(\frac{a}{y} - 1\right)}{y} dy - \int_{(a+\Delta a)/2}^{(a+\Delta a)} \frac{dy}{y} + \int_{a/2}^a \frac{dy}{y}.$$

Hence,

$$\rho(a + \Delta a) - \rho(a) = - \int_1^{(a+\Delta a)/2} \frac{\rho\left(\frac{a+\Delta a}{y} - 1\right)}{y} dy + \int_1^{a/2} \frac{\rho\left(\frac{a}{y} - 1\right)}{y} dy.$$

If we define  $y = (1 + \Delta a/a)z$ , then we have

$$\rho(a + \Delta a) - \rho(a) = - \int_{1/(1+\Delta a/a)}^{(a/2)(1+\Delta a/a)/(1+a/\Delta a)} \frac{\rho\left(\frac{a}{z} - 1\right)}{z} dz + \int_1^{a/2} \frac{\rho\left(\frac{a}{y} - 1\right)}{y} dy.$$

Thus,

$$\rho(a + \Delta a) - \rho(a) = - \int_{1/(1+\Delta a/a)}^1 \frac{\rho\left(\frac{a}{z} - 1\right)}{z} dz.$$

Dividing both sides of the above equation by  $\Delta a$  and letting  $\Delta a$  approach zero, we then obtain

$$\frac{d\rho(a)}{da} = - \frac{\rho(a - 1)}{a}, \quad (47)$$

where  $\rho(a) = 1 - \log(a)$  for  $1 \leq a \leq 2$ . Equation (47) is a first order non-linear delay differential equation that has been extensively analyzed in the literature [10]. The function  $\rho(a)$  is known as the Dickman function. As  $a$  approaches infinity,  $\rho(a)$  can be given by the following estimate [10]

$$\rho(a) = \left( \frac{e + o(1)}{a \log a} \right)^a. \quad (48)$$

For sufficiently large values of  $a$  ( $a > 20$ ), we have  $\rho(a) < a^{-a}$ .

For the remaining of this section, we will use the function  $\rho(a)$  to estimate the value of  $M(s, p_r; 1, p_r^a)$  for  $\Re(s) \geq 1$ . As with the case for  $s = 1$ , we can consider that  $M(s, p_r; 1, p_r^a)$

is comprised of two components. The first component is the regular component defined as  $F(a, \alpha)$  (where  $\alpha = (1 - s) \log p_r$ ) and is given by

$$F(a, \alpha) = \int_1^a p_r^{(s-1)x} d\rho(x) = \int_0^a p_r^{(s-1)x} \rho'(x) dx,$$

or,

$$F(a, \alpha) = \int_0^a e^{-\alpha x} \rho'(x) dx, \quad (49)$$

while the irregular component is given by  $M(s, p_r; 1, p_r^a) - F(a, \alpha)$ . Notice that for  $s = 1$ , we have  $\alpha = 0$  and  $F(a, 0) = \rho(a)$ . We now define  $F(\alpha)$  as

$$F(\alpha) = \lim_{a \rightarrow \infty} F(a, \alpha) = \int_0^\infty e^{-\alpha x} \rho'(x) dx. \quad (50)$$

Thus, for  $\Re(s) \geq 1$ ,  $F(\alpha)$  or the regular component of the series  $M(s, p_r)$  (or,  $M(s, p_r; 1, \infty)$ ) is given by the Laplace transform of the function  $\rho'(x)$ . For  $\Re(s) = 1$ ,  $\alpha$  is given by  $i\omega$  and  $F(i\omega)$  is given by the Fourier transform of the function  $\rho'(x)$ .

Remarkably, these results agree with Theorem 3. In Theorem 3, we have shown that

$$\lim_{r \rightarrow \infty} \{M(s, p_r) \exp(E_1((s-1) \log p_{r+1}))\} = 1.$$

Through simple manipulation, we can also show that, on RH, we have

$$M(s, p_r) = \exp(-E_1(-\alpha)) (1 + \epsilon(p_r, s)), \quad (51)$$

where  $\epsilon(p_r, s) = O\left(\frac{1+t}{(\sigma-0.5)^2} p_r^{1/2-\sigma} \log p_r\right)$ . Since the Laplace transform of the function  $\rho'(x)$  is given by  $\exp(-E_1(-\alpha))$  (where  $\alpha$  is a complex variable given by  $\alpha = (1 - s) \log p_r$ ), hence for  $\Re(s) \geq 1$ , we have

$$M(s, p_r) = F(\alpha) (1 + \epsilon(p_r, s)). \quad (52)$$

It should pointed out that on RH, the previous analysis and Equations (50), (51) and (52) should also hold for  $\Re(s) > 0.5$ .

## 6 The series $M(\sigma, p_r)$ for $\sigma < 1$ and the Riemann Hypothesis.

In this section, we will rewrite Equation (43) to compute the partial sum  $M(s, p_r; 1, p_r^a)$  for  $0.5 < \Re(s) < 1$  as follows

$$M(s, p_r; 1, p_r^a) = 1 - \sum_{p_r \leq p_i < p_r^{a/2}} \frac{1}{p_i^s} M(s, p_i; 1, p_r^a/p_i) - \sum_{p_r^{a/2} \leq p_i < p_r^a} \frac{1}{p_i^s}. \quad (53)$$

Using Stieltjes integral, we can write the above equation as

$$M(s, p_r; 1, p_r^a) = 1 - \int_1^{a/2} \frac{d\pi(p_r^y)}{p_r^{sy}} M(s, p_r^y; 1, p_r^a/p_r^y) - \int_{a/2}^a \frac{d\pi(p_r^y)}{p_r^{sy}}. \quad (54)$$

On the real axis (i.e.  $s = \sigma$ ), we then have

$$M(\sigma, p_r; 1, p_r^a) = 1 - \int_1^{a/2} \frac{d\pi(p_r^y)}{p_r^{\sigma y}} M(\sigma, p_r^y; 1, p_r^{a-y}) - \int_{a/2}^a \frac{d\pi(p_r^y)}{p_r^{\sigma y}}. \quad (55)$$

Using Theorem 3, on RH, the partial sum  $M(\alpha, p_r; 1, p_r^a)$  is convergent as  $a$  approaches infinity and its value is given by,

$$M(\sigma, p_r; 1, p_r^a) = M(\sigma, p_r) = \exp(-E_1(-\alpha)) (1 + \epsilon(p_r, s)), \quad (56)$$

where  $\alpha = (1 - \sigma) \log p_r$ .

