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An approximation for primes

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Abstract

An approximation heuristic for the prime counting function $\pi(x)$ is presented. The presented approximation heuristic is on average as good as $Li(x) - \frac{1}{2}Li(\sqrt{x})$ for x values up to 100,000. The main advantage of the heuristic is, that it does not require an integral to be evaluated. The main disadvantage of the heuristic is, that it gives bad approximations for $x \in \{1, 2, 3\}$. The heuristic is briefly motivated and then directly presented in mathematical and source code form (Matlab/Octave). Its effectiveness is visually illustrated by some plots.

1 Motivation

It can be observed, that the following approximation of $\pi(x)$ and $Li(x) = \int_2^x 1/\log(t)\, dt$ holds well for x up to some thousands (H_x is the x-th harmonic number and γ is Euler's constant):

$$\pi(x) \approx I_e(x) := \frac{x}{H_x - e\gamma}$$

$$Li(x) \approx I_\pi(x) := \frac{x}{H_x - \pi\gamma}$$

Figure 1: Observe how $I_e(x)$ fits $\pi(x)$ and how $I_{pi}(x)$ fits $Li(x)$

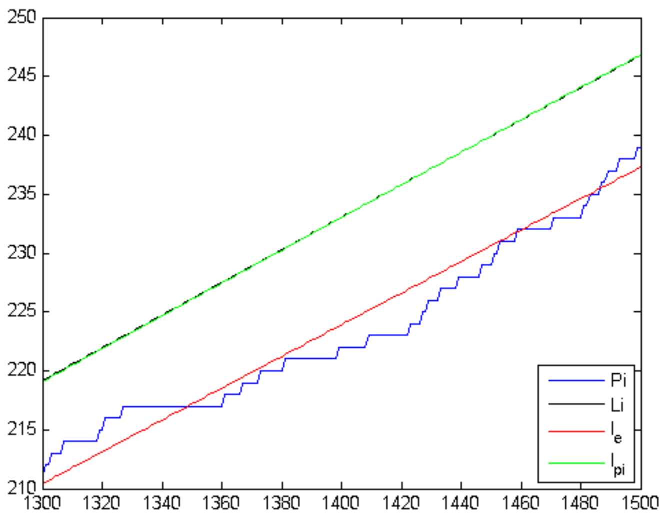
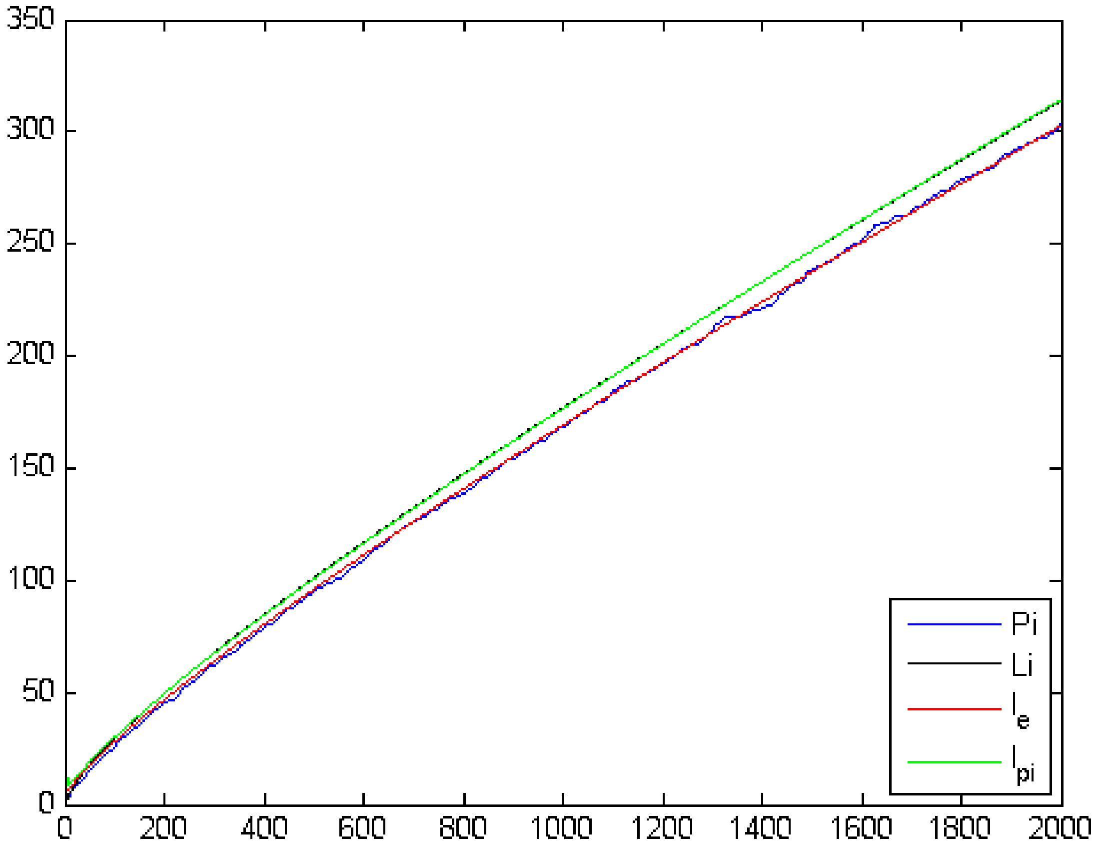


