ON THE VALIDITY OF THE RIEMANN HYPOTHESIS

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Abstract

In this paper, we have used the partial Euler product to examine the validity of the Riemann Hypothesis. The Dirichlet series over the Mobius function $M(s) = \sum_{n=1}^{\infty} 1/n^s$ has been modified and represented in terms of the partial Euler product by progressively eliminating the numbers that first have a prime factor 2, then 3, then 5, ..up to the prime p_r. It is shown that the series $M(s)$ and the new series have the same region of convergence. Unlike the partial sum of $M(s)$ that has irregular behavior, the partial sum of the new series exhibits regular behavior as p_r approaches infinity. This has allowed the use of integration methods to compute the partial sum of the new series to determine its region of convergence and to provide an answer for the validity of the Riemann Hypothesis.

1 Introduction

The Riemann zeta function $\zeta(s)$ satisfies the following functional equation over the complex plain [1]

$$\zeta(1-s) = 2(2\pi)^2 \cos(0.5s\pi)\Gamma(s)\zeta(s), \tag{1}$$

where, $s = \sigma + it$ is a complex variable and $s \neq 0$.

For $\sigma > 1$ (or $\Re(s) > 1$), $\zeta(s)$ can be expressed by the following series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{2}$$

or by the following product over the primes p_i's

$$\frac{1}{\zeta(s)} = \prod_{i=1}^{\infty} \left(1 - \frac{1}{p_i^s}\right). \tag{3}$$

where, $p_1 = 2$, $\prod_{i=1}^{\infty}(1 - 1/{p_i}^s)$ is the Euler product and $\prod_{i=1}^{r}(1 - 1/{p_i}^s)$ is the partial Euler product. The above series and product representations of $\zeta(s)$ are absolutely convergent for $\sigma > 1$.

The region of the convergence can be extended to $\Re(s) > 0$ by using the alternating series $\eta(s)$ where

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}, \tag{4}$$

and

$$\zeta(s) = \frac{1}{1-2^{1-s}}\eta(s). \tag{5}$$

One may notice that the term $1-2^{1-s}$ is zero at $s=1$. This zero cancels the simple pole that $\zeta(s)$ has at $s=1$ enabling the extension (or analog continuation) of the zeta function series representation over the critical strip $0 < \Re(s) < 1$.

It is well known that all the non-trivial zeros of $\zeta(s)$ are located in the critical strip $0 < \Re(s) < 1$. Riemann stated that all the non-trivial zeros were very probably located on the critical line $\Re(s) = 0.5$ [2]. There are many equivalent statements for the Riemann Hypothesis (RH) and one of them involves the Dirichlet series with the Mobius function.

The Mobius function $\mu(n)$ is define as follows
$\mu(n) = 1$, if $n = 1$.
$\mu(n) = (-1)^k$, if $n = \prod_{i=1}^{k} p_i$, p_i's are distinct primes.
$\mu(n) = 0$, if $p^2|n$ for some p.

The Dirichlet series $M(s)$ with the Mobius function is defined as

$$M(s) = \sum_{n=1}^{\infty} \frac{\mu(s)}{n^s}. \tag{6}$$

This series is absolutely convergent to $1/\zeta(s)$ for $\Re(s) > 1$ and conditionally convergent to $1/\zeta(s)$ for $\Re(s) = 1$. The Riemann hypothesis is equivalent to the statement that $M(s)$ is conditionally convergent to $1/\zeta(s)$ for $\Re(s) > 0.5$.

Gonek, Hughes and Keating [3] have done an extensive research into establishing a relationship between $\zeta(s)$ and its partial Euler product for $\Re(s) < 1$. Gonek stated "Analytic number theorists believe that an eventual proof of the Riemann Hypothesis must use both the Euler product and functional equation of the zeta-function. For there are functions with similar functional equations but no Euler product, and functions with an Euler product but no functional equation." In sections 4 and 5, we will present a functional equation for $\zeta(s)$ using its partial Euler product. The method is based on writing the Euler product formula as follows

$$1/\zeta(s) = \prod_{i=1}^{\infty}\left(1-\frac{1}{p_i^s}\right) = \prod_{i=1}^{r}\left(1-\frac{1}{p_i^s}\right)\prod_{r+1}^{\infty}\left(1-\frac{1}{p_i^s}\right).$$

The above equation is valid for $\sigma > 1$. To be able to represent $\zeta(s)$ in term of its partial Euler product for $\sigma \leq 1$, we have to replace the term $\prod_r^\infty (1 - 1/p_i^s)$ with an equivalent one that allows the analytic continuation for the representation of $\zeta(s)$ for $\sigma \leq 1$. Thus, the new term, that we need to introduce to replace $\prod_r^\infty (1 - 1/p_i^s)$, must have a zero that cancels the pole that $\zeta(s)$ has at $s = 1$. In the section 4, we will use the complex analysis to compute this new term. In section 5, we then use the new representation to compute the sum $\sum_{i=1}^{r} {p_i}^{\sigma}$ for $\sigma < 1$. This sum is then used to examine the validity of the Riemann Hypothesis.

In this paper, we claim the the Riemann Hypothesis is invalid. We support our claim by proving that the series $M(\sigma)$ is divergent for $\sigma < 1$. We achieved this results by introducing a method to represent the Dirichlet series $M(s)$ (defined by Equation (6)) in terms of the partial

Euler product. This task is achieved (sections 2) by first eliminating the numbers that have the prime factor 2 to generate the series $M(s,2)$. For the series $M(s,2)$, we then eliminate the numbers with the prime factor 3 to generate the series $M(s,3)$, and so on, up to the prime number p_r. In essence, we have applied the sieving technique to modify the series $M(s)$ to include only the numbers with prime factors greater than p_r. In section 3, we have shown that the series $M(s)$ and the new series $M(s,p_r)$ have the same region of convergence.

So far, the efforts to use the series $M(s)$ to examine the validity of the Riemann Hypothesis have failed due to the irregular behavior of the partial sum of the series $M(s)$. In section 6, we have shown that the partial sum of the new series $M(s,p_r)$ exhibits regular behavior as p_r approaches infinity. This has allowed the use of integration methods to compute the partial sum of the new series and consequently determine its region of convergence. With this analysis, we have shown that non-trivial zeros can be found arbitrary close to the line $s = 1$.

2 Applying the Sieving Method to the Dirichlet Series $M(s)$.

The Dirichlet series $M(s)$ with the Mobius function is defined as

$$M(s) = \sum_{n=1}^{\infty} \frac{\mu(s)}{n^s},$$

where $\mu(n)$ is the Mobius function. Thus,

$$M(s) = 1 - \frac{1}{2^s} - \frac{1}{3^s} + \frac{0}{4^s} - \frac{1}{5^s} + \frac{1}{6^s}....$$

It should be pointed out that our definition of $M(s)$ is different from $M(x)$ that is commonly defined in the literature as $M(x) = \sum_{n \leq x} \mu(n)$.

Now, we introduce the series $M(s,2)$ by eliminating all the numbers that have a prime factor 2. Thus, $M(s,2)$ can be written as

$$M(s,2) = 1 - \frac{1}{3^s} - \frac{1}{5^s} - \frac{1}{7^s} + \frac{0}{9^s} - \frac{1}{11^s} - \frac{1}{13^s} + \frac{1}{15^s}.....$$

The analysis in this paper heavily relay on the testing the convergence of a series by comparing it with another conditionally convergent series. Therefor, rearrangement and permutation of the terms may have a significant impact on the region of convergence of both series. Therefore, it essential to have the same index for both series $M(s)$ and $M(s,2)$ referring to the same term. Hence, the the above series can be re-written as

$$M(s,2) = 1 + \frac{0}{2^s} - \frac{1}{3^s} + \frac{0}{4^s} - \frac{1}{5^s} + \frac{0}{6^s} - \frac{1}{7^s} - \frac{0}{8^s}....,$$

or

$$M(s,2) = \sum_{n=1}^{\infty} \frac{\mu(n,2)}{n^s}, \tag{7}$$

where
$\mu(n,2) = \mu(n)$, if n is an odd number,

$\mu(n,2)=0$, if n is an even number.

The above series $M(s,2)$ can be further modified by eliminating all the numbers that have a prime factor 3 to get the series $M(s,3)$ where

$$M(s,3)=1-\frac{1}{5^s}-\frac{1}{7^s}-\frac{1}{11^s}-\frac{1}{13^s}-\frac{1}{17^s}-\frac{1}{19^s}-\frac{1}{23^s}+\frac{0}{25^s}....,$$

or more conveniently

$$M(s,3)=1+\frac{0}{2^s}-\frac{0}{3^s}+\frac{0}{4^s}-\frac{1}{5^s}+\frac{0}{6^s}-\frac{1}{7^s}-\frac{0}{8^s}....,$$

and so on.

Let $I(p_r)$ represent, in ascending order, the integers with distinct prime factors that belong to the set $\{p_i : p_i > p_r\}$. Let $\{1, I(p_r)\}$ be the set of 1 and $I(p_r)$ (for example, $\{1, I(2)\}$ is the set of square free odd numbers), then we define the series $M(s,p_r)$ as

$$M(s,p_r)=\sum_{n=1}^{\infty}\frac{\mu(n,p_r)}{n^s}, \tag{8}$$

where
$\mu(n,p_r)=\mu(n)$, if $n\in\{1,I(p_r)\}$,
otherwise, $\mu(n,p_r)=0$.

It can be easily shown that $M(s,p_r)$ converges absolutely for $\Re(s)>1$ for every prime number p_r. Furthermore, it can be shown that, for $\Re(s)>1$, $M(s,p_r)$ satisfies the following equation

$$M(s)=M(s,p_r)\prod_{i=1}^{r}\left(1-\frac{1}{p_i^s}\right). \tag{9}$$

Since

$$M(s)=\prod_{i=1}^{\infty}\left(1-\frac{1}{p_i^s}\right),$$

then we conclude that, for $\Re(s)>1$, $M(s,p_r)$ approaches 1 as p_r approaches infinity.

3 Convergence of the series $M(s,p_r)$ within the strip $0.5<\Re(s)\leq 1$.

In this section, we will deal with the question of the conditional convergence of the series $M(s,p_r)$ over the strip $0.5<\Re(s)\leq 1$. This task can be achieved by examining the convergence of the series $M(s,p_r)$ along the real axis (or along the line $0.5<\sigma\leq 1$). Theorems 1 and 2 establishes the relationship between the conditional convergence of the two series $M(s)$ and $M(s,p_r)$ for $0.5<\sigma\leq 1$.

Theorem 1 *For $s=\sigma+i0$, where $0.5<\sigma\leq 1$ and for every prime number p_r, the series $M(\sigma)$ converges conditionally if and only if the series $M(\sigma,p_r)$ converges conditionally. Furthermore, $M(\sigma)$ and $M(\sigma,p_r)$ are related as follows*

$$M(\sigma)=M(\sigma,p_r)\prod_{i=1}^{r}\left(1-\frac{1}{p_i^\sigma}\right). \tag{10}$$

The proof of Theorem 1 is outlined in Appendix 1.

Theorem 2 *For $s = \sigma + it$, where $0.5 < \sigma \leq 1$ and for every prime number p_r, the series $M(s)$ converges conditionally if and only if the series $M(s, p_r)$ converges conditionally. Furthermore, $M(s)$ and $M(s, p_r)$ are related as follows*

$$M(s) = M(s, p_r) \prod_{i=1}^{r} \left(1 - \frac{1}{p_i^s}\right). \tag{11}$$

The proof of the first part of Theorem 2 follows from the fact that $M(s, p_r)$ is a Dirichlet series and consequently this series is conditionally convergent if and only if the series $M(\sigma, p_r)$ is conditionally convergent.

The second part of the theorem can be proved by first defining $M(s, p_r; N_1, N_2)$ as the sum

$$M(s, p_r; N_1, N_2) = \sum_{n=N_1}^{N_2} \frac{\mu(n, p_r)}{n^s}. \tag{12}$$

Then, we have

$$M(s, p_{r-1}; 1, Np_r) = M(s, p_r; 1, Np_r) - \frac{1}{p_r^s} M(s, p_r; 1, N). \tag{13}$$

If both series $M(s, p_{r-1})$ and $M(s, p_r)$ are convergent, then as N approaches infinity, we obtain

$$M(s, p_{r-1}) = M(s, p_r) \left(1 - \frac{1}{p_i^s}\right).$$

By repeating this process $r - 1$ times, we then obtain

$$M(s) = M(s, p_r) \prod_{i=1}^{r} \left(1 - \frac{1}{p_i^s}\right).$$

Note that if we multiply both sides of the above equation by $\prod_{i=1}^{r} \left(1 + {p_i}^{-s}\right)$

$$M(s, p_r) = \frac{1}{\zeta(s) \prod_{i=1}^{r} \left(1 - p_i^{-2s}\right)} \prod_{i=1}^{r} \left(1 + \frac{1}{p_i^s}\right).$$

As p_r approaches infinity, we then have

$$M(s, p_r) = \frac{\zeta(2s)}{\zeta(s)} \prod_{i=1}^{r} \left(1 + \frac{1}{p_i^s}\right).$$

It should be pointed out that the sieving method applied to the Dirichlet series with Mobious function can be also applied to the Dirichlet series with Lioville function. The Dirichlet series L(s) with Lioville Function $\lambda(n)$ is defined as

$$L(s) = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s}, \tag{14}$$

where

$\lambda(n) = 1$, if $n = 1$,
$\lambda(n) = 1$, if n has an even number of prime factors including multiplicities,
$\lambda(n) = -1$, if n has an odd number of prime factors including multiplicities.

Following the same process, we define the series $L(s, p_r)$ as

$$L(s, p_r) = \sum_{n=1}^{\infty} \frac{\lambda(n, p_r)}{n^s}, \tag{15}$$

where

$\lambda(n, p_r) = \lambda(n)$, if $n \in \{1, I(p_r)\}$,
otherwise, $\lambda(n, p_r) = 0$.

It can be easily shown that $L(s, p_r)$ converges absolutely for $\Re(s) > 1$ for every prime number p_r. Furthermore, it can be also shown that, for $\Re(s) > 1$, $L(s, p_r)$ satisfies the following equation

$$L(s, p_r) = L(s) \prod_{i=1}^{r} \left(1 + \frac{1}{p_i^s}\right).$$

It is well known in the literature that, on RH, we have

$$\sum_{n \le x} \lambda(n) = O(x^{1/2+\epsilon}),$$

where ϵ is an arbitrary small number.