Furthermore, referring to Appendix 2, the second integral on the right side of Equation (55) is given by

$$\int_{a/2}^a \frac{d\pi(p_r^y)}{p_r^{\sigma y}} = E_1(-a\alpha) - E_1(-a\alpha/2) + \varepsilon,$$

or

$$\int_{a/2}^a \frac{d\pi(p_r^y)}{p_r^{\sigma y}} < E_1(-a\alpha). \quad (57)$$

The first integral on the right side of Equation (55) can be decomposed into the following two components

$$\begin{aligned} \int_1^{a/2} \frac{d\pi(p_r^y)}{p_r^{\sigma y}} M(\sigma, p_r^y; 1, p_r^{a-y}) &= \int_1^{a/2} \frac{d\text{Li}(p_r^y)}{p_r^{\sigma y}} M(\sigma, p_r^y; 1, p_r^{a-y}) + \\ &\int_1^{a/2} \frac{dO(\sqrt{p_r^y} \log(p_r^y))}{p_r^{\sigma y}} M(\sigma, p_r^y; 1, p_r^{a-y}). \end{aligned}$$

The second integral on the right side of the above equation is negligible compared with the first one, therefore we will be interested in computing only the first integral where

$$\int_1^{a/2} \frac{d\text{Li}(p_r^y)}{p_r^{\sigma y}} M(\sigma, p_r^y; 1, p_r^{a-y}) = \int_1^{a/2} \frac{e^{\alpha y}}{y} M(\sigma, p_r^y; 1, p_r^{a-y}) dy.$$

However, for  $y < a/p_r$ , by the virtue of Equation (49) and the fact that  $\rho(a) < a^{-a}$  (for sufficiently large  $a$ ), one can substitute the value of  $\exp(-E_1(-\alpha y))$  (or  $M(\sigma, p_r^y)$ ) for the partial sum  $M(\sigma, p_r^y; 1, p_r^{a-y})$ . Therefore,

$$\int_1^{a/2} \frac{d\text{Li}(p_r^y)}{p_r^{\sigma y}} M(\sigma, p_r^y; 1, p_r^{a-y}) > \int_1^{\frac{a}{2p_r}} \frac{e^{\alpha y}}{y} \exp(-E_1(-\alpha y)) dy.$$

Hence,

$$\int_1^{a/2} \frac{d\pi(p_r^y)}{p_r^{\sigma y}} M(\sigma, p_r^y; 1, p_r^{a-y}) > \exp(-E_1(-a\alpha/p_r)) \quad (58)$$

Assuming the validity of the Riemann Hypothesis, then the left side of Equation (55) should converge as  $a$  approaches infinity. However, by the virtue of Equations (57) and (58), the right side of Equation (55) diverges to infinity as  $a$  approaches infinity. Therefore, the series  $M(\sigma, p_r)$  and  $M(\sigma)$  diverge for  $\sigma < 1$ . This implies that the Riemann Hypothesis is invalid and the zeros can be found arbitrary close to line  $\Re(s) = 1$ .

## Appendix 1

To prove the first part of Theorem 1 (i.e. for  $s = \sigma + i0$  and  $0.5 < \sigma \leq 1$ , the series  $M(\sigma, p_r)$  converges conditionally if  $M(\sigma)$  converges conditionally), we first start with proving that  $M(\sigma, 2)$  is conditionally convergent if  $M(\sigma)$  is convergent. Since  $M(\sigma)$  is convergent, then for any arbitrary small number  $\delta$ , there exists an integer  $N_0$  such that for every integer  $N > N_0$

$$|M(\sigma; N, \infty)| = \left| \sum_{n=N}^{\infty} \frac{\mu(n)}{n^\sigma} \right| < \delta \quad (59)$$

Let the sums  $M(\sigma; 1, N)$ ,  $M(\sigma; N+1, 2N)$ ,  $M(\sigma; 2N+1, 2^2N)$ ,  $M(\sigma; 2^2N+1, 2^3N)$ , ...,  $M(\sigma; 2^{L-1}N+1, 2^LN)$  be defined as

$$M(\sigma; 1, N) = \sum_{n=1}^N \frac{\mu(n)}{n^\sigma} = A_1,$$

$$M(\sigma; N+1, 2N) = \sum_{n=N+1}^{2N} \frac{\mu(n)}{n^\sigma} = \delta_1,$$

$$M(\sigma; 2N+1, 2^2N) = \sum_{n=2N+1}^{2^2N} \frac{\mu(n)}{n^\sigma} = \delta_2,$$

$$M(\sigma; 2^2N+1, 2^3N) = \sum_{n=2^2N+1}^{2^3N} \frac{\mu(n)}{n^\sigma} = \delta_3,$$

$$M(\sigma; 2^{L-1}N+1, 2^LN) = \sum_{n=2^{L-1}N+1}^{2^LN} \frac{\mu(n)}{n^\sigma} = \delta_L.$$

Throughout the analysis in this appendix,  $N$  will be a fixed number (that is larger than  $N_0$ ) while the test for the convergence will be achieved by letting  $L$  approach infinity.

Let  $\delta(l)$  be defined as the maximum of  $|\delta_l|, |\delta_{l+1}|, |\delta_{l+2}|, \dots, |\delta_L|, |\delta_l + \delta_{l+1}|, |\delta_l + \delta_{l+1} + \delta_{l+2}|, \dots, |\delta_l + \delta_{l+1} + \dots + \delta_L|$ , then by the virtue of the convergence of  $M(\sigma)$ ,

$$|\delta_1|, |\delta_2|, |\delta_3|, \dots, |\delta_L|, |\delta_1 + \delta_2|, |\delta_1 + \delta_2 + \delta_3|, \dots, |\delta_1 + \delta_2 + \delta_3 + \dots + \delta_L| \leq \delta(1) \leq 2\delta.$$

We also have

$$|\delta_l|, |\delta_{l+1}|, |\delta_{l+2}|, \dots, |\delta_L|, |\delta_l + \delta_{l+1}|, |\delta_l + \delta_{l+1} + \delta_{l+2}|, \dots, |\delta_l + \delta_{l+1} + \dots + \delta_L| \leq \delta(l),$$

where by the virtue of the convergence of  $M(\sigma)$ ,  $\delta(l)$  can be set arbitrary close to zero (since  $\delta$ , defined in Equation 59, can be set arbitrary close to zero by setting  $N_0$  arbitrary large).