Figure 2: Observe how $I_e(x)$ fits $\pi(x)$ and how $I_{pi}(x)$ fits $Li(x)$



However, the above relationship does not hold for larger values of x, as $\pi(x)$ and $Li(x)$ appear to *move* towards the inner region defined by $I_e(x)$ and $I_{pi}(x)$. In an attempt to receive a better approximation for larger values of x, a convex combination of $I_e(x)$ and $I_{pi}(x)$ is proposed.

2 Approximation heuristic for $\pi(x)$

The approximation heuristic for $\pi(x)$ is as follows:

$$\pi(x) \quad \approx \quad A(x) \quad := \quad x \cdot \left(\frac{\frac{1}{H_x} + \gamma}{H_x - e\gamma} + \frac{1 - \gamma - \frac{1}{H_x}}{H_x - \pi\gamma}\right) - e$$

where

$$H_x = \sum_{i=1}^{x} \frac{1}{i} \qquad \text{(Harmonic number)}$$

and

$$\begin{aligned} \gamma &= 0.577... && \text{(Euler-Mascheroni constant)} \\ e &= 2.718... && \text{(Euler number)} \\ \pi &= 3.141... && \text{(Circle constant)} \end{aligned}$$

A computational effective implementation is as follows (approximating H_x by $log(x) + \gamma$)

```
function main
for x=4:100
        % Calculate prime counting function Pi(x)
        Pi(x) = size(primes(x),2);

        % Calculate approximation heuristic A(x)
        H = log(x) + 0.577; h = 1/H;
        A(x) = x*((h+0.577)/(H-1.569)+(0.423-h)/(H-1.813))-2.718;
end
% Plot Pi(x) [BLUE] and its Approximation A(x) [RED]
plot(Pi);hold on; plot(A,'r'); legend('Pi(x)','A(x)','Location','SouthEast');hold off
```

3 Graphical illustration of $\pi(x)$ approximation by $A(x)$

Figure 3: Plot of $\pi(x)$ (blue) and $A(x)$ (red) for $x = 4, ..., 100$.