Using the above equation and following similar steps to those used for Theorems (1) and (2), we may obtain the following theorem.

Theorem 3 *For $s = \sigma + it$, where $0.5 < \sigma \le 1$ and for every prime number p_r, the series $L(s)$ converges conditionally if and only if the series $L(s, p_r)$ converges conditionally. Furthermore, $L(s)$ and $L(s, p_r)$ are related as follows*

$$L(s, p_r) = L(s) \prod_{i=1}^{r} \left(1 + \frac{1}{p_i^s}\right). \tag{16}$$

4 Functional representation of $\zeta(s)$ using its partial Euler product.

Theorem 1 of the previous section provides a relationship between $\zeta(s) = 1/M(s)$ and the partial Euler product $\prod_{i=1}^{r}(1 - 1/p_i^s)$. In this section and the following one, we will derive a functional representation for $\zeta(s)$ using its partial Euler product. In this section, we will use the prime counting function to compute this functional representation and in the following section we will use the von Mangoldt function to achieve the same task. This functional representation is then used to compute the sum $\sum_{i=1}^{r} {p_i}^{\sigma}$ for $\sigma < 1$. In section, 6 we will use this sum to show that the series $M(\sigma, p_r)$ is diverges for $\sigma < 1$.

We will start this task by first writing $\zeta(s)$ for $\sigma > 1$ as follows

$$1/\zeta(s) = \prod_{i=1}^{\infty}\left(1 - \frac{1}{p_i^s}\right) = \prod_{i=1}^{r}\left(1 - \frac{1}{p_i^s}\right)\prod_{r+1}^{\infty}\left(1 - \frac{1}{p_i^s}\right). \tag{17}$$

For $\sigma > 0.5$, we have

$$\log \prod_{i=r1}^{r2}\left(1 - \frac{1}{p_i^s}\right) = \sum_{i=r1}^{r2} \log\left(1 - \frac{1}{{p_i}^s}\right),$$

or

$$\log \prod_{i=r1}^{r2}\left(1 - \frac{1}{p_i^s}\right) = \sum_{i=r1}^{r2}\left(-\frac{1}{{p_i}^s} - \frac{1}{2{p_i}^{2s}} - \frac{1}{3{p_i}^{3s}} - ...\right).$$

Let δ be defined as the sum

$$\delta = \sum_{i=r1}^{r2}\left(-\frac{1}{2{p_i}^{2s}} - \frac{1}{3{p_i}^{3s}} - \frac{1}{4{p_i}^{4s}}...\right). \tag{18}$$

Thus,

$$\log \prod_{i=r1}^{r2}\left(1 - \frac{1}{p_i^s}\right) = -\sum_{i=r1}^{r2}\frac{1}{{p_i}^s} + \delta. \tag{19}$$

Since $|\delta| < \sum_{n=p_{r1}}^{\infty}\left(\frac{1}{2n^{2\sigma}} + \frac{1}{3n^{3s}} + \frac{1}{4n^{4s}}...\right)$, thus $\delta = O(p_{r1}^{1-2\sigma}/(2\sigma - 1))$. Furthermore, if $2\sigma - 1$ is a fixed positive number, then $\delta = O(p_{r1}^{1-2\sigma})$. It should be pointed out that for $\sigma = 0.5$ and $t \neq 0$, δ is convergent to a finite number by the virtue of the Prime Number Theorem.

Using the Prime Number Theorem (PNT) with a suitable constant $a > 0$, the number of primes less than x is given by [4, page 43]

$$\pi(x) = \mathrm{Li}(x) + O\left(xe^{-a\sqrt{\log x}}\right), \tag{20}$$

or

$$\pi(x) = \mathrm{Li}(x) + O\left(x/(\log x)^k\right), \tag{21}$$

where $\mathrm{Li}(x)$ is the Logarithmic Integral of x and k is a number greater than zero.

Using Stieltjes integral [5], we may write the sum $\sum_{i=r1}^{r2}\frac{1}{{p_i}^\sigma}$ for $\sigma > 1$ as follows

$$\sum_{i=r1}^{r2}\frac{1}{{p_i}^\sigma} = \int_{x=p_{r1}}^{p_{r2}}\frac{d\pi(x)}{x^\sigma}. \tag{22}$$

Using Equation (21) for the representation of $\pi(x)$, we may then write the integral in Equation (22) as [5, Theorem 2, page 57]

$$\sum_{i=r1}^{r2}\frac{1}{p_i^\sigma} = \int_{p_{r1}}^{p_{r2}}\frac{1}{x^\sigma}\frac{1}{\log x}dx + O\left(\frac{1}{(\log p_{r1})^k}\right), \tag{23}$$

where k is a number greater than zero. Therefore,

$$\sum_{i=r1}^{r2}\frac{1}{p_i^\sigma} = \int_{p_{r1}}^{\infty}\frac{1}{x^\sigma}\frac{1}{\log x}dx - \int_{p_{r2}}^{\infty}\frac{1}{x^\sigma}\frac{1}{\log x}dx + O\left(\frac{1}{(\log p_{r1})^k}\right). \tag{24}$$

Recalling that the Exponential Integral $E_1(r)$ is given by

$$E_1(r) = \int_r^\infty \frac{e^{-u}}{u} du,$$

and using the substitutions $u = (\sigma - 1)\log p_r$, $du = (\sigma - 1)dx/x$ and $x^\sigma/x = e^u$, then for $\sigma > 1$, we may write Equation (24) as

$$\sum_{i=r1}^{r2} \frac{1}{p_i^\sigma} = E_1\left((\sigma - 1)\log p_{r1}\right) - E_1\left((\sigma - 1)\log p_{r2}\right) + O\left(\frac{1}{(\log p_{r1})^k}\right). \tag{25}$$

Combining Equations (19) and ((25)) and noting that, for $\sigma > 1$, $E_1\left((\sigma - 1)\log p_{r2}\right)$ approaches zero as p_{r2} approaches infinity, we may write Equation (17) for $\sigma > 1$ as

$$-\log \zeta(\sigma) = \sum_{i=1}^{r} \log\left(1 - \frac{1}{{p_i}^\sigma}\right) - \sum_{i=r+1}^{\infty} \frac{1}{{p_i}^\sigma} + \delta,$$

or

$$\log \zeta(\sigma) + \sum_{i=1}^{r} \log\left(1 - \frac{1}{{p_i}^\sigma}\right) - E_1\left((\sigma - 1)\log p_{r+1}\right) = \epsilon,$$

where $\epsilon = O(1/(\log p_{r1})^k)$ is an arbitrarily small number attained by setting p_r sufficiently large. Therefore,

$$\zeta(\sigma) \prod_{i=1}^{r} \left(1 - \frac{1}{p_i^\sigma}\right) \exp\left(-E_1((\sigma - 1)\log p_{r+1})\right) = 1 + \epsilon. \tag{26}$$

As p_r approaches infinity, ϵ approaches zero. Hence, the right side of the above equation approaches 1 as p_r approaches infinity.

Similarly, for $\Re(s) > 1$, we can use the following expression for $E_1(s)$

$$E_1(s) = \int_1^\infty \frac{e^{-xs}}{x} dx,$$

to show that

$$\lim_{r\to\infty} \left\{\zeta(s) \prod_{i=1}^{r} \left(1 - \frac{1}{p_i^s}\right) \exp\left(-E_1((s - 1)\log p_{r+1})\right)\right\} = 1. \tag{27}$$

Let the function $G(s, p_r)$ be defined as

$$G(s, p_r) = \zeta(s) \prod_{i=1}^{r} \left(1 - \frac{1}{p_i^s}\right) \exp\left(-E_1((s - 1)\log p_{r+1})\right) \tag{28}$$

where, $G(s, p_r)$ is a regular function for $\Re(s) > 1$. Referring to Equation (27), the function $G(s, p_r)$ approaches 1 as p_r approaches infinity. It should be noted that, for each p_r, the function $\exp\left(-E_1((s - 1)\log p_{r+1})\right)$ is an entire function, the function $\zeta(s)$ is analytic everywhere except at $s = 1$ and the function $\prod_{i=1}^{r}(1 - 1/p_i^s)$ is analytic for $\Re(s) > 0$. Thus, for any $\sigma > 1$, the function $G(s, p_r)$ can be considered as a sequence of analytic functions. Furthermore, as p_r (or r) approaches infinity, this sequence is uniformly convergent over the half plane with $\sigma > 1 + \epsilon$ (where, ϵ is an arbitrary small number). Therefore, by the virtue of the Weiestrass theorem, the limit is also analytic function [6] (Weiestrass theorem states that if the function sequence f_n is analytic over the region Ω and f_n is uniformly convergent to a function f, then f is also analytic on Ω and ${f_n}'$ converges uniformly to f' on Ω). If we define this limit as $G(s)$, where

$$G(s) = \lim_{r\to\infty} G(s, p_r) \tag{29}$$

then, $G(s)$ is analytic over the half plane $\Re(s) > 1$ and it is equal to 1 by the virtue of Equation (27).

The Prime Number Theorem (PNT) allows us to extend the above results to the line $s = 1 + it$. Moreover, we will show that if RH is valid, then for the strip $s = \sigma + it$ where, $0.5 < \sigma < 1$, the above results will also be valid with the limit of $G(s, p_r)$ is 1 as p_r approaches infinity.

We will start this task by showing that although both $\zeta(s)$ and $E_1((s-1)\log p_{r+1})$ have a singularity at $s = 1$, the product $G(s, p_r)$ has a removable singularity at $s = 1$ for every p_r. This can be shown by first expanding $\zeta(s)$ as a Laurent series about its singularity at $s = 1$

$$\zeta(s) = \frac{1}{s-1} + \gamma - \gamma_1(s-1) + \gamma_2\frac{(s-1)^2}{2!} - \gamma_3\frac{(s-1)^3}{3!} + ..., \tag{30}$$

where γ is the Euler-Mascheroni constant and γ_i's are the Stieltjes constants. For $s = 1 + \epsilon$, where $\epsilon = \epsilon_1 + i\epsilon_2$, ϵ_1 and ϵ_2 are arbitrary small numbers, the above equation can be written as

$$\zeta(s) = \frac{1}{\epsilon} + \gamma - \gamma_1\epsilon + \gamma_2\frac{\epsilon^2}{2!} - \gamma_3\frac{\epsilon^3}{3!} + ... \tag{31}$$

Furthermore, for $\sigma > 1$, using the definition of the Exponential Integral, we may write $E_1(s)$ as

$$E_1(s) = -\gamma - \log s + s - \frac{s^2}{2\,2!} + \frac{s^3}{3\,3!} - \frac{s^4}{4\,4!} + ... \tag{32}$$

Thus, for $s = 1 + \epsilon$, we have

$$\exp\left(-E_1((s-1)\log p_r)\right) = e^{\gamma}\epsilon\,\log p_r\,\exp\left(-\epsilon\log p_r + \frac{(\epsilon\log p_r)^2}{2\,2!} - \frac{(\epsilon\log p_r)^3}{3\,3!} + ...\right). \tag{33}$$

By taking the product $\zeta(s)\exp\left(-E_1((s-1)\log p_r)\right)$ and allowing ϵ to approach zero, we then obtain at $s = 1$ (in the same sense as computing $\sin x/x$ at $x = 0$)

$$\zeta(s)\,\exp\left(-E_1((s-1)\log p_r)\right) = e^{\gamma}\,\log p_r. \tag{34}$$

However, it is well known that the partial Euler product at $s = 1$ can be written as [8]

$$\prod_{i=1}^{r}\left(1 - \frac{1}{p_i}\right) = \frac{e^{-\gamma}}{\log p_r} + O\left(\frac{1}{(\log p_r)^2}\right). \tag{35}$$

Multiplying Equations (34) and (35), we may conclude that at $s = 1$, $G(s, p_r)$ approaches 1 as p_r approaches infinity. Furthermore, for $s = 1 + it$ and $t \neq 1$, the value of $\exp(-E_1(it\log p_r))$ approaches 1 as p_r approaches infinity and since

$$\lim_{r\to\infty}\left\{\zeta(s)\prod_{i=1}^{r}\left(1 - \frac{1}{{p_i}^s}\right)\right\} = 1,$$

therefore, for $s = 1 + it$, we have the following

$$\lim_{r\to\infty} G(s, p_r) = \lim_{r\to\infty} \left\{ \zeta(s) \prod_{i=1}^{r} \left(1 - \frac{1}{p_i^s}\right) \exp\left(-E_1((s-1)\log p_{r+1})\right) \right\} = 1.$$

So far, we have shown that the function $G(s, p_r)$ is uniformly convergent to 1 when $\Re(s) > 1$ and using PNT, $G(s, p_r)$ is convergent to 1 for $\Re(s) = 1$. In the following, we will show that, assuming the validity of the Riemann Hypothesis, the function $G(s, p_r)$ is uniformly convergent to 1 for every value of s with $\Re(s) > 0.5 + \epsilon$, where ϵ is an arbitrary small number. Toward this goal, we will first show that the function $G(s, p_r)$ is convergent for any value of s on the real axis with $\sigma > 0.5$. This can be achieved by first writing the expressions for $G(\sigma, p_{r1})$ and $G(\sigma, p_{r2})$ (where $r2$ is an arbitrary large number greater than $r1$)

$$G(\sigma, p_{r1}) = \zeta(\sigma) \exp\left(-E_1((\sigma-1)\log p_{r1+1})\right) \prod_{i=1}^{r1} \left(1 - \frac{1}{p_i^\sigma}\right), \tag{36}$$

$$G(\sigma, p_{r2}) = \zeta(\sigma) \exp\left(-E_1((\sigma-1)\log p_{r2+1})\right) \prod_{i=1}^{r2} \left(1 - \frac{1}{p_i^\sigma}\right). \tag{37}$$

Since the function $G(s, p_r)$ is analytic that is not equal to 0 for $\sigma > 0.5$, hence we can divide Equation (37) by Equation (36) and then take the logarithm to obtain

$$\log\left(\frac{G(\sigma, p_{r2})}{G(\sigma, p_{r1})}\right) = E_1\left((\sigma-1)\log p_{r1+1}\right) - E_1\left((\sigma-1)\log p_{r2+1}\right) + \log\left(\prod_{i=r1+1}^{r2} \left(1 - \frac{1}{{p_i}^\sigma}\right)\right). \tag{38}$$