Furthermore, let the sums  $M(\sigma, 2; 1, N)$ ,  $M(\sigma, 2; N+1, 2N)$ ,  $M(\sigma, 2; 2N+1, 2^2N)$ ,  $M(\sigma, 2; 2^2N+1, 2^3N)$ , ...,  $M(\sigma, 2; 2^{L-1}N+1, 2^LN)$  be defined as

$$M(\sigma, 2; 1, N) = \sum_{n=1}^N \frac{\mu(n, 2)}{n^\sigma} = B_1,$$

$$M(\sigma, 2; N+1, 2N) = \sum_{n=N+1}^{2N} \frac{\mu(n, 2)}{n^\sigma} = \epsilon_1,$$

$$M(\sigma, 2; 2N+1, 2^2N) = \sum_{n=2N+1}^{2^2N} \frac{\mu(n, 2)}{n^\sigma} = \epsilon_2,$$

$$M(\sigma, 2; 2^2N+1, 2^3N) = \sum_{n=2^2N+1}^{2^3N} \frac{\mu(n, 2)}{n^\sigma} = \epsilon_3,$$

$$M(\sigma, 2; 2^{L-1}N+1, 2^LN) = \sum_{n=2^{L-1}N+1}^{2^LN} \frac{\mu(n, 2)}{n^\sigma} = \epsilon_L,$$

Since

$$\sum_{n=1}^{2N} \frac{\mu(n)}{n^\sigma} = \sum_{n=1}^{2N} \frac{\mu(n, 2)}{n^\sigma} - \sum_{n=1}^N \frac{\mu(n, 2)}{(2n)^\sigma},$$

thus

$$M(\sigma; 1, 2N) = M(\sigma, 2; 1, 2N) - \frac{1}{2^\sigma} M(\sigma, 2; 1, N).$$

Similarly, since

$$\sum_{n=2^lN+1}^{2^{l+1}N} \frac{\mu(n)}{n^\sigma} = \sum_{n=2^lN+1}^{2^{l+1}N} \frac{\mu(n, 2)}{n^\sigma} - \sum_{n=2^{l-1}N+1}^{2^lN} \frac{\mu(n, 2)}{(2n)^\sigma},$$

thus

$$M(\sigma; 2^lN+1, 2^{l+1}N) = M(\sigma, 2; 2^lN+1, 2^{l+1}N) - \frac{1}{2^\sigma} M(\sigma, 2; 2^{l-1}N+1, 2^lN).$$

Rearranging the previous equations, we then have

$$A_1 + \delta_1 = B_1 + \epsilon_1 - \frac{1}{2^\sigma} B_1, \tag{60}$$

$$\delta_2 = \epsilon_2 - \frac{1}{2^\sigma} \epsilon_1,$$

$$\delta_3 = \epsilon_3 - \frac{1}{2^\sigma} \epsilon_2,$$

$$\delta_L = \epsilon_L - \frac{1}{2^\sigma} \epsilon_{L-1},$$

where  $|\delta_1|, |\delta_2|, |\delta_3|, \dots, |\delta_L|, |\delta_1 + \delta_2|, |\delta_1 + \delta_2 + \delta_3|, |\delta_1 + \delta_2 + \delta_3 + \dots + \delta_L| \leq \delta(1) \leq 2\delta$  and  $\delta$  can be set arbitrary close to zero. Hence

$$\epsilon_2 = \frac{1}{2^\sigma} \epsilon_1 + \delta_2,$$

$$\epsilon_3 = \frac{1}{2^\sigma} \epsilon_2 + \delta_3 = \frac{1}{2^{2\sigma}} \epsilon_1 + \frac{1}{2^\sigma} \delta_2 + \delta_3,$$

$$\epsilon_4 = \frac{1}{2^\sigma} \epsilon_3 + \delta_4 = \frac{1}{2^{3\sigma}} \epsilon_1 + \frac{1}{2^{2\sigma}} \delta_2 + \frac{1}{2^\sigma} \delta_3 + \delta_4,$$

$$\epsilon_L = \frac{1}{2^\sigma} \epsilon_{L-1} + \delta_L = \frac{1}{2^{(L-1)\sigma}} \epsilon_1 + \frac{1}{2^{(L-2)\sigma}} \delta_2 + \frac{1}{2^{(L-3)\sigma}} \delta_3 + \dots + \delta_L.$$

Therefore,

$$\begin{aligned} \epsilon_1 + \epsilon_2 + \epsilon_3 + \dots + \epsilon_L &= \left(1 + \frac{1}{2^\sigma} + \frac{1}{2^{2\sigma}} + \dots + \frac{1}{2^{(L-1)\sigma}}\right) \epsilon_1 + (\delta_2 + \delta_3 + \dots + \delta_L) + \\ &\quad \frac{1}{2^\sigma} (\delta_2 + \delta_3 + \dots + \delta_{L-1}) + \frac{1}{2^{2\sigma}} (\delta_2 + \delta_3 + \dots + \delta_{L-2}) + \dots + \frac{1}{2^{(L-2)\sigma}} \delta_2. \end{aligned}$$

Since  $|\delta_2| \leq \delta(1), |\delta_2 + \delta_3| \leq \delta(1), \dots, |\delta_1 + \delta_2 + \delta_3 + \dots + \delta_L| \leq \delta(1)$ , hence

$$|\delta_2 + \delta_3 + \dots + \delta_L| + \frac{1}{2^\sigma} |\delta_2 + \delta_3 + \dots + \delta_{L-1}| + \dots + \frac{1}{2^{(L-2)\sigma}} |\delta_2| \leq \left| \delta(1) + \frac{1}{2^\sigma} \delta(1) + \dots + \frac{1}{2^{(L-2)\sigma}} \delta(1) \right|,$$

or

$$|\delta_2 + \delta_3 + \dots + \delta_L| + \frac{1}{2^\sigma} |\delta_2 + \delta_3 + \dots + \delta_{L-1}| + \dots + \frac{1}{2^{(L-2)\sigma}} |\delta_2| \leq \frac{2^\sigma}{2^\sigma - 1} |\delta(1)|.$$