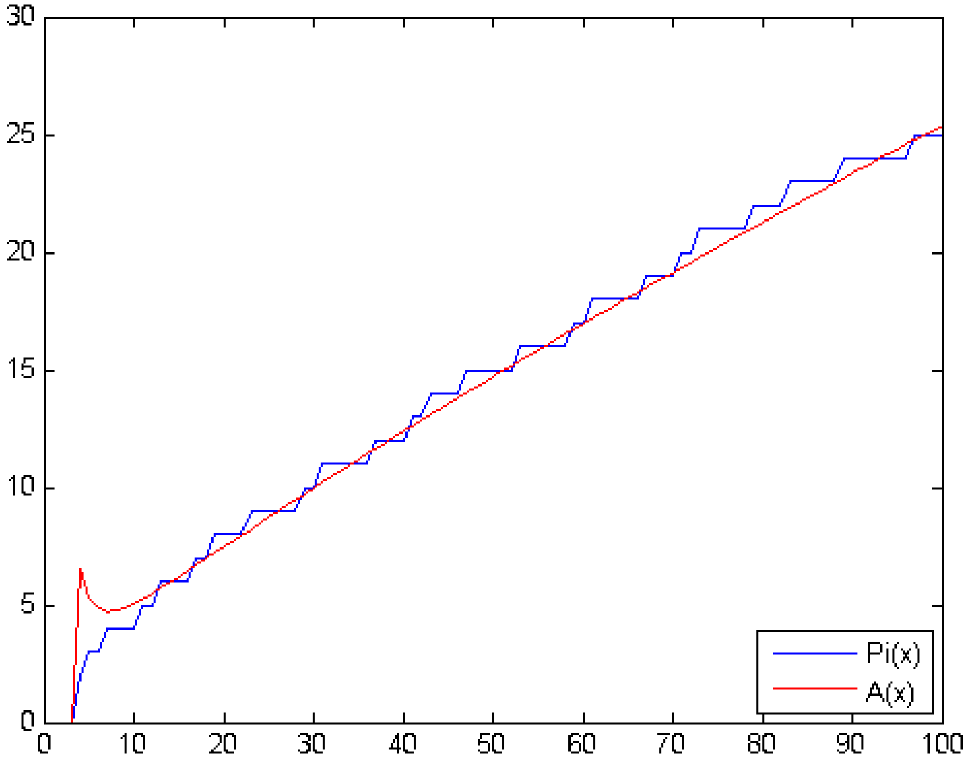


Figure 4: Plot of $\pi(x)$ (blue) and $A(x)$ (red) for $x = 4, ..., 500$.

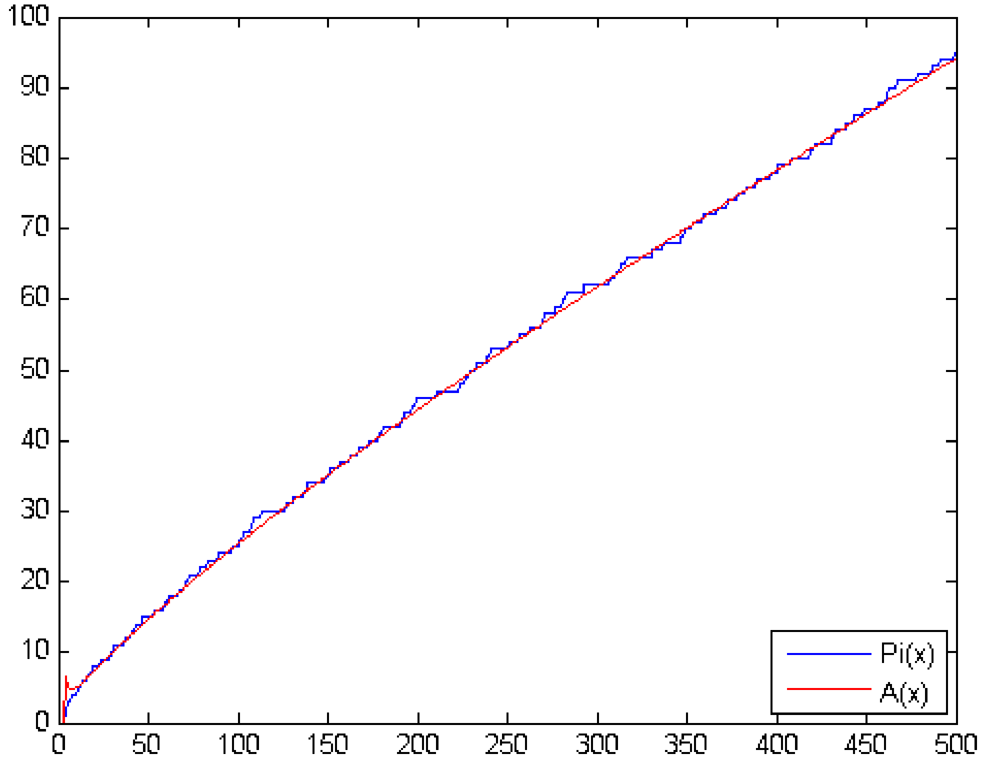


Figure 5: Plot of $\pi(x)$ (blue) and $A(x)$ (red) for $x = 4, ..., 1000$.