To compute the logarithm of the partial Euler product in Equation (38), we recall Equation (19)

$$\log \prod_{r1+1}^{r2} \left(1 - \frac{1}{p_i^s}\right) = -\sum_{i=r1+1}^{r2} \frac{1}{{p_i}^s} + \delta,$$

where $\delta = O(p_{r1}^{1-2\sigma}/(2\sigma-1))$. Furthermore, on RH, we have

$$\pi(x) = \mathrm{Li}(x) + O\left(\sqrt{x}\,\log x\right), \tag{39}$$

where $\mathrm{Li}(x)$ is the Logarithmic Integral of x. Using Equation (39) for the representation of the prime counting function, we may then obtain (Appendix 2)

$$\sum_{i=r1+1}^{r2} \frac{1}{{p_i}^\sigma} = E_1((\sigma-1)\log p_{r1+1}) - E_1((\sigma-1)\log p_{r2}) + \varepsilon,$$

where $\varepsilon = O\left(\frac{t}{(\sigma-0.5)^2}\, {p_{r1}}^{0.5-\sigma} \log p_{r1}\right)$. Hence, Equation (38) can be written as

$$\log\left(\frac{G(\sigma, p_{r2})}{G(\sigma, p_{r1})}\right) = \varepsilon + \delta + E_1((\sigma-1)\log p_{r2}) - E_1((\sigma-1)\log p_{r2+1}).$$

Since, for $\sigma > 0.5+\epsilon$, $\varepsilon+\delta$ and $E_1((\sigma-1)\log p_{r2}) - E_1((\sigma-1)\log p_{r2+1})$ can be made arbitrary small by choosing p_{r1} arbitrary large, thus the limit of $G(\sigma, p_r)$ exists as p_r approaches infinity and it is given by

$$G(\sigma) = \lim_{r\to\infty} G(\sigma, p_r) \tag{40}$$

This proves that, on RH, $G(\sigma, p_r)$ is convergent as p_r approaches infinity and thus $G(\sigma)$ exists for $\sigma > 0.5$. In Appendix 3, we have shown that, on RH and for $\Re(s) > 0.5$, we have

$$\sum_{i=r1}^{r2} \frac{1}{{p_i}^s} = E_1((s-1)\log p_{r1}) - E_1((s-1)\log p_{r2}) + \varepsilon,$$

where $\varepsilon = O\left(\frac{t+1}{(\sigma-0.5)^2}\, {p_{r1}}^{0.5-\sigma} \log p_{r1}\right)$. Thus, we can follow the same steps and show that $G(s, p_r)$ is convergent as p_r approaches infinity and thus $G(s)$ exists for $\Re(s) > 0.5$.

It should be noted that, while the function sequence $G(s, p_r)$ is not uniformly convergent when the region of convergence is extended all the way to the line $\sigma = 0.5$, it is however uniformly convergence for any strip with $\sigma > 0.5 + \epsilon$, where ϵ is an arbitrary small number. This follows from the fact that ε (or, the O term) is bounded for any $\sigma > 0.5 + \epsilon$. Since $G(s, p_r)$ is analytic for $\Re(s) > 0$ and it is uniformly convergent for $\Re(s) > 0.5 + \epsilon$, thus $G(s)$ is analytic for the half right complex plain with $\Re(s) > 0.5 + \epsilon$ (Weiestrass theorem [6]). Since we have shown that $G(s) = 1$ for $\Re(s) \geq 1$, thus on RH, $G(s) = 1$ for $\Re(s) > 0.5 + \epsilon$. Hence, we have the following theorem

Theorem 4 *For s = $\sigma + it$ and $\sigma > 0.5$, the following holds if RH is valid*

$$\lim_{r\to\infty} \left\{ \zeta(s) \prod_{i=1}^{r} \left(1 - \frac{1}{p_i^s}\right) \exp\left(-E_1((s-1)\log p_{r+1})\right) \right\} = 1. \tag{41}$$

$$\lim_{r\to\infty} \left\{ M(s, p_r) \exp\left(E_1((s-1)\log p_{r+1})\right) \right\} = 1. \tag{42}$$

It should be pointed out that Theorem 4 can be generalized to the case where there are no non-trivial zeros for values of s with $\Re(s) > a$ (where, $a > 0.5$). For this case, Equation (41) is valid for every s with $\Re(s) > a$ and ε in Appendix 3 is given by $O\left(\frac{t+1}{(\sigma-a)^2}\, {p_{r1}}^{a-\sigma} \log p_{r1}\right)$.

Equation (41) of Theorem 4 can be written as follows

$$\log \zeta(s) + \log \prod_{i=1}^{r2} \left(1 - \frac{1}{p_i^s}\right) - E_1\left((s-1)\log p_{r2+1}\right) = 0,$$

where the equality of both sides is attained as r_2 (or p_{r2}) approaches infinity. It should be pointed out that both functions $\log \zeta(s)$ and $E_1((s-1)\log p_{r2+1})$ have a branch cut along the real axis where $0.5 \leq \sigma < 1$, while the difference (i.e. $\log \zeta(s) - E_1((s-1)\log p_{r2+1})$) does not have a branch cut. For $r < r2$, the above equation can be then written as

$$\log \zeta(s) = E_1\left((s-1)\log p_{r2+1}\right) - \sum_{i=1}^{r} \log\left(1 - \frac{1}{p_i^s}\right) - \sum_{i=r+1}^{r2} \log\left(1 - \frac{1}{p_i^s}\right).$$

Since, on RH and for $\Re(s) > 0.5$, (refer to Appendix 3)

$$-\sum_{i=r+1}^{r2} \log\left(1 - \frac{1}{p_i^s}\right) = \sum_{i=r+1}^{r2} \frac{1}{{p_i}^s} + \delta = E_1\left((s-1)\log p_{r+1}\right) - E_1\left((s-1)\log p_{r2}\right) + \varepsilon + \delta$$

where $\varepsilon = O\left(\frac{t+1}{(\sigma-0.5)^2}\, {p_r}^{0.5-\sigma}\log p_r\right)$ and $\delta = O(p_r^{1-2\sigma}/(1-2\sigma))$, therefore

$$\log\zeta(s) = -\sum_{i=1}^{r}\log\left(1-\frac{1}{p_i^s}\right) + E_1\left((s-1)\log p_{r+1}\right) + O\left(\frac{t+1}{(\sigma-0.5)^2}\,{p_{r1}}^{0.5-\sigma}\log p_r\right). \quad (43)$$

Equation (43) represents well the singularity of $\log\zeta(s)$ at $s = 1$ and it allows analytic continuation for values of s with $\Re(s) < 1$. This analytic continuation should extend all the way to the non-trivial zeros with the highest value of σ. Unfortunately, Equation (43) poorly represents $\zeta(s)$ in the vicinity of the non-trivial zeros as the O term grows much faster than the growth of $\log\zeta(s)$ in the vicinity of the simple non-trivial zeros. In the next section, we will use the von Mangoldt function to provide a better representation for $\log\zeta(s)$ in the vicinity of the no-trivial zeros.

5 Partial Euler product functional representation of $\zeta(s)$ using von Mangoldt function.

The derivation of Equation (43) was based on computing the sum $\sum_{i=r1}^{r2} 1/p_i^s$ (Appendix 3) as follows

$$\sum_{i=r1}^{r2}\frac{1}{p_i^s} = \int_{p_{r1}}^{p_{r2}}\frac{d\pi(x)}{x^s} = \int_{p_{r1}}^{p_{r2}}\frac{1}{x^s\log x}dx + \int_{p_{r1}}^{p_{r2}}\frac{1}{x^s}dO\left(\sqrt{x}\log x\right)dx.$$

The above sum can be also computed using the von Mangoldt function $\Lambda(n)$ (where $\Lambda(n) = \log p$, if $n = p^k$ for some prime p and integer $k \geq 1$, otherwise, $\Lambda(n) = 0$) to obtain

$$\sum_{i=r1}^{r2}\frac{1}{p_i^s} = \sum_{n=r1}^{r2}\frac{1}{n^s\log n}\Lambda(n) + \Delta, \quad (44)$$

where Δ is added to eliminate the contribution by the terms of the form m^{-s}, where $m = p^k$ and $2 \leq k < \lfloor\log_2 p_{r2}\rfloor + 1$. In other words, Δ is given by

$$\Delta = \sum_{p_i=\lfloor\sqrt{p_{r1}}\rfloor}^{\lfloor\sqrt{p_{r2}}\rfloor}\frac{1}{2{p_i}^{2s}} + \sum_{p_i=\lfloor\sqrt[3]{p_{r1}}\rfloor}^{\lfloor\sqrt[3]{p_{r2}}\rfloor}\frac{1}{3{p_i}^{3s}} + \ldots + \sum_{p_i=\lfloor\sqrt[L]{p_{r1}}\rfloor}^{\lfloor\sqrt[L]{p_{r2}}\rfloor}\frac{1}{L{p_i}^{Ls}}, \quad (45)$$

where $L = \lfloor\log_2 p_{r2}\rfloor + 1$ and $\lfloor x\rfloor$ is the integer value of x. The order of Δ is determined by the order of the first term $\sum_{p_i=\lfloor\sqrt{p_{r1}}\rfloor}^{\lfloor\sqrt{p_{r2}}\rfloor} 0.5/{p_i}^{2s}$. Thus, the order of Δ can be computed (in the same way the order of δ was computed) to obtain $\Delta = O((\sqrt{p_{r1}})^{1-2\sigma}/(2\sigma-1)) = O(p_{r1}^{0.5-\sigma}/(2\sigma-1))$. Furthermore, if $2\sigma - 1$ is a fixed positive number, then $\Delta = O(p_{r1}^{0.5-\sigma})$. It should be pointed out that for $\sigma = 0.5$ and $t \neq 0$, Δ is convergent to a finite number by the virtue of PNT.

Since the Chebyshev function $\psi(x)$ is given by the following sum

$$\psi(x) = \sum_{n=1}^{x}\Lambda(n)$$

therefore, using the Stieltjes integral, one may write the sum of Equation (44) as the following integral

$$\sum_{i=r1}^{r2}\frac{1}{p_i^s} = \int_{p_{r1}}^{p_{r2}}\frac{1}{x^s\log x}d\psi(x) + \Delta, \quad (46)$$

where $\psi(x)$ is also given by [1]

$$\psi(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} + \sum_{n} \frac{x^{-2n}}{2n} - \frac{\zeta'(0)}{\zeta(0)} \tag{47}$$

It should be pointed out that the first term x in Equation (47) is attributed to the pole of $\zeta(s)$ at $s = 1$, the sum over ρ (or non-travail zeros) is attributed to the non-trivial zeros in the critical strip and the sum over n is attributed to the trivial zeros. Hence, Equation (46) can be written as

$$\sum_{i=r1}^{r2} \frac{1}{p_i^s} = \int_{p_{r1}}^{p_{r2}} \frac{1}{x^s \log x} dx - \int_{p_{r1}}^{p_{r2}} \frac{1}{x^s \log x} d\left(\sum_{\rho} \frac{x^{\rho}}{\rho}\right) + \Delta \tag{48}$$

where the contribution by the last two terms of Equation (47) is negligible compared with the term Δ. In Appendix (3), we have shown that

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^s \log x} dx = E_1((s-1)\log p_{r1}) - E_1((s-1)\log p_{r2}). \tag{49}$$

For the integral with the sum over ρ, we first compute the integral over the ρ's with $|\Im(\rho)| < T$. Thus, we have

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^s \log x} d\left(\sum_{|\Im(\rho)|<T} \frac{x^{\rho}}{\rho}\right) = \sum_{|\Im(\rho)|<T} \left(\int_{p_{r1}}^{p_{r2}} \frac{1}{x^s \log x} d\left(\frac{x^{\rho}}{\rho}\right)\right). \tag{50}$$

For the above integral, for each ρ, $|x^{\rho}/\rho|$ is a continuous function and bounded over the range $p_{r1} \le x \le p_{r2}$, therefore the interchange between the differentiation and summation is justified (alternatively, one may integrate by parts to get the same results, where the sum becomes the integrand and the differentiation is applied to the term $1/(x^s \log x)$ instead of the sum). Furthermore, for each ρ, $\Re(s)$ is higher than $\Re(\rho)$, therefore $\int_{p_{r1}}^{p_{r2}} |x^{\rho-1}/(x^s \log x)| dx$ is convergent as p_{r2} approaches infinity. Hence, the interchange between the integral and the sum is justified. Therefore, Equation (50) can be written as

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^s \log x} d\left(\sum_{|\Im(\rho)|<T} \frac{x^{\rho}}{\rho}\right) = \sum_{|\Im(\rho)|<T} \left(E_1((s-\rho)\log p_{r1}) - E_1((s-\rho)\log p_{r2})\right). \tag{51}$$

In Appendix 4, we have shown that the sum on the right side of (51) is convergent as T approaches infinity. Thus,

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^s \log x} d\left(\sum_{\rho} \frac{x^{\rho}}{\rho}\right) = \sum_{\rho} \left(E_1((s-\rho)\log p_{r1}) - E_1((s-\rho)\log p_{r2})\right). \tag{52}$$

Consequently,

$$\sum_{i=r1}^{r2} \frac{1}{p_i^s} = E_1((s-1)\log p_{r1}) - E_1((s-1)\log p_{r2}) - \sum_{\rho} \left(E_1((s-\rho)\log p_{r1}) - E_1((s-\rho)\log p_{r2})\right) + \Delta, \tag{53}$$

where $\Delta = O(p_{r1}^{0.5-\sigma})$. If the function $J(s, p_{r1}, p_{r2})$ is defined as follows

$$J(s, p_{r1}, p_{r2}) = \sum_{i=r_1}^{r_2} \frac{1}{p_i^s} - E_1((s-1)\log p_{r1}) + E_1((s-1)\log p_{r2}), \tag{54}$$

then

$$J(s,p_{r1},p_{r2}) = \sum_{\rho} \left(E_1((s-\rho)\log p_{r1}) - E_1((s-\rho)\log p_{r2})\right) + \Delta. \tag{55}$$

We notice that the function $J(s,p_{r1},p_{r2})$ is analytic for every p_{r1}, p_{r2} and s. This follows from the fact that although the functions $E_1((s-1)\log p_{r1})$ and $E_1((s-1)\log p_{r2})$ have a branch cut on the negative real axis, the difference does not have a branch cut. Moreover, although the functions $E_1((s-1)\log p_{r1})$ and $E_1((s-1)\log p_{r2})$ have a singularity at $s=1$, the difference has a removable singularity at $s=1$. This follows from the fact that as s approaches 1, the difference can be written as

$$E_1((s-1)\log p_{r1}) - E_1((s-1)\log p_{r2}) = -\log\left((1-s)\log p_{r1}\right) - \gamma + \log\left((1-s)\log p_{r2}\right) + \gamma$$

or,

$$E_1((s-1)\log p_{r1}) - E_1((s-1)\log p_{r2}) = -\log\log p_{r1} + \log\log p_{r2} \tag{56}$$

Therefore, the function $J(s,p_{r1},p_{r2})$ is analytic for every p_{r1}, p_{r2} and s.