Therefore

$$\epsilon_1 + \epsilon_2 + \epsilon_3 + \dots + \epsilon_L = \left(1 + \frac{1}{2^\sigma} + \frac{1}{2^{2\sigma}} + \dots + \frac{1}{2^{L\sigma}}\right) \epsilon_1 + \gamma_1,$$

where  $\gamma_1$  is of the same order as that of  $\delta(1)$  (where  $\delta(1)$  can be set arbitrary close to zero by setting  $\delta$ , defined in Equation 59, arbitrary close to zero).

As  $L$  approaches infinity, we then obtain

$$\sum_{i=1}^{\infty} \epsilon_i = \frac{2^\sigma}{2^\sigma - 1} \epsilon_1 + \gamma_1.$$

Therefore, if the series  $M(\sigma)$  is convergent, then the sum  $M(\sigma, 2; N+1, \infty)$  (which is equal to  $\epsilon_1 + \epsilon_2 + \epsilon_3 + \dots$ ) is bounded by the sum  $M(\sigma, 2; N+1, 2N)$  (which is equal to  $\epsilon_1$ ).

The previous process can be repeated with the substitution of  $A_1$  and  $B_1$  in Equation (60) with  $A_2$  and  $B_2$ , where  $A_2 = M(\sigma; 1, 2N)$  and  $B_2 = M(\sigma, 2; 1, 2N)$ , to obtain

$$A_2 + \delta_2 = B_2 + \epsilon_2 - \frac{1}{2^\sigma} B_2.$$

Thus,

$$A_2 = B_2 - \frac{1}{2^\sigma} B_2 + \frac{1}{2^\sigma} \epsilon_1.$$

Following the same process, we can show that the sum  $M(\sigma, 2; 2N + 1, \infty)$  is given by

$$\sum_{i=2}^{\infty} \epsilon_i = \frac{1}{2^\sigma - 1} \epsilon_1 + \gamma_2.$$

where  $\gamma_2$  is of the same order as that of  $\delta(2)$  (where  $\delta(2)$  can be set arbitrary close to zero by setting  $\delta$ , defined in Equation 59, arbitrary close to zero).

If we repeat the process  $l$  times, we obtain

$$A_l = B_l - \frac{1}{2^\sigma} B_l + \frac{1}{2^{(l-1)\sigma}} \epsilon_1,$$

where  $A_l = M(\sigma; 1, 2^l N)$  and  $B_l = M(\sigma, 2; 1, 2^l N)$  and the sum  $M(\sigma, 2; 2^l N + 1, \infty)$  is given by

$$\sum_{i=l}^{\infty} \epsilon_i = \frac{1}{2^{(l-2)\sigma}} \frac{1}{2^\sigma - 1} \epsilon_1 + \gamma_l.$$

where  $\gamma_l$  is of the same order as that of  $\delta(l)$ . Since by the virtue of the convergence of  $M(\sigma)$ ,  $\delta(l)$  tends to zero as  $l$  approaches infinity, therefore  $\gamma_l$  and the above sum approach zero as  $l$  approaches infinity.

Thus, we conclude that  $M(\sigma, 2; 2^l N + 1, \infty)$  (given by  $\sum_{i=l}^{\infty} \epsilon_i$ ) approaches zero as  $l$  approaches infinity. Furthermore, as  $l$  approaches infinity,  $B = \lim_{l \rightarrow \infty} B_l$  approaches its limit given by

$$\left(1 - \frac{1}{2^\sigma}\right) B = M(\sigma; 1, \infty).$$

Hence,

$$\left(1 - \frac{1}{2^\sigma}\right) M(\sigma, 2) = M(\sigma).$$

Similarly, following the same steps, we can show that

$$\left(1 - \frac{1}{3^\sigma}\right) M(\sigma, 3; 1, \infty) = M(\sigma, 2; 1, \infty).$$

or

$$\left(1 - \frac{1}{2^\sigma}\right) \left(1 - \frac{1}{3^\sigma}\right) M(\sigma, 3; 1, \infty) = M(\sigma; 1, \infty).$$

This task can be achieved by first defining

$$M(\sigma, 2; 1, N) = \sum_{n=1}^N \frac{\mu(n, 2)}{n^\sigma} = A_1,$$

$$M(\sigma, 2; N+1, 3N) = \sum_{n=N+1}^{3N} \frac{\mu(n, 2)}{n^\sigma} = \delta_1,$$

$$M(\sigma, 2; 3N+1, 3^2N) = \sum_{n=3N+1}^{3^2N} \frac{\mu(n, 2)}{n^\sigma} = \delta_2,$$

$$M(\sigma, 2; 3^{L-1}N+1, 3^LN) = \sum_{n=3^{L-1}N+1}^{3^LN} \frac{\mu(n, 2)}{n^\sigma} = \delta_L,$$

and

$$M(\sigma, 3; 1, N) = \sum_{n=1}^N \frac{\mu(n, 3)}{n^\sigma} = B_1,$$

$$M(\sigma, 3; N+1, 3N) = \sum_{n=N+1}^{3N} \frac{\mu(n, 3)}{n^\sigma} = \epsilon_1,$$

$$M(\sigma, 3; 3N+1, 3^2N) = \sum_{n=3N+1}^{3^2N} \frac{\mu(n, 3)}{n^\sigma} = \epsilon_2,$$

$$M(\sigma, 3; 3^{L-1}N+1, 3^LN) = \sum_{n=3^{L-1}N+1}^{3^LN} \frac{\mu(n, 3)}{n^\sigma} = \epsilon_L,$$

Since

$$\sum_{n=1}^{3N} \frac{\mu(n, 2)}{n^\sigma} = \sum_{n=1}^{3N} \frac{\mu(n, 3)}{n^\sigma} - \sum_{n=1}^N \frac{\mu(n, 3)}{(3n)^\sigma},$$

thus

$$M(\sigma, 2; 1, 3N) = M(\sigma, 3; 1, 3N) - \frac{1}{3^\sigma} M(\sigma, 3; 1, N)$$