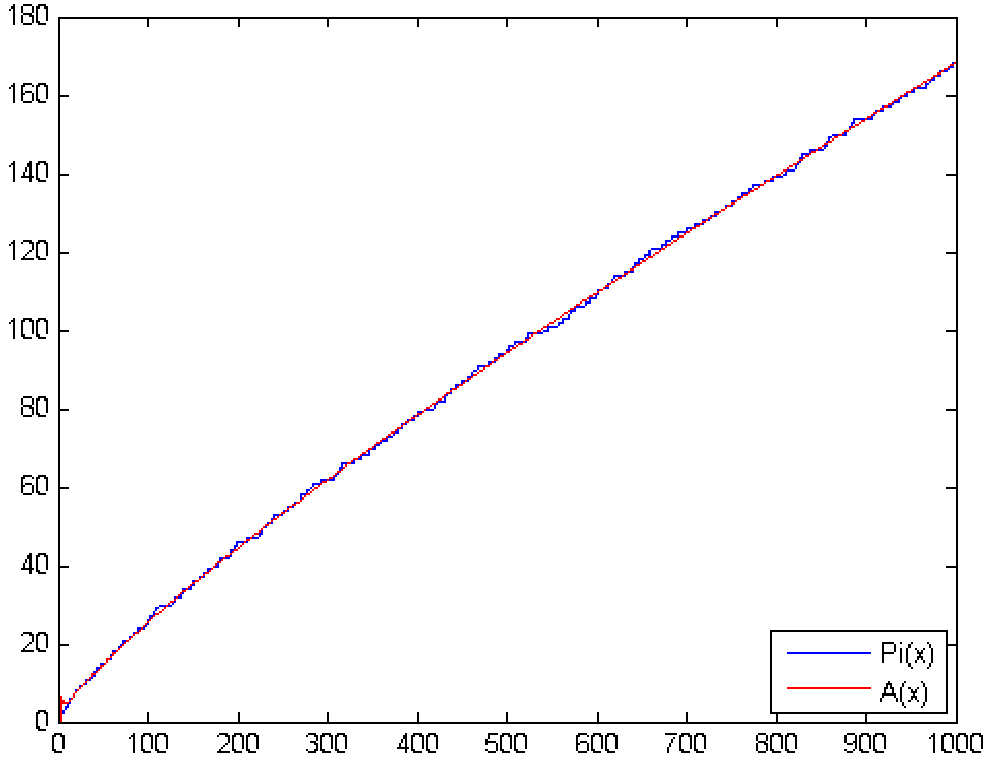


Figure 6: Plot of $\pi(x)$ (blue) and $A(x)$ (red) for $x = 4, ..., 10000$.

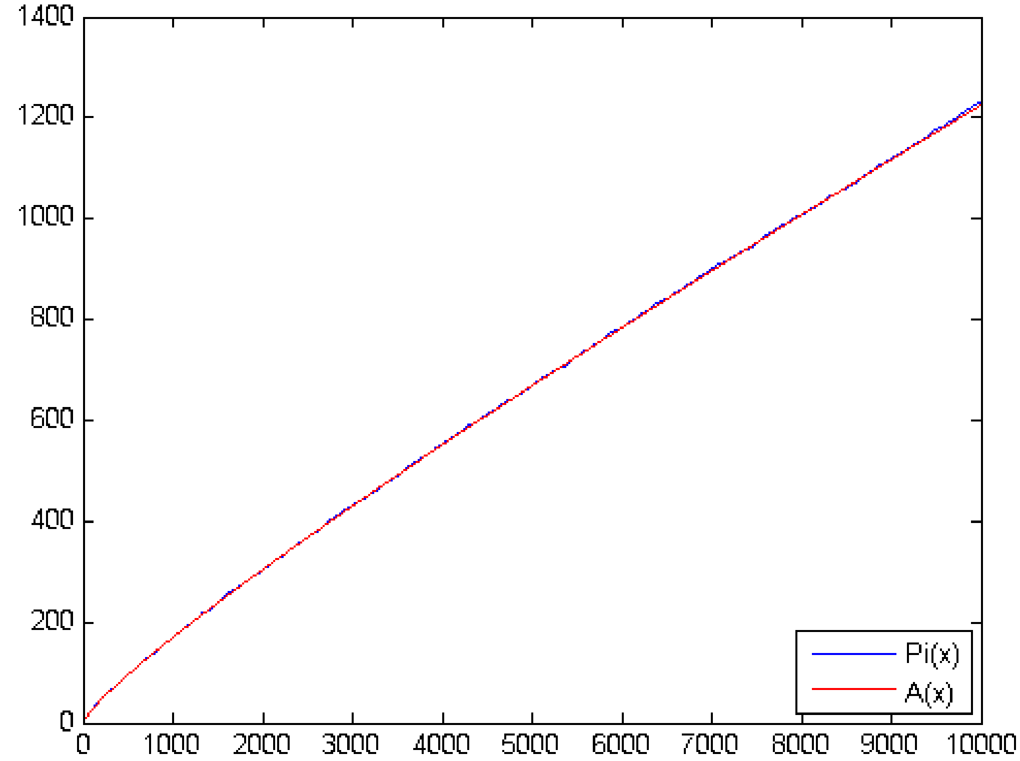


Figure 7: Plot of $\pi(x)$ (blue) and $A(x)$ (red) for $x = 4, ..., 100000$.