Referring to Appendix (4), we notice that for every s with $\Re(s) > \max\Re(\rho)$, the term $\sum_\rho \left(E_1((s-\rho)\log p_{r1}) - E_1((s-\rho)\log p_{r2})\right)$ approaches zero as p_{r1} approaches infinity. Thus, for $\Re(s) > \max\Re(\rho)$, we have

$$\sum_{i=r1}^{r2} \frac{1}{p_i^s} = E_1((s-1)\log p_{r1}) - E_1((s-1)\log p_{r2}) + O(p_{r1}^{-\sigma+\max\Re(\rho)}). \tag{57}$$

To compute $\log\zeta(s)$ using Equation (47), we recall Equation (41) of Theorem 1. Thus, for every s with $\Re(s) > \max\Re(\rho)$, we have

$$\log\zeta(s) = E_1\left((s-1)\log p_{r2+1}\right) - \sum_{i=1}^{r2} \log\left(1-\frac{1}{p_i^s}\right),$$

where the equality of both sides is attained as p_{r2} approaches infinity. Alternatively,

$$\log\zeta(s) = E_1\left((s-1)\log p_{r2+1}\right) - \sum_{i=1}^{r} \log\left(1-\frac{1}{p_i^s}\right) - \sum_{i=r+1}^{r2} \log\left(1-\frac{1}{p_i^s}\right).$$

Hence,

$$\log\zeta(s) = E_1\left((s-1)\log p_{r2+1}\right) - \sum_{i=1}^{r} \log\left(1-\frac{1}{p_i^s}\right) + \sum_{i=r+1}^{r2} \frac{1}{{p_i}^s} + \delta.$$

Consequently, using Equations (46), (48), (49) and (52) (and noting that when $\Re(s-\rho) > 0$ for every ρ, the sum $\sum_\rho E_1\left((s-\rho)\log p_{r2}\right)$ approaches zero as p_{r2} approaches infinity), we have the following theorem

Theorem 5 *If $\Re(s-\rho) > 0$ for every non-trivial zero ρ, then*

$$\log\zeta(s) = -\sum_{i=1}^{r} \log\left(1-\frac{1}{p_i^s}\right) + E_1\left((s-1)\log p_{r+1}\right) - \sum_{\rho} E_1\left((s-\rho)\log p_{r+1}\right) + O\left({p_r}^{0.5-\sigma}\right). \tag{58}$$

where $\sigma = \Re(s)$ and the O term is given by $\delta + \Delta$.

The differentiation of $\log\zeta(s)$ or $\zeta'(s)/\zeta(s)$ has been extensively used in the analysis of the Riemann zeta function. Using Equation (58), we may obtain a functional representation of $\zeta'(s)/\zeta(s)$ in terms of the partial Euler product of $\zeta(s)$.

Theorem 6 *If $\Re(s-\rho) > 0$ for every non-trivial zero ρ, then*

$$\frac{\zeta'(s)}{\zeta(s)} = -\frac{\mathrm{d}}{\mathrm{d}s}\left(\log\prod_{i=1}^{r-1}\left(1-\frac{1}{p_i^s}\right)\right) - \frac{{p_r}^{-(s-1)}}{s-1} + \sum_{\rho}\frac{{p_r}^{-(s-\rho)}}{s-\rho} + O\left({p_r}^{0.5-\sigma}\right). \qquad (59)$$

where $\sigma = \Re(s)$ and the O term is given by $d(\delta+\Delta)/ds$.

Although Theorems (4), (5) and (6) provide a functional representation for $\zeta(s)$ in terms of it partial Euler product, our attempts to prove or disprove the Riemann hypothesis using these representations in conjunction with other properties (such as the growth of $\zeta(1+iT)$ with T) have failed. However, the sum $\sum_{p_{r1}\le p_i\le p_{r2}} 1/{p_i}^{\sigma}$ for $\sigma < 1$ (that was computed using these theorems) has been successfully used to examine the convergence of the series $M(\sigma)$ for $\sigma < 1$ as described in the next section.

6 The convergence of the series $M(\sigma, p_r)$ and $M(\sigma)$ for $\sigma \le 1$.

In this section, we will first provide an estimate for the partial sum $M(1, p_r; 1, {p_r}^a)$ as a approaches infinity. This estimate will be computed by using Equation (57) and noting that $M(1, p_r)$ equals zero for every p_r. Therefore for every p_r, $M(1, p_r; 1, {p_r}^a)$ approaches zero as a approaches infinity. We also notice that, for every p_r and N, we have (see Appendix (5)),

$$\left|\sum_{n=1}^{N}\frac{\mu(n,p_r)}{n}\right| \le 2$$

The estimation of the partial sum $M(1, p_r; 1, {p_r}^a)$ as a function of a will then be used to establish a relationship between $M(1, p_r; 1, {p_r}^a)$ and $M(\sigma, p_r; 1, {p_r}^a)$. This relationship is then used to show that $M(\sigma, p_r)$ and $M(\sigma)$ diverge for $\sigma < 1$. We will describe the details of our method in the following steps.

- In the first step, we will show that, for every a and as p_r approaches infinity, the partial sum $M(1, p_r; 1, {p_r}^a)$ is as a function of only a (independent of p_r).

Toward this end, we define the function $f(a, p_r)$ as

$$f(a,p_r) = M(1,p_r;1,{p_r}^a) = \sum_{n=1}^{{p_r}^a}\frac{\mu(n,p_r)}{n}.$$

We will then show that, for every a and as p_r approaches infinity, the function $f(a, p_r)$ approaches a deterministic function $F(a)$. In other words; if we plot $M(1, p_r; 1, N)$ (where $N = {p_r}^a$) as a function of $a = \log N/\log p_r$, then for each value of a and as p_r approaches infinity, $f(a, p_r)$ approaches a unique value $F(a)$. This result can be achieved by first dividing the prime numbers that are in the range $p_r < x \le {p_r}^2$ into N sections. The first section comprises of all the prime numbers that are in the range $p_r < x \le {p_r}^{1+\delta}$ (where, $\delta << 1$ and it is given by $\delta = 1/(\log p_r)^\alpha, \alpha > 1$ and $(\log p_r)^\alpha << p_r$). The second section comprises of all the prime numbers that are in the range in the range ${p_r}^{1+\delta} < x \leq {p_r}^{1+2\delta}$ and so on (where the j-th section comprises of all the prime numbers that are in the range ${p_r}^{1+(j-1)\delta} < x \leq {p_r}^{1+j\delta}$). Hence,

$$N\delta = 1. \tag{60}$$

The process of dividing the prime numbers into sections continues for primes greater than ${p_r}^2$. Thus, the total number of sections L over the range $p_r < x \leq {p_r}^a$ is given by $(a-1)N$.

If we define K_i as the sum of the reciprocals of the prime numbers in section j (where $i = j + N$), then by Mertens' Theorem, K_i is given by

$$K_i = \log\log {p_r}^{(i+1)\delta} - \log\log {p_r}^{i\delta} + \frac{O(1/\log p_r)}{i},$$

where $1 \leq i\delta \leq a$. Hence, for sufficiently large p_r, we then have

$$K_i = \frac{1}{i} + \frac{1}{i}O(1/\log p_r) + O\left(1/i^2\right), \tag{61}$$

where $O(1/\log p_r)$ can be made arbitrary small by selecting p_r arbitrary large. Therefore, we may consider that each K_i is comprised of two terms. The first one is a deterministic or regular term defined as G_i and it is given by $1/i$. The second one is an irregular term defined R_i and it is the remaining part of K_i (i.e. the irregular term R_i is given by $K_i - D_i$). Hence,

$$K_i = G_i + R_i, \tag{62}$$

where

$$G_i = \frac{1}{i}, \tag{63}$$

and

$$R_i = K_i - G_i = \frac{1}{i}O(1/\log p_r) + O\left(1/i^2\right). \tag{64}$$

Notice that although $O\left(1/i^2\right)$ behaves regularly, we lumped it with the irregular term R_i due to its negligible effect on the partial sum $M(1, p_r; 1, {p_r}^a)$ as p_r (and consequently N) approaches infinity.

Next, we will device an algorithm to construct a series that is equivalent to the series $M(1, p_r; 1, {p_r}^a)$ from these $(a-1)N$ sections (that are comprised of the prime numbers with their associated values of K_i's) and the products of K_i's (with the appropriate signs). This series starts with the number 1. Then, instead of subtracting the terms $1/p_r, 1/p_{r+1}, \ldots$, we subtract the values of K_i's for the first N sections. These sections are ordered based on the value of the largest member within each section. It can be easily shown that the value of $M(1, p_r; 1, {p_r}^2)$ constructed by this method is given $1 - \log 2$ plus a factor that is determined by the sum of N terms of the form $(1/i)O(1/\log p_r) + O(1/i^2)$ and this factor (as mentioned earlier) can be made arbitrary small by selecting p_r arbitrary large. In other words;

$$M(1, p_r; 1, {p_r}^2) = 1 - \sum_{i=N}^{2N} K_i,$$

and

$$\lim_{p_r,N\to\infty} M(1,p_r;1,{p_r}^2) = 1 - \lim_{N\to\infty}\sum_{i=N}^{2N}\frac{1}{i} = 1 - \lim_{N\to\infty}\sum_{j=0}^{N}\frac{1}{N+j}.$$

Thus

$$\lim_{p_r,N\to\infty} M(1,p_r;1,{p_r}^2) = 1 - \int_0^1 \frac{1}{1+x}dx = 1-\log 2$$

The terms of the series $M(1,p_r;1,{p_r}^a)$ in the range $p_r \leq x < {p_r}^3$ are either a reciprocal of a prime or a reciprocal of the product of two primes. To reconstruct these terms, we start with 1 and subtract the sum of K_i's for the sections of primes in the range $p_r \leq x < {p_r}^3$ and then add to it the sum of the terms that are the product of K_{i1}'s and K_{i2}'s for any two sections of the prime numbers (where the product of any member of the one section with any member of the second section is less than ${p_r}^3$). Hence

$$M(1,p_r;1,{p_r}^3) = 1 - \sum_{i=N}^{3N} K_i + \frac{1}{2}\sum_{i=N}^{2N}\left(K_{3N-i}\sum_{j=N}^{i} K_j\right)$$

where the factor of 1/2 was added to the last term since each term of the form $1/(p_{j1}p_{j2})$ is repeated twice. As p_r (and consequently N) approach infinity, we have

$$\lim_{p_r,N\to\infty} M(1,p_r;1,{p_r}^3) = 1 - \lim_{N\to\infty}\sum_{i=N}^{3N}\frac{1}{i} + \frac{1}{2}\lim_{N\to\infty}\sum_{i=N}^{2N}\left(\frac{1}{3N-i}\sum_{j=N}^{i}\frac{1}{j}\right).$$

The above equation can be easily computed using integration methods. Hence, as p_r (and consequently N) approaches infinity, the partial sum $M(1,p_r;1,{p_r}^a)$ for $a \leq 3$ is independent of p_r

Similarly, the terms of the series $M(1,p_r;1,{p_r}^a)$ in the range $p_r \leq x < {p_r}^4$ are either a reciprocal of a prime, a reciprocal of the product of two primes or a reciprocal of the product of three primes. To reconstruct these terms, we start with 1 and subtract the sum of K_i's for the sections of primes in the range $p_r \leq x < {p_r}^4$ and then add to it the sum of the terms that are the product of K_i's and K_j's for any two sections of the prime numbers (where the product of any member of the one section with any member of the second section is less than ${p_r}^4$). Finally, we subtract the sum of the terms that are the product of K_{i1}'s, K_{i2}'s and K_{i3}'s for any three sections of the prime numbers (where the product of any member of the one section with any member of the second section and any member of the third section is less than ${p_r}^4$). Following the same previous method, we can easily show that , as p_r (and N) approaches infinity, the partial sum $M(1,p_r;1,{p_r}^a)$ for $a \leq 4$ is independent of p_r.

This process is repeated $a-1$ times to show that, as p_r (and consequently N) approaches infinity, the partial sum $M(1,p_r;1,{p_r}^a)$ is dependent on only a. Hence for every a, we have

$$F(a) = \lim_{p_r\to\infty} M(1,p_r;1,{p_r}^a) \tag{65}$$

It should be pointed out that the series constructed by this algorithm includes both square-free terms (that form $M(1,p_r;1,{p_r}^a)$) as well as non square-free terms. In the following, we will show that, for every a and as p_r approaches infinity, the contribution by the non square-free terms to the partial sum $M(1, p_r; 1, {p_r}^a)$ approaches zero as well. Toward this end, let S_0 be the sum of the terms with the factor $1/p_r^2$. Let S_1 be the sum of the remaining terms with the factor $1/(p_{r+1})^2$, S_2 be the sum of the remaining terms with the factor $1/(p_{r+2})^2$, and so on. Let H be sum of all the terms associated with non square-free terms. Thus, H is given by

$$H = \frac{1}{{p_r}^2}S_0 + \frac{1}{{p_{r+1}}^2}S_1 + ... + \frac{1}{{p_{r+l}}^2}S_L,$$

where p_{r+l} is the largest prime that its square is less than ${p_r}^a$. However,

$$|S_0|, |S_1|, ..., |S_l| < 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{{p_r}^a}.$$

Thus,

$$|S_0|, |S_1|, ..., |S_l| = O(a \log p_r).$$

Therefore

$$H = \left(\frac{1}{{p_r}^2} + \frac{1}{{p_{r+1}}^2} + ... + \frac{1}{{p_{r+l}}^2}\right) O(a \log p_r).$$

Hence, the contribution by the non square free terms H is given by,

$$H = O(a \log p_r / p_r).$$

Consequently, for every a and as p_r approaches infinity, H (or the contribution by the non-square free terms to the partial sum $M(1, p_r; 1, {p_r}^a)$) approaches zero.