Similarly,

$$M(\sigma, 2; 3^lN+1, 3^{l+1}N) = M(\sigma, 3; 3^lN+1, 3^{l+1}N) - \frac{1}{3^\sigma} M(\sigma, 3; 3^{l-1}N+1, 3^lN)$$

Following the same process, we can show that

$$\sum_{i=1}^{\infty} \epsilon_i = \frac{3^\sigma}{3^\sigma - 1} \epsilon_1 + \gamma_1,$$

where  $\gamma_1$  is of the same order as that of  $\delta(1)$  ( $\delta(l)$  is defined as the maximum of  $|\delta_l|, |\delta_{l+1}|, |\delta_{l+2}|, \dots, |\delta_L|, |\delta_l + \delta_{l+1}|, |\delta_l + \delta_{l+1} + \delta_{l+2}|, \dots, |\delta_l + \delta_{l+1} + \dots + \delta_L|$ ).

Similarly, if we define  $A_2 = M(\sigma, 2; 1, 3N)$  and  $B_2 = M(\sigma, 3; 1, 3N)$ , then

$$A_2 = B_2 - \frac{1}{3^\sigma} B_2 + \frac{1}{3^\sigma} \epsilon_1.$$

Therefore

$$\sum_{i=2}^{\infty} \epsilon_i = \frac{1}{3^\sigma - 1} \epsilon_1 + \gamma_2.$$

where  $\gamma_2$  is of the same order as that of  $\delta(2)$ .

Repeating the steps  $l$  times, we then obtain

$$\sum_{i=l}^{\infty} \epsilon_i = \frac{1}{3^{(l-2)\sigma}} \frac{1}{3^\sigma - 1} \epsilon_1 + \gamma_l.$$

where  $\gamma_l$  is of the same order as that of  $\delta(l)$ . Hence the above sum approaches zero as  $l$  approaches infinity

Thus, we conclude that  $M(\sigma, 3; 3^l N + 1, \infty)$  (given by  $\sum_{i=l}^{\infty} \epsilon_i$ ) approaches zero as  $l$  approaches infinity. Furthermore, as  $l$  approaches infinity,  $B = \lim_{l \rightarrow \infty} B_l$  approaches its limit given by

$$\left(1 - \frac{1}{3^\sigma}\right) B = M(\sigma, 2; 1, \infty).$$

Hence,

$$\left(1 - \frac{1}{3^\sigma}\right) M(\sigma, 3) = M(\sigma, 2).$$

Repeating the process  $r$  times, we then conclude

$$M(\sigma) = M(\sigma, p_r) \prod_{i=1}^r \left(1 - \frac{1}{p_i^\sigma}\right).$$

The second part of the theorem can be proved by recalling

$$M(s, p_{r-1}; 1, N p_r) = M(s, p_r; 1, N p_r) - \frac{1}{p_r^s} M(s, p_r; 1, N).$$

If both series  $M(s, p_{r-1})$  and  $M(s, p_r)$  are convergent, then as  $N$  approaches infinity, we obtain

$$M(s, p_{r-1}) = M(s, p_r) \left(1 - \frac{1}{p_r^s}\right).$$

Repeating the process  $r$  times, we then conclude

$$M(\sigma) = M(\sigma, p_r) \prod_{i=1}^r \left(1 - \frac{1}{p_i^\sigma}\right).$$

## Appendix 2

Assuming RH is valid and for  $\sigma > 0.5$ , to show that

$$\sum_{i=r_1}^{r_2} \frac{1}{p_i^\sigma} = E_1((\sigma - 1) \log p_{r_1}) - E_1((\sigma - 1) \log p_{r_2}) + \varepsilon$$

where,  $\varepsilon = O\left(\frac{t}{(\sigma-0.5)^2} p_{r_1}^{1/2-\sigma} \log p_{r_1}\right)$ , we first recall that

$$\sum_{i=r_1}^{r_2} \frac{1}{p_i^\sigma} = \int_{p_{r_1}}^{p_{r_2}} \frac{d\pi(x)}{x^\sigma} = \int_{p_{r_1}}^{p_{r_2}} \frac{1}{x^\sigma \log x} dx + \int_{p_{r_1}}^{p_{r_2}} \frac{1}{x^\sigma} dO(\sqrt{x} \log x).$$

We will first compute the integral with the  $O$  notation. This can be done by integration by parts to obtain

$$\int_{p_{r_1}}^{p_{r_2}} \frac{1}{x^\sigma} dO(\sqrt{x} \log x) = \frac{O(\sqrt{p_{r_2}} \log p_{r_2})}{p_{r_2}^\sigma} - \frac{O(\sqrt{p_{r_1}} \log p_{r_1})}{p_{r_1}^\sigma} - \int_{p_{r_1}}^{p_{r_2}} O(\sqrt{x} \log x) d\left(\frac{1}{x^\sigma}\right)$$

Since  $x > 0$ , thus

$$\int_{p_{r_1}}^{p_{r_2}} \frac{1}{x^\sigma} dO(\sqrt{x} \log x) = \frac{O(\sqrt{p_{r_2}} \log p_{r_2})}{p_{r_2}^\sigma} - \frac{O(\sqrt{p_{r_1}} \log p_{r_1})}{p_{r_1}^\sigma} - O\left(\int_{p_{r_1}}^{p_{r_2}} \sqrt{x} \log x d\left(\frac{1}{x^\sigma}\right)\right)$$

With the substitution of variables  $y = \log x$ , we then obtain

$$\int_{p_{r_1}}^{p_{r_2}} \sqrt{x} \log x d\left(\frac{1}{x^\sigma}\right) = - \int_{p_{r_1}}^{p_{r_2}} \sigma y e^{(\frac{1}{2}-\sigma)y} dy.$$