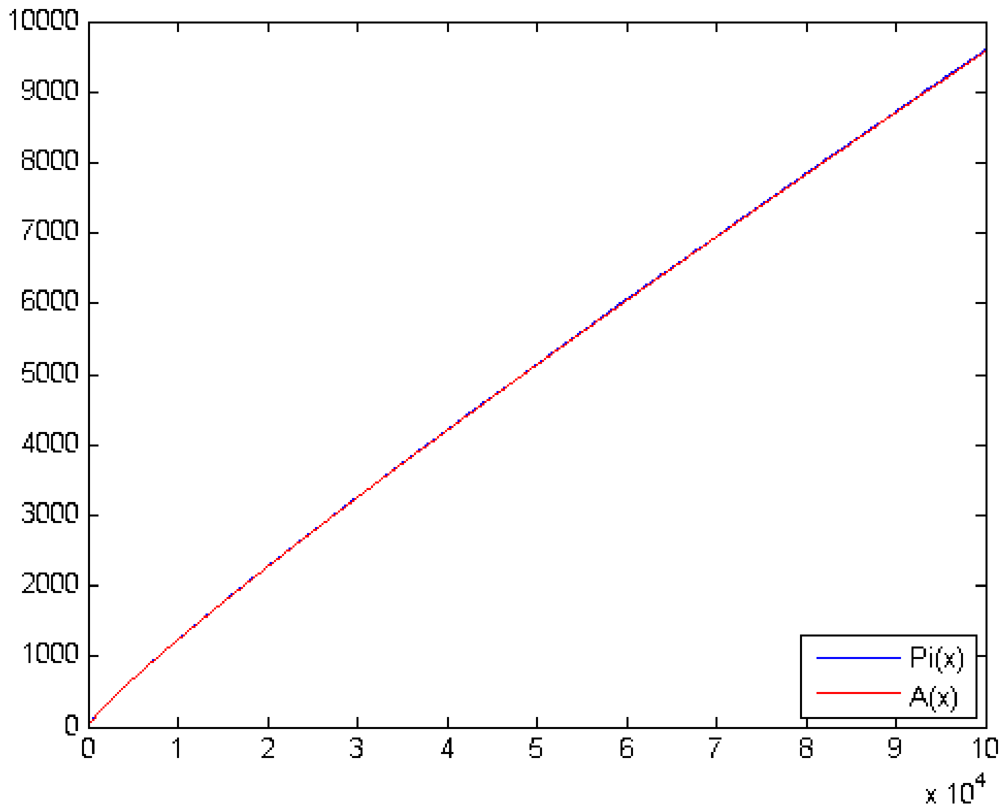


Figure 8: Comparison with $Li(x) - \frac{1}{2}Li(\sqrt{x})$ for $x = 4, ..., 100$.