- In the second step, we write the partial sum $M(1, p_r; 1, {p_r}^a)$ as the sum of two components. The first is the deterministic or regular component and it is given by $F(a)$. The second one is the irregular component $M_r(1, p_r; 1, {p_r}^a)$ given by $M(1, p_r; 1, {p_r}^a) - F(a)$. We will show that $F(a)$ decays slower that $1/a$ (this is the key step to disproving the Riemann Hypotheses as RH requires an exponential decay of $F(a)$).

  Toward this end, we write the partial sum $M(1, p_r; 1, {p_r}^a)$ as follows

$$M(1, p_r; 1, {p_r}^a) = 1 - \frac{1}{2}K_N M(1, p_r; 1, {p_r}^{a-1}) - \frac{1}{2}K_N - \frac{1}{2}K_{N+1}M(1, p_r; 1, {p_r}^{a-1-\delta}) - \frac{1}{2}K_{N+1}$$
$$-\frac{1}{2}K_{N+2}M(1, p_r; 1, {p_r}^{a-1-2\delta}) - \frac{1}{2}K_{N+2} - ... - \frac{1}{2}K_{(a-1)N-1}M(1, p_r; 1, {p_r}^{1+\delta}) - \frac{1}{2}K_{(a-1)N-1}$$
$$-K_{(a-1)N} - K_{(a-1)N+1} - ... - K_{aN} - H,$$

or

$$M(1, p_r; 1, {p_r}^a) = 1 - \frac{1}{2}\left(\sum_{i=N}^{(a-1)N-1} K_i M(1, p_r; 1, {p_r}^{a-i/N}) + \sum_{i=N}^{(a-1)N-1} K_i\right) - \sum_{i=(a-1)N}^{aN} K_i - H \tag{66}$$

where the factor of 1/2 was added to each of the products $K_i M(1, p_r; 1, {p_r}^{a-i\delta})$'s since each term (except those of the form $1/p_j$) is repeated twice. Since the terms $1/p_j$'s are not repeated, therefore we added the terms $\frac{1}{2}K_i$'s (for $N \le i < (a-1)N$). We have also added the term H to account for the non square free terms generated by the products $K_i M(1, p_r; 1, {p_r}^{a-i\delta})$'s (where H approaches zero as p_r approaches infinity). As p_r approaches infinity, $M(1, p_r; 1, {p_r}^x)$ approaches the function $F(x)$. We also have

$$\lim_{p_r, N\to\infty} \sum_{i=N}^{(a-1)N-1} K_i = \lim_{N\to\infty} \sum_{i=N}^{(a-1)N-1} \frac{1}{i} = \int_0^{a-2} \frac{1}{1+x} dx = \log(a-1),$$

$$\lim_{p_r, N\to\infty} \sum_{i=(a-1)N}^{aN} K_i = \lim_{N\to\infty} \sum_{i=(a-1)N}^{aN} \frac{1}{i} = \int_{a-2}^{a-1} \frac{1}{1+x} dx = \log a - \log(a-1) = \frac{1}{a} + O(1/a^2),$$

and

$$\lim_{p_r, N\to\infty} \sum_{i=N}^{(a-1)N-1} K_i M(1, p_r; 1, {p_r}^{a-i/N}) = \lim_{N\to\infty} \sum_{i=N}^{(a-1)N-1} \frac{F(a-i/N)}{i} = \int_{a-1}^{1} \frac{F(x)}{a-x} dx.$$

Hence, for sufficiently large a, we have

$$F(a) = 1 - \frac{1}{2} \int_{a-1}^{1} \frac{F(x)}{a-x} dx - \frac{1}{2} \log(a-1) - \frac{1}{a} + O(1/a^2), \tag{67}$$

In the following we will show that $F(a) << \log(a)$. This task is achieved by noting that $F(a)$ is also given by

$$F(a) = M(1, p_r; 1, {p_r}^a) - M_r(1, p_r; 1, {p_r}^a)$$

where $M_r(1, p_r; 1, {p_r}^a)$ is given by

$$M_r(1, p_r; 1, {p_r}^a) = \sum_{i=N}^{(a-1)N-1} R_i M(1, p_r; 1, {p_r}^{a-i/N}).$$

Since R_i is given by $O(1/(i \log p_r))$ and $|M_r(1, p_r; 1, {p_r}^x)| \le 2$, hence

$$M_r(1, p_r; 1, {p_r}^a) = O(\log a/p_r) \tag{68}$$

Thus, for sufficiently large a we have

$$\int_1^{a-1} \frac{F(x)}{a-x} dx = \log(a-1). \tag{69}$$

Hence, we conclude that $F(a)$ approaches zero at a rate that is no faster than the rate at which $1/a$ approaches zero. In other words;. $F(a)$ decays no faster than $1/a$. Therefore, for some constant C, we have

$$F(a) > C/a$$

or

$$F(a) = \Omega(1/a)$$

Thus, if we write the partial sum of $M(1, p_r)$ as $M(1, p_r, 1, n)$ (where $n = p_r^a$), then the regular component of $M(1, p_r, 1, n)$ decays slower than $C/\log n$. This suggests (as we will show in the next step) that not only is the Riemann Hypothesis invalid but also the non trivial zeros can be found arbitrary close to the line $s = 1$

- For the third step, we will compute the partial sum $M(1, p_r; 1, {p_r}^a)$ for $\sigma < 1$ and show that it diverges as a approaches infinity.

Toward this end, we follow the same algorithm (that we used for the construction of $M(1, p_r; 1, {p_r}^a)$) to construct the series $M(\sigma, p_r; 1, {p_r}^a)$. For this case, on RH, K_i is given by (refer to Equation (57))

$$K_i = \sum_{{p_r}^{i\delta} < p_j < {p_r}^{(i+1)\delta}} \frac{1}{p_j^\sigma} = E_1\left((\sigma-1)\log {p_r}^{i\delta}\right) - E_1\left((\sigma-1)\log {p_r}^{(i+1)\delta}\right) + O(p_r^{-\sigma+0.5}).$$

Using the following asymptotic representation of the Exponential Integral

$$E_1(z) = \frac{e^{-z}}{z}\left(1 + O\left(\frac{1}{z}\right)\right),$$

we then obtain

$$E_1\left((\sigma-1)\log {p_r}^{i\delta}\right) = -\frac{e^{(1-\sigma)i\delta\log p_r}}{(1-\sigma)i\delta\log p_r}\left(1 + O\left(\frac{1}{i\log p_r}\right)\right),$$

and

$$E_1\left((\sigma-1)\log {p_r}^{(i+1)\delta}\right) = -\frac{e^{(1-\sigma)(i+1)\delta\log p_r}}{(1-\sigma)(i+1)\delta\log p_r}\left(1 + O\left(\frac{1}{i\log p_r}\right)\right).$$

Hence,

$$K_i = C\frac{{p_r}^{(1-\sigma)i\delta}}{i}\left(1 + O\left(\frac{1}{i\log p_r}\right)\right) + O(p_r^{-\sigma+0.5}),$$

where

$$C = \frac{{p_r}^{(1-\sigma)\delta} - 1}{(1-\sigma)\delta\log p_r}\left(1 + O\left(\frac{1}{i\log p_r}\right)\right).$$

To simplify the analysis and without loss of generality, we set $C = 1$ (one way to achieve this is by setting $\delta = 1/(\log p_r)^2$. As p_r approaches infinity, $(1-\sigma)\delta\log p_r$ approaches zero and C approaches 1. The same analysis described below can be also used for the general case to obtain the same results). Hence, as p_r approaches infinity, we have

$$K_i = \frac{{p_r}^{(1-\sigma)i\delta}}{i} \tag{70}$$

Next, we will show that,

$$M\left(\sigma; {p_r}^{J\delta}, {p_r}^{(J+1)\delta}\right) = {p_r}^{J(1-\sigma)\delta} M\left(1; {p_r}^{J\delta}, {p_r}^{(J+1)\delta}\right)$$

where, $J > N$ and $N\delta = 1$. Toward this end, we define $M_1\left(1; {p_r}^{J\delta}, {p_r}^{(J+1)\delta}\right)$ as the sum of the terms of the form $1/p_j$ in the interval $[{p_r}^{J\delta}, {p_r}^{(J+1)\delta}]$ and we define $M_1\left(\sigma; {p_r}^{J\delta}, {p_r}^{(J+1)\delta}\right)$ as the sum of the terms of the form $1/{p_i}^{\sigma}$ in the same interval. We also define $M_2\left(1; {p_r}^{J\delta}, {p_r}^{(J+1)\delta}\right)$ as the sum of the terms of the form $1/(p_{j1}p_{j2})$ in the interval $[{p_r}^{J\delta}, {p_r}^{(J+1)\delta}]$ and we define $M_2\left(\sigma; {p_r}^{J\delta}, {p_r}^{(J+1)\delta}\right)$ as the sum of the terms of the form $1/(p_{j1}p_{j2})^{\sigma}$ in the same interval and so on. Hence,

$$M\left(1; {p_r}^{J\delta}, {p_r}^{(J+1)\delta}\right) = -M_1\left(1; {p_r}^{J\delta}, {p_r}^{(J+1)\delta}\right) + M_2\left(1; {p_r}^{J\delta}, {p_r}^{(J+1)\delta}\right) - M_3\left(1; {p_r}^{J\delta}, {p_r}^{(J+1)\delta}\right) + ...$$

and

$$M\left(\sigma; {p_r}^{J\delta}, {p_r}^{(J+1)\delta}\right) = -M_1\left(\sigma; {p_r}^{J\delta}, {p_r}^{(J+1)\delta}\right) + M_2\left(\sigma; {p_r}^{J\delta}, {p_r}^{(J+1)\delta}\right) - M_3\left(\sigma; {p_r}^{J\delta}, {p_r}^{(J+1)\delta}\right) + ...$$

The term $M_1\left(1; {p_r}^{J\delta}, {p_r}^{(J+1)\delta}\right)$ is given by

$$M_1\left(1; {p_r}^{J\delta}, {p_r}^{(J+1)\delta}\right) = K_i = \frac{1}{J}.$$

Similarly,

$$M_1\left(\sigma; {p_r}^{J\delta}, {p_r}^{(J+1)\delta}\right) = \frac{{p_r}^{J(1-\sigma)\delta}}{J}.$$

Hence,

$$M_1\left(\sigma; {p_r}^{J\delta}, {p_r}^{(J+1)\delta}\right) = {p_r}^{J(1-\sigma)\delta} M_1\left(\sigma; {p_r}^{J\delta}, {p_r}^{(J+1)\delta}\right).$$

The term $M_2\left(1; {p_r}^{i\delta}, {p_r}^{(i+1)\delta}\right)$ is given by

$$M_2\left(1; {p_r}^{J\delta}, {p_r}^{(J+1)\delta}\right) = \frac{1}{2}\sum_{j=N}^{J} K_{J-j}K_j = \frac{1}{2}\sum_{j=N}^{J} \frac{1}{J-j}\frac{1}{j}$$

where the factor of 1/2 is added since each term of the form $1/p_{j1}p_{j2}$ is repeated twice in the above sum. The term $M_2\left(\sigma; {p_r}^{i\delta}, {p_r}^{(i+1)\delta}\right)$ is also given by

$$M_2\left(\sigma; {p_r}^{J\delta}, {p_r}^{(J+1)\delta}\right) = \frac{1}{2}\sum_{j=N}^{J} \frac{{p_r}^{(J-j)(1-\sigma)\delta}}{J-j}\frac{{p_r}^{j(1-\sigma)\delta}}{j}.$$

Hence

$$M_2\left(\sigma; {p_r}^{J\delta}, {p_r}^{(J+1)\delta}\right) = {p_r}^{J(1-\sigma)\delta} M_2\left(\sigma; {p_r}^{J\delta}, {p_r}^{(J+1)\delta}\right).$$

The term $M_3\left(1; {p_r}^{i\delta}, {p_r}^{(i+1)\delta}\right)$ is given by

$$M_3\left(1; {p_r}^{J\delta}, {p_r}^{(J+1)\delta}\right) = \frac{1}{2}\sum_{j=N}^{J} K_{J-j} M_2\left(1; {p_r}^{j\delta}, {p_r}^{(j+1)\delta}\right),$$

or

$$M_3\left(1; {p_r}^{J\delta}, {p_r}^{(J+1)\delta}\right) = \frac{1}{2}\sum_{j=N}^{J} \frac{1}{J-j} M_2\left(1; {p_r}^{j\delta}, {p_r}^{(j+1)\delta}\right).$$

Similarly, $M_3\left(\sigma; {p_r}^{i\delta}, {p_r}^{(i+1)\delta}\right)$ is given by

$$M_3\left(\sigma; {p_r}^{J\delta}, {p_r}^{(J+1)\delta}\right) = \frac{{p_r}^{J(1-\sigma)\delta}}{2}\sum_{j=N}^{J} \frac{1}{J-j} M_2\left(\sigma; {p_r}^{j\delta}, {p_r}^{(j+1)\delta}\right).$$

Hence,

$$M_3\left(\sigma; {p_r}^{J\delta}, {p_r}^{(J+1)\delta}\right) = {p_r}^{J(1-\sigma)\delta} M_3\left(1; {p_r}^{J\delta}, {p_r}^{(J+1)\delta}\right).$$

Repeating the process i times, we then obtain

$$M_i\left(\sigma; {p_r}^{J\delta}, {p_r}^{(J+1)\delta}\right) = {p_r}^{J(1-\sigma)\delta} M_i\left(1; {p_r}^{J\delta}, {p_r}^{(J+1)\delta}\right).$$

Consequently,

$$M\left(\sigma; {p_r}^{J\delta}, {p_r}^{(J+1)\delta}\right) = {p_r}^{J(1-\sigma)\delta} M\left(1; {p_r}^{J\delta}, {p_r}^{(J+1)\delta}\right). \tag{71}$$

The term $M\left(1; {p_r}^{J\delta}, {p_r}^{(J+1)\delta}\right)$ converges to zero by the virtue of the convergence of $M(1, p_r)$. However, the term ${p_r}^{J(1-\sigma)\delta}$ grows at a rate faster than the rate $M\left(1; {p_r}^{J\delta}, {p_r}^{(J+1)\delta}\right)$ decays to zero (where $M(1; 1, {p_r}^{(J+1)\delta}) > C/(J\delta)$). Therefore, the term $M\left(\sigma; {p_r}^{J\delta}, {p_r}^{(J+1)\delta}\right)$ does not converge to zero as J approaches infinity. Consequently, the series $M(\sigma, p_r)$ and $M(\sigma)$ diverge for $\sigma < 1$. This implies that the Riemann Hypothesis is invalid and the zeros can be found arbitrary close to line $\Re(s) = 1$.