Since

$$\int x e^{ax} dx = \left(\frac{x}{a} - \frac{1}{a^2}\right) e^{ax},$$

therefore

$$\int_{p_{r_1}}^{p_{r_2}} \sqrt{x} \log x d\left(\frac{1}{x^\sigma}\right) = -\sigma \left(\frac{\log p_{r_2}}{0.5 - \sigma} - \frac{1}{(0.5 - \sigma)^2}\right) p_{r_2}^{0.5-\sigma} + \sigma \left(\frac{\log p_{r_1}}{0.5 - \sigma} - \frac{1}{(0.5 - \sigma)^2}\right) p_{r_1}^{0.5-\sigma}.$$

Hence, for  $\sigma > 0.5$ , we have

$$\int_{p_{r_1}}^{p_{r_2}} \frac{1}{x^\sigma} dO(\sqrt{x} \log x) = O\left(\frac{p_{r_1}^{0.5-\sigma} \log p_{r_1}}{(\sigma - 0.5)^2}\right) \quad (61)$$

For  $\sigma \geq 1$ , the integral  $\int_{p_{r_1}}^{p_{r_2}} \frac{1}{x^\sigma \log x} dx$  can be computed directly from the definition of the Exponential Integral  $E_1(r) = \int_r^\infty \frac{e^{-u}}{u} du$  (where  $r \geq 0$ ) to obtain

$$\int_{p_{r_1}}^{p_{r_2}} \frac{1}{x^\sigma \log x} dx = E_1((\sigma - 1) \log p_{r_1}) - E_1((\sigma - 1) \log p_{r_2})$$

It should be pointed out that although the functions  $E_1((\sigma - 1) \log p_{r_1})$  and  $E_1((\sigma - 1) \log p_{r_2})$

have a singularity at  $\sigma = 1$ , the difference has a removable singularity at  $\sigma = 1$ . This follows from the fact that as  $\sigma$  approaches 1, the difference can be written as

$$E_1((\sigma - 1) \log p_{r_1}) - E_1((\sigma - 1) \log p_{r_2}) = -\log((1 - \sigma) \log p_{r_1}) - \gamma + \log((1 - \sigma) \log p_{r_2}) + \gamma$$

or,

$$\lim_{\sigma \rightarrow 1} \int_{p_{r_1}}^{p_{r_2}} \frac{1}{x^\sigma \log x} dx = \lim_{\sigma \rightarrow 1} E_1((\sigma - 1) \log p_{r_1}) - E_1((\sigma - 1) \log p_{r_2}) = -\log \log p_{r_1} + \log \log p_{r_2}$$

To compute the integral  $\int_{p_{r_1}}^{p_{r_2}} \frac{1}{x^\sigma \log x} dx$  for  $\sigma < 0$ , we first use the substitution  $y = \log x$  to obtain

$$\int_{p_{r_1}}^{p_{r_2}} \frac{1}{x^\sigma \log x} dx = \int_{\log p_{r_1}}^{\log p_{r_2}} \frac{e^{(1-\sigma)y}}{y} dy = \int_{\epsilon}^{\log p_{r_2}} \frac{e^{(1-\sigma)y}}{y} dy - \int_{\epsilon}^{\log p_{r_1}} \frac{e^{(1-\sigma)y}}{y} dy$$

where,  $\epsilon$  is an arbitrary small positive number. With the variable substitutions  $z_1 = y/\log p_{r_1}$  and  $z_2 = y/\log p_{r_2}$ , we then obtain

$$\int_{p_{r_1}}^{p_{r_2}} \frac{1}{x^\sigma \log x} dx = \int_{\epsilon/\log p_{r_2}}^1 \frac{e^{(1-\sigma)(\log p_{r_2})z_2}}{z_2} dz_2 - \int_{\epsilon/\log p_{r_1}}^1 \frac{e^{(1-\sigma)(\log p_{r_1})z_1}}{z_1} dz_1.$$

With the variable substitutions  $w_1 = (1 - \sigma)(\log p_{r_1})z_1$  and  $w_2 = (1 - \sigma)(\log p_{r_2})z_2$  and by adding and subtracting the terms  $-\int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r_2}} \frac{dw_2}{w_2} + \int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r_1}} \frac{dw_1}{w_1}$ , we then have

$$\begin{aligned} \int_{p_{r_1}}^{p_{r_2}} \frac{1}{x^\sigma \log x} dx &= \int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r_2}} \frac{e^{w_2} - 1}{w_2} dw_2 - \int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r_1}} \frac{e^{w_1} - 1}{w_1} dw_1 + \\ &\quad \int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r_2}} \frac{dw_2}{w_2} - \int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r_1}} \frac{dw_1}{w_1}. \end{aligned}$$

Using the following identity [9, page 230]

$$\int_0^a \frac{e^t - 1}{t} dt = -E_1(-a) - \log(a) - \gamma$$

where  $a > 0$ , we then obtain for  $\sigma < 1$ ,

$$\int_{p_{r_1}}^{p_{r_2}} \frac{1}{x^\sigma \log x} dx = E_1((\sigma - 1) \log p_{r_1}) - E_1((\sigma - 1) \log p_{r_2})$$

Hence, for  $\sigma > 0.5$ , we have

$$\sum_{i=r_1}^{r_2} \frac{1}{p_i^\sigma} = E_1((\sigma - 1) \log p_{r_1}) - E_1((\sigma - 1) \log p_{r_2}) + \varepsilon$$

It should be pointed out that in general, if there are no non-trivial zeros for values of  $s$  with  $\Re(s) > a$ , then by following the same steps, we may also show that for  $\sigma > a$ , we have

$$\sum_{i=r_1}^{r_2} \frac{1}{p_i^\sigma} = E_1((\sigma - 1) \log p_{r_1}) - E_1((\sigma - 1) \log p_{r_2}) + \varepsilon$$

where,  $\varepsilon = O(p_{r_1}^{a-\sigma} \log p_{r_1}/(\sigma - a)^2)$ .