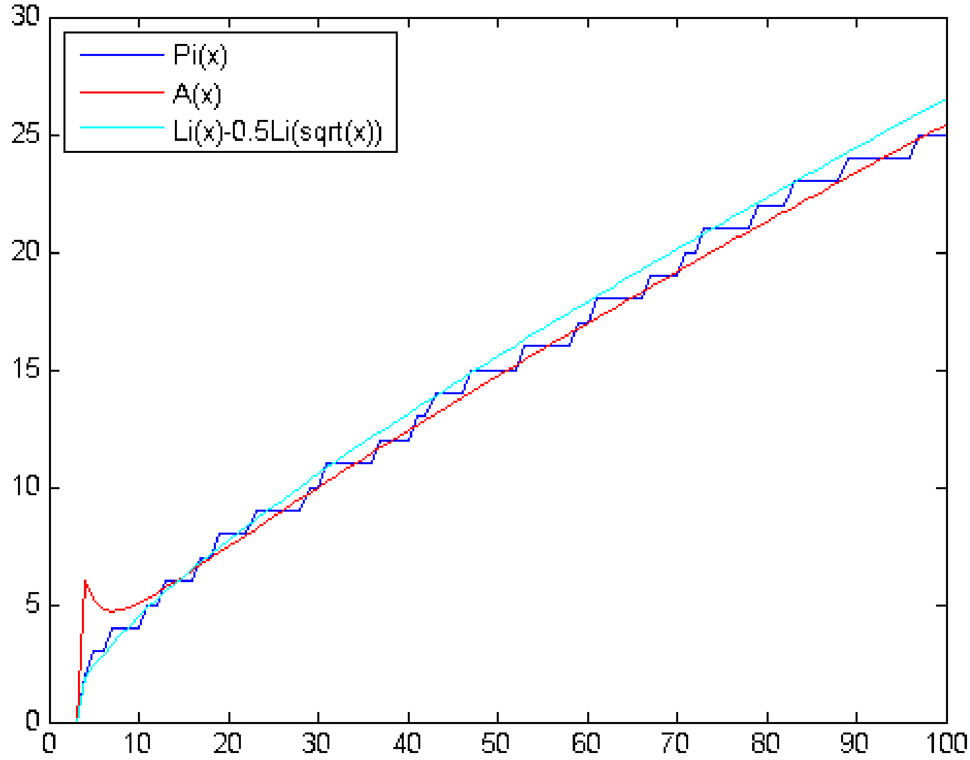


Figure 9: Comparison with $Li(x) - \frac{1}{2}Li(\sqrt{x})$ for $x = 500, ..., 1000$.

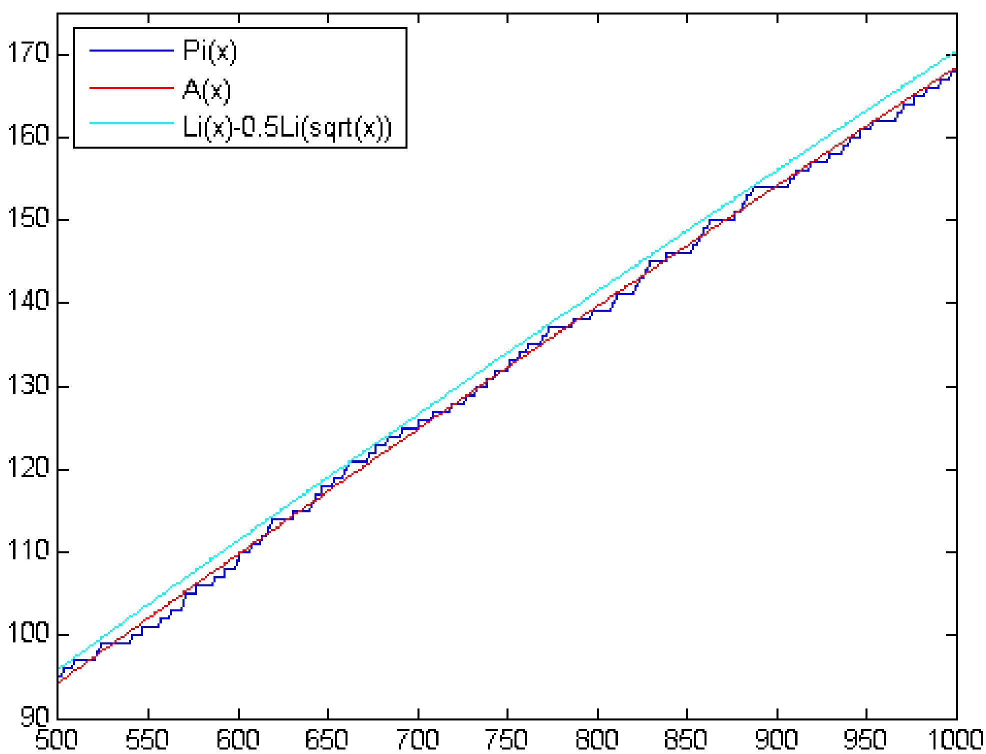


Figure 10: Comparison with $Li(x) - \frac{1}{2}Li(\sqrt{x})$ for $x = 48000, ..., 50000$.