Appendix 1

To prove the first part of Theorem 1 (i.e. for $s = \sigma + i0$ and $0.5 < \sigma \leq 1$, the series $M(\sigma, p_r)$ converges conditionally if $M(\sigma)$ converges conditionally), we first start with proving that $M(\sigma, 2)$ is convergent if $M(\sigma)$ is convergent. Since $M(\sigma)$ is convergent, then for any arbitrary small number δ, there exists an integer N_0 such that for every integer $N > N_0$

$$|M(\sigma; N, \infty)| = \left|\sum_{n=N}^{\infty} \frac{\mu(n)}{n^\sigma}\right| < \delta$$

Let the sums $M(\sigma; 1, N)$, $M(\sigma; N+1, 2N)$, $M(\sigma; 2N+1, 2^2N)$, $M(\sigma; 2^2N+1, 2^3N)$, ..., $M(\sigma; 2^{L-1}N+1, 2^LN)$ be defined as

$$M(\sigma; 1, N) = \sum_{n=1}^{N} \frac{\mu(n)}{n^\sigma} = A_1,$$

$$M(\sigma; N+1, 2N) = \sum_{n=N+1}^{2N} \frac{\mu(n)}{n^\sigma} = \delta_1,$$

$$M(\sigma; 2N+1, 2^2N) = \sum_{n=2N+1}^{2^2N} \frac{\mu(n)}{n^\sigma} = \delta_2,$$

$$M(\sigma; 2^2N+1, 2^3N) = \sum_{n=2^2N+1}^{2^3N} \frac{\mu(n)}{n^\sigma} = \delta_3,$$

$$M(\sigma; 2^{L-1}N+1, 2^LN) = \sum_{n=2^{L-1}N+1}^{2^LN} \frac{\mu(n)}{n^\sigma} = \delta_{L-1},$$

If we define $\delta(l)$ as the maximum of $|\delta_l|, |\delta_{l+1}|, |\delta_{l+2}|, ..., |\delta_{L-1}|, |\delta_l+\delta_{l+1}|, |\delta_l+\delta_{l+1}+\delta_{l+2}|, ..., |\delta_l+\delta_{l+1}+...+\delta_{L-1}|$, then by the virtue of the convergence of $M(\sigma)$,

$$|\delta_1|, |\delta_2|, |\delta_3|, ..., |\delta_{L-1}|, |\delta_1+\delta_2|, |\delta_1+\delta_2+\delta_3|, ..., |\delta_1+\delta_2+\delta_3+...+\delta_{L-1}| \leq \delta(1) \leq 2\delta.$$

We also have

$$|\delta_l|, |\delta_{l+1}|, |\delta_{l+2}|, ..., |\delta_{L-1}|, |\delta_l+\delta_{l+1}|, |\delta_l+\delta_{l+1}+\delta_{l+2}|, ..., |\delta_l+\delta_{l+1}+...+\delta_{L-1}| \leq \delta(l),$$

where by the virtue of the convergence of $M(\sigma)$, $\delta(l)$ approaches zero as l approaches infinity.

Furthermore, let the sums $M(\sigma, 2; 1, N)$, $M(\sigma, 2; N+1, 2N)$, $M(\sigma, 2; 2N+1, 2^2N)$, $M(\sigma, 2; 2^2N+1, 2^3N)$, ..., $M(\sigma, 2; 2^{L-1}N+1, 2^LN)$ be defined as

$$M(\sigma, 2; 1, N) = \sum_{n=1}^{N} \frac{\mu(n,2)}{n^\sigma} = B_1,$$

$$M(\sigma, 2; N+1, 2N) = \sum_{n=N+1}^{2N} \frac{\mu(n,2)}{n^\sigma} = \epsilon_1,$$

$$M(\sigma, 2; 2N+1, 2^2N) = \sum_{n=2N+1}^{2^2N} \frac{\mu(n,2)}{n^\sigma} = \epsilon_2,$$

$$M(\sigma, 2; 2^2N+1, 2^3N) = \sum_{n=2^2N+1}^{2^3N} \frac{\mu(n,2)}{n^\sigma} = \epsilon_3,$$

$$M(\sigma, 2; 2^{L-1}N+1, 2^LN) = \sum_{n=2^{L-1}N+1}^{2^LN} \frac{\mu(n,2)}{n^\sigma} = \epsilon_{L-1},$$

Since

$$\sum_{n=1}^{2N} \frac{\mu(n)}{n^\sigma} = \sum_{n=1}^{2N} \frac{\mu(n,2)}{n^\sigma} - \sum_{n=1}^{N} \frac{\mu(n,2)}{(2n)^\sigma},$$

thus

$$M(\sigma; 1, 2N) = M(\sigma, 2; 1, 2N) - \frac{1}{2^\sigma} M(\sigma, 2; 1, N).$$

Similarly, since

$$\sum_{n=2^lN+1}^{2^{l+1}N}\frac{\mu(n)}{n^\sigma}=\sum_{n=2^lN+1}^{2^{l+1}N}\frac{\mu(n,2)}{n^\sigma}-\sum_{n=2^{l-1}N+1}^{2^lN}\frac{\mu(n,2)}{(2n)^\sigma},$$

thus

$$M(\sigma;2^lN+1,2^{l+1}N)=M(\sigma,2;2^lN+1,2^{l+1}N)-\frac{1}{2^\sigma}M(\sigma,2;2^{l-1}N+1,2^lN).$$

Rearranging the previous equations, we then have

$$A_1+\delta_1=B_1+\epsilon_1-\frac{1}{2^\sigma}B_1, \tag{72}$$

$$\delta_2=\epsilon_2-\frac{1}{2^\sigma}\epsilon_1,$$

$$\delta_3=\epsilon_3-\frac{1}{2^\sigma}\epsilon_2,$$

$$\delta_{L-1}=\epsilon_{L-1}-\frac{1}{2^\sigma}\epsilon_{L-2},$$

where $|\delta_1|,|\delta_2|,|\delta_3|,...,|\delta_{L-1}|,|\delta_1+\delta_2|,|\delta_1+\delta_2+\delta_3|,|\delta_1+\delta_2+\delta_3+...+\delta_{L-1}|\le\delta(1)\le 2\delta$ and δ is arbitrary small. Hence

$$\epsilon_2=\frac{1}{2^\sigma}\epsilon_1+\delta_2,$$

$$\epsilon_3=\frac{1}{2^\sigma}\epsilon_2+\delta_3=\frac{1}{2^{2\sigma}}\epsilon_1+\frac{1}{2^\sigma}\delta_2+\delta_3,$$

$$\epsilon_{L-1}=\frac{1}{2^\sigma}\epsilon_{L-2}+\delta_{L-1}=\frac{1}{2^{(L-2)\sigma}}\epsilon_1+\frac{1}{2^{(L-3)\sigma}}\delta_2+\frac{1}{2^{(L-4)\sigma}}\epsilon_3+...+\delta_{L-1}.$$

Therefore,

$$\epsilon_1+\epsilon_2+\epsilon_2+...+\epsilon_{L-1}=\left(1+\frac{1}{2^\sigma}+\frac{1}{2^{2\sigma}}+...+\frac{1}{2^{(L-1)\sigma}}\right)\epsilon_1+(\delta_2+\delta_3+...+\delta_{L-1})+$$

$$\frac{1}{2^\sigma}(\delta_2+\delta_3+...+\delta_{L-2})+\frac{1}{2^{2\sigma}}(\delta_2+\delta_3+...+\delta_{L-3})+...+\frac{1}{2^{(L-3)\sigma}}\delta_2.$$

Since $|\delta_2|\le\delta(1)$, $|\ |\delta_2+\delta_3|\le\delta(1)$, ..., $|\delta_1+\delta_2+\delta_3+...+\delta_{L-1}|\le\delta(1)$, hence

$$|\delta_2+\delta_3+...+\delta_{L-1}|+\frac{1}{2^\sigma}|\delta_2+\delta_3+...+\delta_{L-2}|+...+\frac{1}{2^{(L-3)\sigma}}|\delta_2|\le\left|\delta(1)+\frac{1}{2^\sigma}\delta(1)+...+\frac{1}{2^{(L-3)\sigma}}\delta(1)\right|,$$

or

$$|\delta_2+\delta_3+...+\delta_{L-1}|+\frac{1}{2^\sigma}|\delta_2+\delta_3+...+\delta_{L-2}|+...+\frac{1}{2^{(L-3)\sigma}}|\delta_2|\le\frac{2^\sigma}{2^\sigma-1}|\delta(1)|.$$

Therefore

$$\epsilon_1+\epsilon_2+\epsilon_3+...+\epsilon_{L-1}=\left(1+\frac{1}{2^\sigma}+\frac{1}{2^{2\sigma}}+...+\frac{1}{2^{(L-2)\sigma}}\right)\epsilon_1+\gamma_1,$$

where γ_1 is of the same order as that of $\delta(1)$.

As L approaches infinity, we then obtain

$$\sum_{i=1}^{\infty}\epsilon_i=\frac{2^\sigma}{2^\sigma-1}\epsilon_1+\gamma_1.$$

Therefore, the sum $M(\sigma,2;N+1,\infty)$ (which is equal to $\epsilon_1+\epsilon_2+\epsilon_3+...$) is bounded by the sum $M(\sigma,2;N+1,2N)$ (which is equal to ϵ_1).

The previous process can be repeated with $2N$ is substituted for N and Equation (72) becomes

$$A_2+\delta_2=B_2+\epsilon_2-\frac{1}{2^\sigma}B_2,$$

where $A_2=M(\sigma;1,2N)$ and $B_2=M(\sigma,2;1,2N)$. Thus,

$$A_2=B_2-\frac{1}{2^\sigma}B_2+\frac{1}{2^\sigma}\epsilon_1.$$

Following the same process, we can show that that the sum $M(\sigma,2;2N+1,\infty)$ is given by

$$\sum_{i=2}^{\infty}\epsilon_i=\frac{1}{2^\sigma-1}\epsilon_1+\gamma_2.$$

where γ_2 is of the same order as that of $\delta(2)$.

If we repeat the process l times, we obtain

$$A_l=B_l-\frac{1}{2^\sigma}B_l+\frac{1}{2^{(l-1)\sigma}}\epsilon_1,$$

where $A_l=M(\sigma;1,2^lN)$ and $B_l=M(\sigma,2;1,2^lN)$ and the sum $M(\sigma,2;2^lN+1,\infty)$ is given by

$$\sum_{i=l}^{\infty}\epsilon_i=\frac{1}{2^{(l-2)\sigma}}\frac{1}{2^\sigma-1}\epsilon_1+\gamma_l.$$

where γ_l is of the same order as that of $\delta(l)$. Since by the virtue of the convergence of $M(\sigma)$, $\delta(l)$ tends to zero as l approaches infinity, therefore γ_l and the above sum approach zero as l approaches infinity.

Thus, we conclude that $M(\sigma,2;2^lN+1,\infty)$ (given by $\sum_{i=l}^{\infty}\epsilon_i$) approaches zero as l approaches infinity. Furthermore, as l approaches infinity, $B=\lim_{l\to\infty}B_l$ approaches its limit given by

$$\left(1-\frac{1}{2^\sigma}\right)B=M(\sigma;1,\infty).$$

Hence,

$$\left(1-\frac{1}{2^\sigma}\right)M(\sigma,2)=M(\sigma).$$

Similarly, following the same steps, we can show that

$$\left(1-\frac{1}{3^\sigma}\right)M(\sigma,3;1,\infty)=M(\sigma,2;1,\infty).$$

or

$$\left(1-\frac{1}{2^\sigma}\right)\left(1-\frac{1}{3^\sigma}\right)M(\sigma,3;1,\infty)=M(\sigma;1,\infty).$$

This task can be achieved by first defining

$$M(\sigma,2;1,N)=\sum_{n=1}^{N}\frac{\mu(n,2)}{n^\sigma}=A_1,$$

$$M(\sigma,2;N+1,3N)=\sum_{n=N+1}^{3N}\frac{\mu(n,2)}{n^\sigma}=\delta_1,$$

$$M(\sigma,2;3N+1,3^2N)=\sum_{n=3N+1}^{3^2N}\frac{\mu(n,2)}{n^\sigma}=\delta_2,$$

$$M(\sigma,2;3^{L-1}N+1,3^LN)=\sum_{n=3^{L-1}N+1}^{3^LN}\frac{\mu(n,2)}{n^\sigma}=\delta_{L-1},$$

and

$$M(\sigma,3;1,N)=\sum_{n=1}^{N}\frac{\mu(n,3)}{n^\sigma}=B_1,$$

$$M(\sigma,3;N+1,3N)=\sum_{n=N+1}^{3N}\frac{\mu(n,3)}{n^\sigma}=\epsilon_1,$$

$$M(\sigma,3;3N+1,3^2N)=\sum_{n=3N+1}^{3^2N}\frac{\mu(n,3)}{n^\sigma}=\epsilon_2,$$

$$M(\sigma,3;3^{L-1}N+1,3^LN)=\sum_{n=3^{L-1}N+1}^{3^LN}\frac{\mu(n,3)}{n^\sigma}=\epsilon_{L-1},$$

Since

$$\sum_{n=1}^{3N}\frac{\mu(n,2)}{n^\sigma}=\sum_{n=1}^{3N}\frac{\mu(n,3)}{n^\sigma}-\sum_{n=1}^{N}\frac{\mu(n,3)}{(3n)^\sigma},$$

thus

$$M(\sigma,2;1,3N) = M(\sigma,3;1,3N) - \frac{1}{3^\sigma}M(\sigma,3;1,N)$$

Similarly,

$$M(\sigma,2;3^lN+1,3^{l+1}N) = M(\sigma,3;3^lN+1,3^{l+1}N) - \frac{1}{3^\sigma}M(\sigma,3;3^{l-1}N+1,3^lN)$$

Following the same process, we can show that

$$\sum_{i=1}^{\infty}\epsilon_i = \frac{3^\sigma}{3^\sigma-1}\epsilon_1 + \gamma_1,$$

where γ_1 is of the same order as that of $\delta(1)$ ($\delta(l)$ is defined as the maximum of $|\delta_l|, |\delta_{l+1}|, |\delta_{l+2}|, ..., |\delta_{L-1}|, |\delta_l + \delta_{l+1}|, |\delta_l + \delta_{l+1} + \delta_{l+2}|, ..., |\delta_l + \delta_{l+1} + ... + \delta_{L-1}|$).