### Appendix 3

Assuming RH is valid and for  $\sigma > 0.5$ , to show that

$$\sum_{i=r_1}^{r_2} \frac{1}{p_i^s} = E_1((s-1) \log p_{r_1}) - E_1((s-1) \log p_{r_2}) + \varepsilon$$

where,  $\varepsilon = O\left(\frac{|s|+1}{(\sigma-0.5)^2} p_{r_1}^{1/2-\sigma} \log p_{r_1}\right)$ , we first recall that

$$\sum_{i=r_1}^{r_2} \frac{1}{p_i^s} = \int_{p_{r_1}}^{p_{r_2}} \frac{d\pi(x)}{x^s} = \int_{p_{r_1}}^{p_{r_2}} \frac{1}{x^s \log x} dx + \int_{p_{r_1}}^{p_{r_2}} \frac{1}{x^s} dO(\sqrt{x} \log x).$$

We will first compute the integral with the  $O$  notation. This can be done by integration by parts to obtain

$$\int_{p_{r_1}}^{p_{r_2}} \frac{1}{x^s} dO(\sqrt{x} \log x) = \frac{O(\sqrt{p_{r_2}} \log p_{r_2})}{p_{r_2}^s} - \frac{O(\sqrt{p_{r_1}} \log p_{r_1})}{p_{r_1}^s} - \int_{p_{r_1}}^{p_{r_2}} O(\sqrt{x} \log x) d\left(\frac{1}{x^s}\right)$$

The integral on the right side of the above equation can be then written as

$$\int_{p_{r_1}}^{p_{r_2}} O(\sqrt{x} \log x) d\left(\frac{1}{x^s}\right) = -s \int_{p_{r_1}}^{p_{r_2}} O(\sqrt{x} \log x) x^{-s-1} dx.$$

Hence,

$$\left| \int_{p_{r_1}}^{p_{r_2}} O(\sqrt{x} \log x) d\left(\frac{1}{x^s}\right) \right| \leq |s| \int_{p_{r_1}}^{p_{r_2}} O(\sqrt{x} \log x) |x^{-s-1}| dx.$$

Consequently,

$$\int_{p_{r_1}}^{p_{r_2}} \frac{1}{x^s} dO(\sqrt{x} \log x) = O\left((|s|+1) \frac{p_{r_1}^{0.5-\sigma} \log p_{r_1}}{(\sigma-0.5)^2}\right).$$

For  $\Re(s) \geq 1$ , the integral  $\int_{p_{r_1}}^{p_{r_2}} \frac{1}{x^s \log x} dx$  can be computed directly from the definition of the Exponential Integral  $E_1(z) = \int_1^\infty \frac{e^{-tz}}{t} dt$  (where  $\Re(z) \geq 0$ ) to obtain

$$\int_{p_{r_1}}^{p_{r_2}} \frac{1}{x^s \log x} dx = E_1((s-1) \log p_{r_1}) - E_1((s-1) \log p_{r_2})$$

To compute the integral  $\int_{p_{r_1}}^{p_{r_2}} \frac{1}{x^s \log x} dx$  for  $\Re(z) < 1$ , we first write the integral as follows

$$\int_{p_{r_1}}^{p_{r_2}} \frac{1}{x^s \log x} dx = \int_{p_{r_1}}^{p_{r_2}} \frac{e^{-\sigma \log x} \cos(t \log x)}{\log x} dx - i \int_{p_{r_1}}^{p_{r_2}} \frac{e^{-\sigma \log x} \sin(t \log x)}{\log x} dx.$$

The first integral on the right side  $\int_{p_{r_1}}^{p_{r_2}} \frac{e^{-\sigma \log x} \cos(t \log x)}{\log x} dx$  can be computed by using the substitution  $y = \log x$  to obtain

$$\int_{p_{r_1}}^{p_{r_2}} \frac{e^{-\sigma \log x} \cos(t \log x)}{\log x} dx = \int_{p_{r_1}}^{p_{r_2}} \frac{e^{(1-\sigma)y} \cos(ty)}{y} dy,$$

or

$$\int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma \log x} \cos(t \log x)}{\log x} dx = \int_{p_{r1}}^{p_{r2}} \frac{e^{(1-\sigma)y} \cos(ty)}{y} dy + \int_{p_{r1}}^{p_{r2}} \frac{e^{(1-\sigma)y}}{y} dy - \int_{p_{r1}}^{p_{r2}} \frac{e^{(1-\sigma)y}}{y} dy.$$

Hence,

$$\begin{aligned} \int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma \log x} \cos(t \log x)}{\log x} dx &= \int_{\epsilon}^{p_{r1}} \frac{e^{(1-\sigma)y} (1 - \cos(ty))}{y} dy - \int_{\epsilon}^{p_{r2}} \frac{e^{(1-\sigma)y} (1 - \cos(ty))}{y} dy - \\ &\quad \int_{\epsilon}^{p_{r1}} \frac{e^{(1-\sigma)y}}{y} dy + \int_{\epsilon}^{p_{r2}} \frac{e^{(1-\sigma)y}}{y} dy \end{aligned}$$

where,  $\epsilon$  is an arbitrary small positive number. With the variable substantiations  $z_1 = y/\log p_{r1}$  and  $z_2 = y/\log p_{r2}$ , we then obtain

$$\begin{aligned} \int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma \log x} \cos(t \log x)}{\log x} dx &= \int_{\epsilon/\log p_{r1}}^1 \frac{e^{(1-\sigma)(\log p_{r1})z_1} (1 - \cos(t(\log p_{r1})z_1))}{z_1} dz_1 - \\ &\quad \int_{\epsilon/\log p_{r2}}^1 \frac{e^{(1-\sigma)(\log p_{r2})z_2} (1 - \cos(t(\log p_{r2})z_2))}{z_2} dz_2 - \\ &\quad \int_{\epsilon/\log p_{r1}}^1 \frac{e^{(1-\sigma)(\log p_{r1})z_1}}{z_1} dz_1 + \int_{\epsilon/\log p_{r2}}^1 \frac{e^{(1-\sigma)(\log p_{r2})z_2}}{z_2} dz_2 \end{aligned}$$