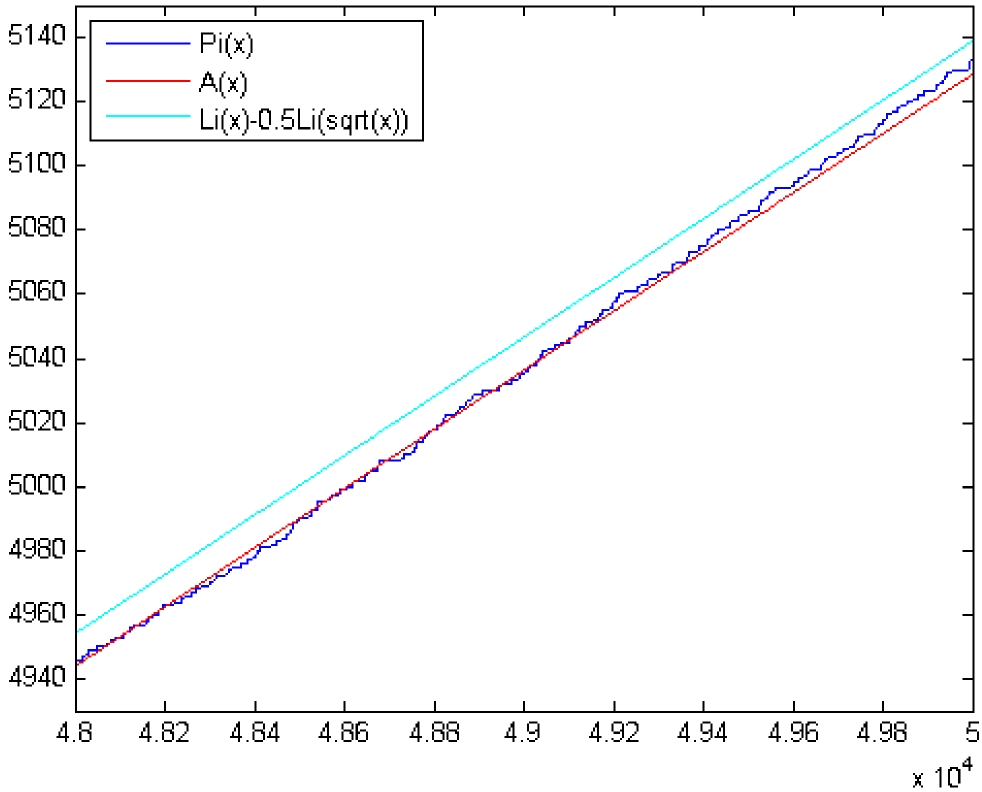


Figure 11: Comparison with $Li(x) - \frac{1}{2}Li(\sqrt{x})$ for $x = 990000, ..., 1000000$.

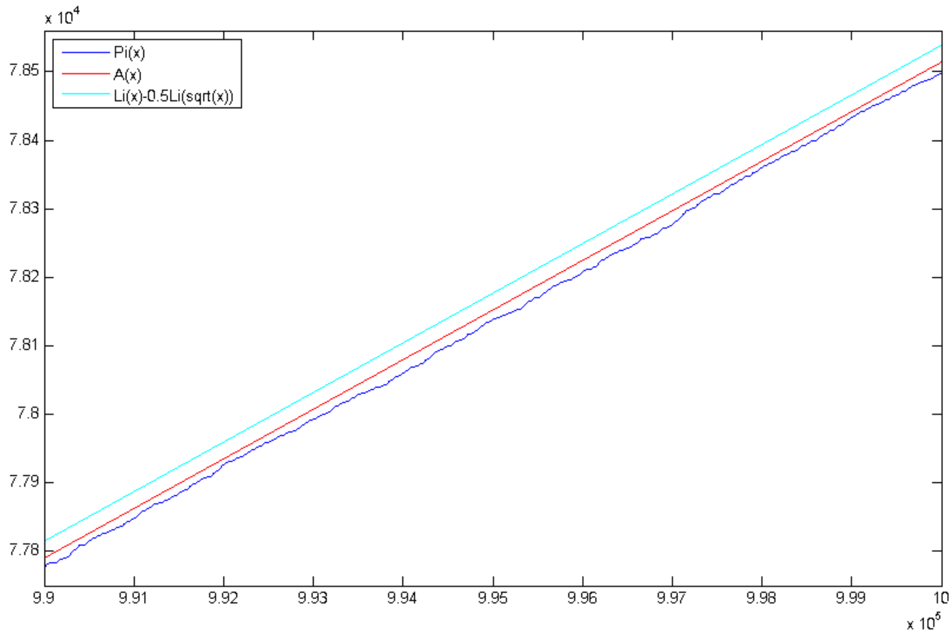


Figure 12: Comparison with $Li(x)$, $Li(x) - \frac{1}{2}Li(\sqrt{x})$ and $x/(log(x) - 1)$.