Similarly, if we define $A_2 = M(\sigma,2;1,3N)$ and $B_2 = M(\sigma,3;1,3N)$, then

$$A_2 = B_2 - \frac{1}{3^\sigma}B_2 + \frac{1}{3^\sigma}\epsilon_1.$$

Therefore

$$\sum_{i=2}^{\infty}\epsilon_i = \frac{1}{3^\sigma-1}\epsilon_1 + \gamma_2.$$

where γ_2 is of the same order as that of $\delta(2)$.

Repeating the steps l times, we then obtain

$$\sum_{i=l}^{\infty}\epsilon_i = \frac{1}{3^{(l-2)\sigma}}\frac{1}{3^\sigma-1}\epsilon_1 + \gamma_l.$$

where γ_l is of the same order as that of $\delta(l)$. Hence the above sum approaches zero as l approaches infinity

Thus, we conclude that $M(\sigma,3;3^lN+1,\infty)$ (given by $\sum_{i=l}^{\infty}\epsilon_i$) approaches zero as l approaches infinity. Furthermore, as l approaches infinity, $B = \lim_{l\to\infty} B_l$ approaches its limit given by

$$\left(1-\frac{1}{3^\sigma}\right)B = M(\sigma,2;1,\infty).$$

Hence,

$$\left(1-\frac{1}{3^\sigma}\right)M(\sigma,3) = M(\sigma,2).$$

Repeating the process r times, we then conclude

$$M(\sigma) = M(\sigma, p_r) \prod_{i=1}^{r} \left(1 - \frac{1}{{p_i}^{\sigma}}\right).$$

The second part of the theorem can be proved by recalling

$$M(s, p_{r-1}; 1, Np_r) = M(s, p_r; 1, Np_r) - \frac{1}{p_r^s} M(s, p_r; 1, N).$$

If both series $M(s, p_{r-1})$ and $M(s, p_r)$ are convergent, then as N approaches infinity, we obtain

$$M(s, p_{r-1}) = M(s, p_r) \left(1 - \frac{1}{p_i^s}\right).$$

Repeating the process r times, we then conclude

$$M(\sigma) = M(\sigma, p_r) \prod_{i=1}^{r} \left(1 - \frac{1}{{p_i}^{\sigma}}\right).$$

Appendix 2

Assuming RH is valid and for $\sigma > 0.5$, to show that

$$\sum_{i=r1}^{r2} \frac{1}{{p_i}^{\sigma}} = E_1((\sigma - 1)\log p_{r1}) - E_1((\sigma - 1)\log p_{r2}) + \varepsilon$$

where, $\varepsilon = O\left(\frac{t}{(\sigma-0.5)^2} {p_{r1}}^{1/2-\sigma} \log p_{r1}\right)$, we first recall that

$$\sum_{i=r1}^{r2} \frac{1}{p_i^{\sigma}} = \int_{p_{r1}}^{p_{r2}} \frac{d\pi(x)}{x^{\sigma}} = \int_{p_{r1}}^{p_{r2}} \frac{1}{x^{\sigma} \log x} dx + \int_{p_{r1}}^{p_{r2}} \frac{1}{x^{\sigma}} dO\left(\sqrt{x} \log x\right).$$

We will first compute the integral with the O notation. This can be done by integration by parts to obtain

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^{\sigma}} dO\left(\sqrt{x}\log x\right) = \frac{O\left(\sqrt{p_{r2}}\log p_{r2}\right)}{{p_{r2}}^{\sigma}} - \frac{O\left(\sqrt{p_{r1}}\log p_{r1}\right)}{{p_{r1}}^{\sigma}} - \int_{p_{r1}}^{p_{r2}} O\left(\sqrt{x}\log x\right) d\left(\frac{1}{x^{\sigma}}\right)$$

Since $x > 0$, thus

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^{\sigma}} dO\left(\sqrt{x}\log x\right) = \frac{O\left(\sqrt{p_{r2}}\log p_{r2}\right)}{{p_{r2}}^{\sigma}} - \frac{O\left(\sqrt{p_{r1}}\log p_{r1}\right)}{{p_{r1}}^{\sigma}} - O\left(\int_{p_{r1}}^{p_{r2}} \sqrt{x}\log x \, d\left(\frac{1}{x^{\sigma}}\right)\right)$$

With the substitution of variables $y = \log x$, we then obtain

$$\int_{p_{r1}}^{p_{r2}} \sqrt{x}\log x \, d\left(\frac{1}{x^{\sigma}}\right) = -\int_{p_{r1}}^{p_{r2}} \sigma y e^{(\frac{1}{2}-\sigma)y} dy.$$

Since

$$\int xe^{ax}dx = \left(\frac{x}{a} - \frac{1}{a^2}\right)e^{ax},$$

therefore

$$\int_{p_{r1}}^{p_{r2}} \sqrt{x}\log x\, d\left(\frac{1}{x^\sigma}\right) = -\sigma\left(\frac{\log p_{r2}}{0.5-\sigma} - \frac{1}{(0.5-\sigma)^2}\right){p_{r2}}^{0.5-\sigma} + \sigma\left(\frac{\log p_{r1}}{0.5-\sigma} - \frac{1}{(0.5-\sigma)^2}\right){p_{r1}}^{0.5-\sigma}.$$

Hence, for $\sigma > 0.5$, we have

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma} dO\left(\sqrt{x}\log x\right) = O\left(\frac{{p_{r1}}^{0.5-\sigma}\log p_{r1}}{(\sigma-0.5)^2}\right) \tag{73}$$

For $\sigma \geq 1$, the integral $\int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma \log x}dx$ can be computed directly from the definition of the Exponential Integral $E_1(r) = \int_r^\infty \frac{e^{-u}}{u}du$ (where $r \geq 0$) to obtain

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma \log x}dx = E_1((\sigma-1)\log p_{r1}) - E_1((\sigma-1)\log p_{r2})$$

To compute the integral $\int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma \log x}dx$ for $\sigma < 0$, we first use the substantiation $y = \log x$ to obtain

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma \log x}dx = \int_{\log p_{r1}}^{\log p_{r2}} \frac{e^{(1-\sigma)y}}{y}dy = \int_{\epsilon}^{\log p_{r2}} \frac{e^{(1-\sigma)y}}{y}dy - \int_{\epsilon}^{\log p_{r1}} \frac{e^{(1-\sigma)y}}{y}dy$$

where, ϵ is an arbitrary small positive number. With the variable substantiations $z_1 = y/\log p_{r1}$ and $z_2 = y/\log p_{r2}$, we then obtain

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma \log x}dx = \int_{\epsilon/\log p_{r2}}^{1} \frac{e^{(1-\sigma)(\log p_{r2})z_2}}{z_2}dz_2 - \int_{\epsilon/\log p_{r1}}^{1} \frac{e^{(1-\sigma)(\log p_{r1})z_1}}{z_1}dz_1.$$

With the variable substantiations $w_1 = (1-\sigma)(\log p_{r1})z_1$ and $w_2 = (1-\sigma)(\log p_{r2})z_1$ and by adding and subtracting the terms $-\int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r2}} \frac{dw_2}{w_2} + \int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r1}} \frac{dw_1}{w_1}$, we then have

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma \log x}dx = \int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r2}} \frac{e^{w_2}-1}{w_2}dw_2 - \int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r1}} \frac{e^{w_1}-1}{w_1}dw_1 +$$

$$\int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r2}} \frac{dw_2}{w_2} - \int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r1}} \frac{dw_1}{w_1}.$$

Using the following identity [9, page 230]

$$\int_0^a \frac{e^t-1}{t}dt = -E_1(-a) - \log(a) - \gamma$$

where $a > 0$, we then obtain for $\sigma < 1$,

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma \log x}dx = E_1((\sigma-1)\log p_{r1}) - E_1((\sigma-1)\log p_{r2})$$

Hence, for $\sigma > 0.5$, we have

$$\sum_{i=r1}^{r2} \frac{1}{{p_i}^{\sigma}} = E_1((\sigma-1)\log p_{r1}) - E_1((\sigma-1)\log p_{r2}) + \varepsilon$$

It should be pointed out that in general, if there are no non-trivial zeros for values of s with $\Re(s) > a$, then by following the same steps, we may also show that for $\sigma > a$, we have

$$\sum_{i=r1}^{r2} \frac{1}{{p_i}^{\sigma}} = E_1((\sigma-1)\log p_{r1}) - E_1((\sigma-1)\log p_{r2}) + \varepsilon$$

where, $\varepsilon = O\left(\frac{t}{(\sigma-a)^2} {p_{r1}}^{a-\sigma} \log p_{r1}\right)$.

Appendix 3

Assuming RH is valid and for $\sigma > 0.5$, to show that

$$\sum_{i=r1}^{r2} \frac{1}{{p_i}^{s}} = E_1((s-1)\log p_{r1}) - E_1((s-1)\log p_{r2}) + \varepsilon$$

where, $\varepsilon = O\left(\frac{t+1}{(\sigma-0.5)^2} {p_{r1}}^{1/2-\sigma} \log p_{r1}\right)$, we first recall that

$$\sum_{i=r1}^{r2} \frac{1}{{p_i}^{s}} = \int_{p_{r1}}^{p_{r2}} \frac{d\pi(x)}{x^s} = \int_{p_{r1}}^{p_{r2}} \frac{1}{x^s \log x} dx + \int_{p_{r1}}^{p_{r2}} \frac{1}{x^s} dO\left(\sqrt{x}\log x\right).$$

We will first compute the integral with the O notation. This can be done by integration by parts to obtain

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^s} dO\left(\sqrt{x}\log x\right) = \frac{O\left(\sqrt{p_{r2}}\log p_{r2}\right)}{{p_{r2}}^s} - \frac{O\left(\sqrt{p_{r1}}\log p_{r1}\right)}{{p_{r1}}^s} - \int_{p_{r1}}^{p_{r2}} O\left(\sqrt{x}\log x\right) d\left(\frac{1}{x^s}\right)$$

The integral on the right side of the above equation can be then written as

$$\int_{p_{r1}}^{p_{r2}} O\left(\sqrt{x}\log x\right) d\left(\frac{1}{x^s}\right) = -s\int_{p_{r1}}^{p_{r2}} O\left(\sqrt{x}\log x\right) x^{-s-1} dx.$$

Hence,

$$\left|\int_{p_{r1}}^{p_{r2}} O\left(\sqrt{x}\log x\right) d\left(\frac{1}{x^s}\right)\right| \le |s| \int_{p_{r1}}^{p_{r2}} O\left(\sqrt{x}\log x\right) |x^{-s-1}| dx.$$

For sufficiently large t, we can write $|s| = t$ and consequently

$$\left|\int_{p_{r1}}^{p_{r2}} O\left(\sqrt{x}\log x\right) d\left(\frac{1}{x^s}\right)\right| = O\left(t\ \frac{{p_{r1}}^{0.5-\sigma}\log p_{r1}}{(\sigma-0.5)^2}\right).$$

Hence,

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^s} dO\left(\sqrt{x}\log x\right) = O\left((t+1)\ \frac{{p_{r1}}^{0.5-\sigma}\log p_{r1}}{(\sigma-0.5)^2}\right).$$

For $\Re(s) \geq 1$, the integral $\int_{p_{r1}}^{p_{r2}} \frac{1}{x^s \log x} dx$ can be computed directly from the definition of the Exponential Integral $E_1(z) = \int_1^{\infty} \frac{e^{-tz}}{t} dt$ (where $\Re(z) \geq 0$) to obtain

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^s \log x} dx = E_1((s-1)\log p_{r1}) - E_1((s-1)\log p_{r2})$$

To compute the integral $\int_{p_{r1}}^{p_{r2}} \frac{1}{x^s \log x} dx$ for $\Re(z) < 1$, we first write the integral as follows

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^s \log x} dx = \int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma \log x} \cos(t \log x)}{\log x} dx - i \int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma \log x} \sin(t \log x)}{\log x} dx.$$

The first integral on the right side $\int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma \log x} \cos(t \log x)}{\log x} dx$ can be computed by using the substitution $y = \log x$ to obtain

$$\int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma \log x} \cos(t \log x)}{\log x} dx = \int_{p_{r1}}^{p_{r2}} \frac{e^{(1-\sigma)y} \cos(ty)}{y} dy,$$

or

$$\int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma \log x} \cos(t \log x)}{\log x} dx = \int_{p_{r1}}^{p_{r2}} \frac{e^{(1-\sigma)y} \cos(ty)}{y} dy + \int_{p_{r1}}^{p_{r2}} \frac{e^{(1-\sigma)y}}{y} dy - \int_{p_{r1}}^{p_{r2}} \frac{e^{(1-\sigma)y}}{y} dy.$$

Hence,

$$\int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma \log x} \cos(t \log x)}{\log x} dx = \int_{\epsilon}^{p_{r1}} \frac{e^{(1-\sigma)y}(1-\cos(ty))}{y} dy - \int_{\epsilon}^{p_{r2}} \frac{e^{(1-\sigma)y}(1-\cos(ty))}{y} dy -$$

$$\int_{\epsilon}^{p_{r1}} \frac{e^{(1-\sigma)y}}{y} dy + \int_{\epsilon}^{p_{r2}} \frac{e^{(1-\sigma)y}}{y} dy$$

where, ϵ is an arbitrary small positive number. With the variable substantiations $z_1 = y/\log p_{r1}$ and $z_2 = y/\log p_{r2}$, we then obtain

$$\int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma \log x} \cos(t \log x)}{\log x} dx = \int_{\epsilon/\log p_{r1}}^{1} \frac{e^{(1-\sigma)(\log p_{r1})z_1}(1-\cos(t(\log p_{r1})z_1))}{z_1} dz_1 -$$

$$\int_{\epsilon/\log p_{r2}}^{1} \frac{e^{(1-\sigma)(\log p_{r2})z_2}(1-\cos(t(\log p_{r2})z_2))}{z_2} dz_2 -$$

$$\int_{\epsilon/\log p_{r1}}^{1} \frac{e^{(1-\sigma)(\log p_{r1})z_1}}{z_1} dz_1 + \int_{\epsilon/\log p_{r2}}^{1} \frac{e^{(1-\sigma)(\log p_{r2})z_2}}{z_2} dz_2$$