By the virtue of the following identity ([9], page 230)

$$\int_0^1 \frac{e^{at} (1 - \cos(bt))}{t} dt = \frac{1}{2} \log(1 + b^2/a^2) + \text{Li}(a) + \Re[E_1(-a + ib)],$$

where  $a > 0$ , we then obtain the following

$$\begin{aligned} \int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma \log x} \cos(t \log x)}{\log x} dx &= \Re[E_1((s-1) \log p_{r1})] + \text{Li}((1-\sigma) \log p_{r1}) - \\ &\quad \Re[E_1((s-1) \log p_{r2})] - \text{Li}((1-\sigma) \log p_{r2}) - \\ &\quad \int_{\epsilon/\log p_{r1}}^1 \frac{e^{(1-\sigma)(\log p_{r1})z_1}}{z_1} dz_1 + \int_{\epsilon/\log p_{r2}}^1 \frac{e^{(1-\sigma)(\log p_{r2})z_2}}{z_2} dz_2 \end{aligned}$$

With the variable substantiations  $w_1 = (1-\sigma)(\log p_{r1})z_1$  and  $w_2 = (1-\sigma)(\log p_{r2})z_2$  and by adding and subtracting the terms  $-\int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r2}} \frac{dw_2}{w_2} + \int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r1}} \frac{dw_1}{w_1}$ , we then have

$$\begin{aligned} \int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma \log x} \cos(t \log x)}{\log x} dx &= \Re[E_1((s-1) \log p_{r1})] + \text{Li}((1-\sigma) \log p_{r1}) - \\ &\quad \Re[E_1((s-1) \log p_{r2})] - \text{Li}((1-\sigma) \log p_{r2}) + \\ &\quad \int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r2}} \frac{e^{w_2} - 1}{w_2} dw_2 - \int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r1}} \frac{e^{w_1} - 1}{w_1} dw_1 + \\ &\quad \int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r2}} \frac{dw_2}{w_2} - \int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r1}} \frac{dw_1}{w_1}. \end{aligned}$$

Using the following identity [9, page 230]

$$\int_0^a \frac{e^t - 1}{t} dt = \text{Ei}(a) - \log(a) - \gamma$$

where  $a > 0$ , we then obtain for  $\sigma < 1$ ,

$$\int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma \log x} \cos(t \log x)}{\log x} dx = \Re[E_1((s-1) \log p_{r1})] - \Re[E_1((s-1) \log p_{r2})]$$

Similarly, using the identity [9, page 230]

$$\int_{p_0}^1 \frac{e^{at} \sin(bt)}{t} dt = \pi - \arctan(b/a) + \Im[E_1(-a + ib)],$$

where  $a > 0$ , we can show that for  $\sigma < 1$ , we have

$$- \int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma \log x} \sin(t \log x)}{\log x} dx = \Im[E_1((s-1) \log p_{r1})] - \Im[E_1((s-1) \log p_{r2})].$$

Therefore, for  $\Re(s) > 0.5$ , we have

$$\sum_{i=r1}^{r2} \frac{1}{p_i^s} = E_1((s-1) \log p_{r1}) - E_1((s-1) \log p_{r2}) + \varepsilon$$

where,  $\varepsilon = O\left(\frac{|s|+1}{(\sigma-0.5)^2} p_{r1}^{1/2-\sigma} \log p_{r1}\right)$ .

## Appendix 4

To show that

$$\left| \sum_{n=1}^N \frac{\mu(n, p_r)}{n} \right| \leq 2$$

we first note that

$$\begin{aligned} \sum_{d|n} \mu(d, p_r) &= 1, \text{ if } n = 1, \\ \sum_{d|n} \mu(d, p_r) &= 1, \text{ if all the prime factors of } n \text{ are less than } p_r, \\ \sum_{d|n} \mu(d, p_r) &= 0, \text{ if any of the prime factors of } n \text{ is greater than } p_r. \end{aligned}$$

Adding all the terms  $\sum_{d|n} \mu(d, p_r)$  for  $1 \leq n \leq N$ , we then obtain

$$0 < \sum_{n=1}^N \mu(n, p_r) \left\lfloor \frac{N}{n} \right\rfloor \leq N,$$

where  $\lfloor x \rfloor$  refers to the integer value of  $x$ . Define  $r_n$  as

$$r_n = \frac{N}{n} - \left\lfloor \frac{N}{n} \right\rfloor,$$

where  $0 \leq r_n < 1$ . Hence, we have

$$\sum_{n=1}^N \mu(n, p_r) r_n < \sum_{n=1}^N \mu(n, p_r) \left\lfloor \frac{N}{n} \right\rfloor + \sum_{n=1}^N \mu(n, p_r) r_n \leq \sum_{n=1}^N \mu(n, p_r) r_n$$

Since

$$-N \leq \sum_{n=1}^N \mu(n, p_r) r_n \leq N,$$

thus, for every  $p_r$  we have

$$-N < \sum_{n=1}^N \mu(n, p_r) \frac{N}{n} \leq 2N,$$

or

$$-1 < \sum_{n=1}^N \frac{\mu(n, p_r)}{n} \leq 2.$$

## References

- [1] Edwards H. M., *Riemann Zeta Function.*, Academic Press, 1974.
- [2] Riemann R., *On the Number of Prime Numbers less than a Given Quantity.*, Nov. 1856. Translated by Wilkins, D. R., Dec. 1998.
- [3] Gonek S. M., Hughes C.P., and Keating J.P, *A Hybrid Euler-Hadamard Product for the Riemann Zeta Function*, Duke Mathematical Journal, Vol. 136, No. 3,c 2007
- [4] Gerald Tenenbaum and Michel Mendes France, Translated by Philip G Spain, *The prime numbers and their distribution.*, Student Medical Library, Volume 6, AMS, USA, 2000.
- [5] Daniel H. Green and Donald E. Knuth, *Mathematics for the analysis of Algorithms.*, Third Edition, Birkhauser. 1990.
- [6] Eberhard Freitag and Rolf Busam, *Complex Analysis*, Second Edition, Springer, 2005.
- [7] Titchmarsh E. C., *The Theory of the Riemann Zeta-function.*, Second Revised Edition (Heath-Brown), Oxford Science Publications, 1986, Reprinted 2007.
- [8] William J. LeVeque, *Fundamentals of Number Theory.*, LeVeque, W. L., Dover Duplications, 1996.
- [9] Abramowitz and Stegun , *Handbook of Mathematical Functions.*, Tenth Edition, United States Printing Office, Washington DC, 1972.
- [10] A. Granville, *Smooth numbers: Computational number theory and beyond.*, Algorithmic Number Theory, Vol. 44, 267-323, 2008.