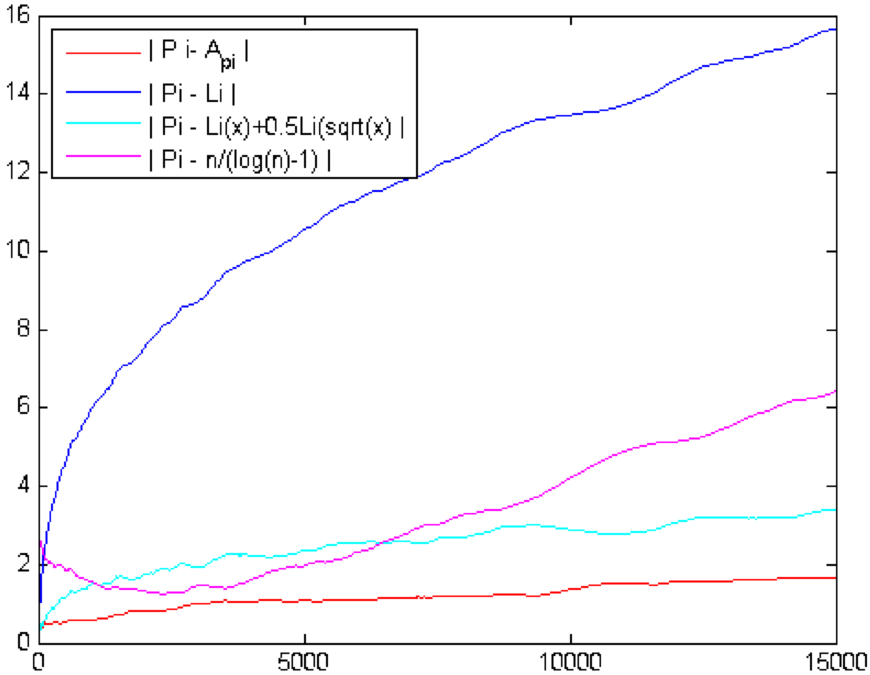
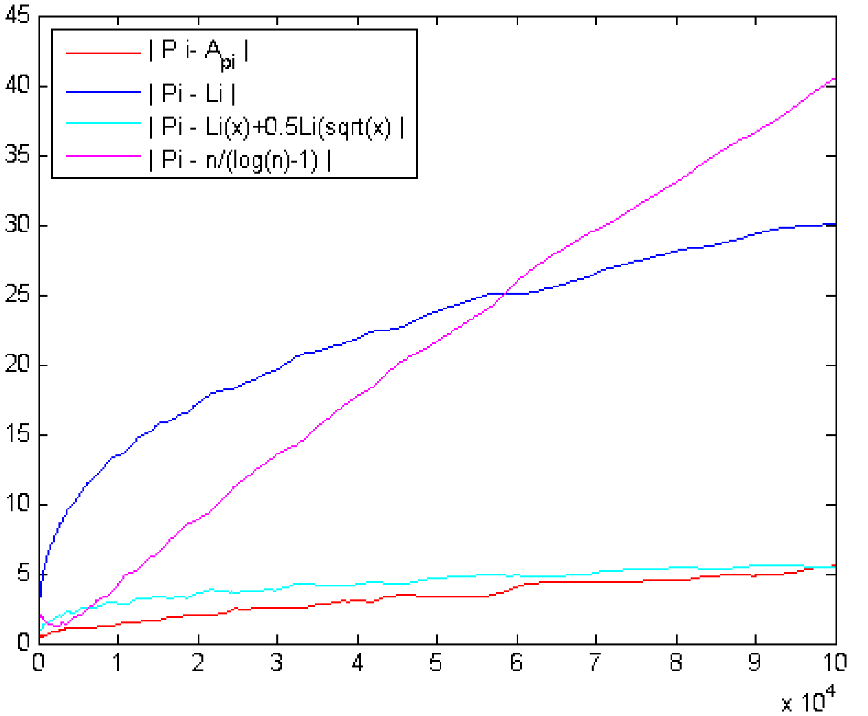


Figure 13: Comparison with $Li(x)$, $Li(x) - \frac{1}{2}Li(\sqrt{x})$ and $x/(log(x) - 1)$.



References

[1] Schlueter, M.: *Some formulas and pattern.* Preprint (2013), available at http://vixra.org/abs/1307.0078