By the virtue of the following identity ([9], page 230)

$$\int_0^1 \frac{e^{at}(1-\cos(bt))}{t} dt = \frac{1}{2}\log(1 + b^2/a^2) + \mathrm{Li}(a) + \Re[E_1(-a+ib)],$$

where $a > 0$, we then obtain the following

$$\int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma \log x} \cos(t \log x)}{\log x} dx = \Re[E_1((s-1)\log p_{r1})] + \mathrm{Li}((1-\sigma)\log p_{r1}) -$$

$$\Re[E_1((s-1)\log p_{r2})] - \mathrm{Li}((1-\sigma)\log p_{r2}) -$$
$$\int_{\epsilon/\log p_{r1}}^{1} \frac{e^{(1-\sigma)(\log p_{r1})z_1}}{z_1} dz_1 + \int_{\epsilon/\log p_{r2}}^{1} \frac{e^{(1-\sigma)(\log p_{r2})z_2}}{z_2} dz_2$$

With the variable substantiations $w_1 = (1-\sigma)(\log p_{r1})z_1$ and $w_1 = (1-\sigma)(\log p_{r1})z_1$ and by adding and subtracting the terms $-\int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r2}} \frac{dw_2}{w_2} + \int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r1}} \frac{dw_1}{w_1}$, we then have

$$\int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma \log x}\cos(t\log x)}{\log x} dx = \Re[E_1((s-1)\log p_{r1})] + \mathrm{Li}((1-\sigma)\log p_{r1}) -$$
$$\Re[E_1((s-1)\log p_{r2})] - \mathrm{Li}((1-\sigma)\log p_{r2}) +$$
$$\int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r2}} \frac{e^{w_2}-1}{w_2} dw_2 - \int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r1}} \frac{e^{w_1}-1}{w_1} dw_1 +$$
$$\int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r2}} \frac{dw_2}{w_2} - \int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r1}} \frac{dw_1}{w_1}.$$

Using the following identity [9, page 230]

$$\int_0^a \frac{e^t - 1}{t} dt = \mathrm{Ei}(a) - \log(a) - \gamma$$

where $a > 0$, we then obtain for $\sigma < 1$,

$$\int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma \log x}\cos(t\log x)}{\log x} dx = \Re[E_1((s-1)\log p_{r1})] - \Re[E_1((s-1)\log p_{r2})]$$

Similarly, using the identity [9, page 230]

$$\int_{p_0}^{1} \frac{e^{at}\sin(bt)}{t} dt = \pi - \arctan(b/a) + \Im[E_1(-a+ib)],$$

where $a > 0$, we can show that for $\sigma < 1$, we have

$$-\int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma \log x}\sin(t\log x)}{\log x} dx = \Im[E_1((s-1)\log p_{r1})] - \Im[E_1((s-1)\log p_{r2})].$$

Therefore, for $\Re(s) > 0.5$, we have

$$\sum_{i=r1}^{r2} \frac{1}{{p_i}^s} = E_1((s-1)\log p_{r1}) - E_1((s-1)\log p_{r2}) + \varepsilon$$

where, $\varepsilon = O\left(\frac{t+1}{(\sigma-0.5)^2} {p_{r1}}^{1/2-\sigma} \log p_{r1}\right)$.

Appendix 4

In Appendix 4, we will show that the sum $\sum_\rho E_1\left((s-\rho)\log p_r\right)$ is convergent if $|s-\rho|>0$ for every ρ. Furthermore, we will show that the sum approaches zeros as p_r approaches infinity. this task will be achieved by noting that, for sufficiently large p_r, $E_1\left((s-\rho)\log p_r\right)$ can be written as

$$E_1\left((s-\rho)\log p_r\right)=\frac{e^{-(s-\rho)\log p_r}}{(s-\rho)\log p_r}\left(1+O\left(\frac{1}{|s-\rho|\log p_r}\right)\right) \tag{74}$$

Therefore, if the sum $\sum_\rho E_1\left((s-\rho)\log p_r\right)$ is convergent, then it will be given by

$$\sum_\rho E_1\left((s-\rho)\log p_r\right)=\sum_\rho \frac{e^{-(s-\rho)\log p_r}}{(s-\rho)\log p_r}+\epsilon, \tag{75}$$

where ϵ is the contribution by the sum of the O terms in Equation (74). It can be easily shown that if $|s-\rho|\geq\varepsilon>0$ for every ρ, then ϵ in Equation (75) tends to zero as p_r approaches infinity. This result can be deduced by noting that $O(\epsilon)=(p_r^{\min\Re(s-\rho)}/(\log p_r)^2)\sum_\rho 1/|s-\rho|^2$. Since the sum $\sum_\rho 1/|s-\rho|^2$ is bounded, therefore Equation (75) can be further simplified to

$$\sum_\rho E_1\left((s-\rho)\log p_r\right)=\frac{{p_r}^{-s}}{\log p_r}\sum_\rho\frac{{p_r}^{\rho}}{s-\rho}+O(p_r^{\min\Re(s-\rho)}/(\log p_r)^2). \tag{76}$$

To show the sum $\sum_\rho E_1\left((s-\rho)\log p_r\right)$ is convergent, let $s=\sigma+iT$ and $\rho_i=\beta_i+i\gamma_i$. We split ρ_i's into two groups. The first group comprises of the non-trivial zeros with γ_i's less than or equal to mT, where $m>1$. The rest of the non-trivial zeros belong to the second group. Since the first group has a finite number of ρ_i's, thus the sum $\sum_{|\gamma_i|\leq mT} E_1\left((s-\rho)\log p_r\right)$ is bounded. Since $|{p_r}^{-s}{p_r}^{\rho}|<1$ for every ρ, therefore

$$\left|\sum_{|\gamma_i|\leq mT} E_1\left((s-\rho)\log p_r\right)\right|=(1/\log p_r)\sum_{|\gamma_i|\leq mT}\frac{1}{|s-\rho|}.$$

Hence

$$\sum_{|\gamma_i|\leq mT} E_1\left((s-\rho)\log p_r\right)=O(1/\log p_r).$$

The sum over the second group can be expanded as follows

$$\sum_{|\gamma_i|>mT} E_1\left((s-\rho)\log p_r\right)=-\frac{{p_r}^{-s}}{\log p_r}\left(\sum_{|\gamma_i|>mT}\frac{{p_r}^{\rho_i}}{\rho_i}+s\sum_{|\gamma_i|>mT}\frac{{p_r}^{\rho_i}}{{\rho_i}^2}+s^2\sum_{|\gamma_i|>mT}\frac{{p_r}^{\rho}}{{\rho_i}^3}+...\right)+\epsilon.$$

The first sum $\sum_{|\gamma_i|>mT}{p_r}^{\rho_i}/\rho_i$ is convergent by the virtue of Equation (47). The upper bound for the second term $({p_r}^{-s}/\log p_r)\ s\ \sum_{|\gamma_i|>mT}{p_r}^{\rho_i}/{\rho_i}^2$can be determined as follows

$$\left|\frac{{p_r}^{-s}s}{\log p_r}\sum_{|\gamma_i|>mT}\frac{{p_r}^{\rho_i}}{{\rho_i}^2}\right|\leq\frac{|{p_r}^{-s}||s|}{\log p_r}\sum_{|\gamma_i|>mT}\frac{|{p_r}^{\rho_i}|}{|{\rho_i}^2|}.$$

Since for sufficiently large T, $|s|$ is given by T and the density of the non-trivial zeros is given by $O(\log t)$ (note that if there are roots off the critical line then their density is given by Bohr Landau theorem [1] and it is less than $O(\log t)$), thus

$$\left|\frac{{p_r}^{-s} s}{\log p_r} \sum_{|\gamma_i|>mT} \frac{{p_r}^{\rho_i}}{{\rho_i}^2}\right| \leq \frac{{p_r}^{-\sigma+\max\beta_i} T}{\log p_r} \int_{mT}^{\infty} \frac{O(\log t)}{t^2} dt.$$

Hence

$$\left|\frac{{p_r}^{-s} s}{\log p_r} \sum_{|\gamma_i|>mT} \frac{{p_r}^{\rho_i}}{{\rho_i}^2}\right| \leq \frac{{p_r}^{-\sigma+\max\beta_i}}{\log p_r} \frac{O(\log T))}{m}.$$

Similarly, we can show that

$$\left|\frac{{p_r}^{-s} s^2}{\log p_r} \sum_{|\gamma_i|>mT} \frac{{p_r}^{\rho_i}}{{\rho_i}^3}\right| \leq \frac{{p_r}^{-\sigma+\max\beta_i}}{\log p_r} \frac{O(\log T)}{m^2},$$

and,

$$\left|\frac{{p_r}^{-s} s^i}{\log p_r} \sum_{|\gamma_i|>mT} \frac{{p_r}^{\rho_i}}{{\rho_i}^{i+1}}\right| \leq \frac{{p_r}^{-\sigma+\max\beta_i}}{\log p_r} \frac{O(\log T)}{m^i}.$$

Therefore,

$$\left|\frac{{p_r}^{-s}}{\log p_r}\left(s \sum_{|\gamma_i|>mT} \frac{{p_r}^{\rho_i}}{{\rho_i}^2} + s^2 \sum_{|\gamma_i|>mT} \frac{{p_r}^{\rho}}{{\rho_i}^3} + ...\right)\right| \leq \frac{{p_r}^{-\sigma+\max\beta_i} O(\log T)}{\log p_r} \sum_{i=1}^{\infty} \frac{1}{m^i}.$$

Since $\sum_{i=1}^{\infty} 1/m^i$ is convergent, hence $({p_r}^{-\sigma+\max\beta_i} O(\log T)/\log p_r) \sum_{i=1}^{\infty} 1/m^i$ is convergent and it is given by

$$\left|\frac{{p_r}^{-s}}{\log p_r}\left(s \sum_{|\gamma_i|>mT} \frac{{p_r}^{\rho_i}}{{\rho_i}^2} + s^2 \sum_{|\gamma_i|>mT} \frac{{p_r}^{\rho}}{{\rho_i}^3} + ...\right)\right| = O({p_r}^{-\sigma+\max\beta_i} \log(T)/\log p_r).$$

Hence

$$\sum_{|\gamma_i|>mT} E_1\left((s-\rho)\log p_r\right) = -\frac{{p_r}^{-s}}{\log p_r}\left(\sum_{|\gamma_i|>mT} \frac{{p_r}^{\rho_i}}{\rho_i}\right) + O({p_r}^{-\sigma+\max\beta_i} \log(T)/\log p_r).$$

Thus

$$\sum_{|\gamma_i|>mT} E_1\left((s-\rho)\log p_r\right) = -\frac{{p_r}^{-s}}{\log p_r}\left(\sum_{|\gamma_i|>mT} \frac{{p_r}^{\rho_i}}{\rho_i}\right) + O({p_r}^{-\sigma+\max\beta_i} \log(T)/\log p_r).$$

Consequently, $\sum_{\rho} E_1\left((s-\rho)\log p_r\right)$ is convergent and it is given by

$$\sum_{\rho} E_1\left((s-\rho)\log p_r\right) = \frac{{p_r}^{-s}}{\log p_r} \sum_{\rho} \frac{{p_r}^{\rho}}{s-\rho} + O(1/\log p_r).$$

In the remaining of this Appendix, we will derive a formula to show the dependence of the sum $\sum_\rho E_1\left((s-\rho)\log p_r\right)$ on T (where, $s=\sigma+iT$). On RH, we have

$$\sum_{|\gamma_i|>mT} E_1\left((s-\rho)\log p_r\right) = -\frac{{p_r}^{-s}}{\log p_r}\left(\sum_{|\gamma_i|>mT}\frac{{p_r}^{\rho_i}}{\rho_i}\right) + O\left({p_r}^{0.5-\sigma}\log(T)/\log p_r\right).$$

Thus

$$\left|\sum_{|\gamma_i|>mT} E_1\left((s-\rho)\log p_r\right)\right| = O\left(p_r^{0.5-\sigma}\log p_r\right) + O\left({p_r}^{0.5-\sigma}\log(T)/\log p_r\right).$$

Since the density of the roots on the critical line is given by $\log T$, thus the sum over the roots with $|\gamma_i|\leq mT$ can be given by the following integral

$$\left|\sum_{|\gamma_i|\leq mT} E_1\left((s-\rho)\log p_r\right)\right| = \frac{p_r^{0.5-\sigma}}{\log p_r}\int_{-mT}^{mT}\frac{O(\log t)}{\sqrt{(t-T)^2+(\sigma-0.5)^2}}dt.$$

Thus, for fixed $\sigma>0.5+\epsilon$, we have

$$\left|\sum_{|\gamma_i|\leq mT} E_1\left((s-\rho)\log p_r\right)\right| = p_r^{0.5-\sigma}O((m\log T)^2)/\log p_r.$$

Therefore, on RH, we have

$$\left|\sum_{\rho} E_1\left((s-\rho)\log p_r\right)\right| = O\left({p_r}^{0.5-\sigma}\log p_r(\log T)^2\right). \tag{77}$$

Appendix 5

To show that

$$\left|\sum_{n=1}^{N}\frac{\mu(n,p_r)}{n}\right| \leq 2$$

we first note that

$\sum_{d/n}\mu(d,p_r)=1$, if $n=1$,
$\sum_{d/n}\mu(d,p_r)=1$, if all the prime factor of n are less than p_r,
$\sum_{d/n}\mu(d,p_r)=0$, if any of the prime factor of n is greater than p_r.

Adding all the terms $\sum_{d/n}\mu(d,p_r)$ for $1\leq n\leq N$, we then obtain

$$0<\sum_{n=1}^{N}\mu(n,p_r)\left[\frac{N}{n}\right]\leq N,$$

where $[x]$ refers to the integer value of x. Define r_n as

$$r_n=\frac{N}{n}-\left[\frac{N}{n}\right],$$

where $0 \leq r_n < 1$. Hence, we have

$$\sum_{n=1}^{N} \mu(n,p_r) r_n < \sum_{n=1}^{N} \mu(n,p_r) \left[\frac{N}{n}\right] + \sum_{n=1}^{N} \mu(n,p_r) r_n \leq \sum_{n=1}^{N} \mu(n,p_r) r_n$$

Since

$$-N \leq \sum_{n=1}^{N} \mu(n,p_r) r_n \leq N,$$

thus, for every p_r we have

$$-N < \sum_{n=1}^{N} \mu(n,p_r) \frac{N}{n} \leq 2N,$$

or

$$-1 < \sum_{n=1}^{N} \frac{\mu(n,p_r)}{n} \leq 2.$$

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