The Position-Momentum Commutator

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It is shown that the position-momentum commutator is a diagonal matrix.

Key words: position-momentum commutator, diagonal matrix.

In a recent paper [1], we have implied that the position-momentum commutator is a diagonal matrix

$$[x, p] = xp - px = i\hbar I \tag{1}$$

where x and p are the position and the momentum of the atomic electron, respectively, $i^2 = -1$, $\hbar = h/2\pi$, and h is the Planck's constant, and I the unit (or identity) matrix.

And where:

$x_{ij}(t) = x_{ij}(0)\ exp(i2\pi f_{ij}t)$; f_{ij} being the oscillation frequency of the atomic electron, and therefore also the frequency of the light emitted by it: $f_{ij} = (E_i - E_j)/h = E_{ij}/h$, when it goes from the level i to the level j ($i > j$), and where E_i and E_j are the energies of the electron corresponding to these levels and E_{ij} the energy of the light emitted, and t is the time

$p_{ij}(t) = m\ dx_{ij}(t)/dt = m\ x_{ij}(0)\ exp(i2\pi f_{ij}t)\ i2\pi f_{ij} = m\ x_{ij}(t)\ i2\pi f_{ij} = a\ x_{ij}(t)\ f_{ij}$; m being the electron moving mass and $a = m\ i2\pi$

$x = [x_{ij}]$, $p = [p_{ij}]$, $x_{ji} = x_{ij}^*$, $f_{ji} = (E_j - E_i)/h = -f_{ij}$

$$\begin{aligned}
[x, p] &= xp - px = [\Sigma_k\, x_{ik}\, p_{kj}] - [\Sigma_k\, p_{ik}\, x_{kj}] = [\Sigma_k\, x_{ik}\, a\, x_{kj}\, f_{kj}] - [\Sigma_k\, a\, x_{ik}\, f_{ik}\, x_{kj}] \\
&= a\, [\Sigma_k\, (f_{kj} - f_{ik})\, x_{ik}\, x_{kj}] = a\, [(\Sigma_k\, (f_{kj} - f_{ik})\, x_{ik}\, x_{kj})_{i\neq j} + (\Sigma_k\, (f_{kj} - f_{ik})\, x_{ik}\, x_{kj})_{i=j}] \\
&= a\, [0 + \Sigma_k\, (f_{kj} - f_{jk})\, x_{jk}\, x_{kj}] = a\, [\Sigma_k\, 2\, f_{kj}\, x_{kj}^*\, x_{kj}] = a\, [\Sigma_k\, 2\, f_{kj}\, |x_{kj}|^2] \\
&= i\, [\Sigma_k\, 2\, m\, 2\pi f_{kj}\, |x_{kj}|^2] = i\hbar I
\end{aligned}$$

That is

$$[x, p] = i\, [\Sigma_k\, 2\, m\, 2\pi f_{kj}\, |x_{kj}|^2] = i\hbar I \tag{2}$$

For the last relation, note that the stationary orbit condition of Bohr for the atomic electron was: $m\ v\ r = n\hbar$; then, $n\hbar = m\ v\ r = m\ \omega r\ r = m\ 2\pi f\ r^2$, where n is a positive integer, $\omega = 2\pi f$ the angular frequency and r the radius of the orbit.

Note also that it would be

$$(\Sigma_k\ (f_{kj} - f_{ik})\ x_{ik}\ x_{kj})_{i \neq j} = 0 \qquad (3)$$

which implies that $[x, p] = xp - px$ is a diagonal matrix.

To demonstrate this, Jordan [2] used the Hamilton's equations: $\dot{q} = dq/dt = \partial H/\partial p$ and $\dot{p} = -\partial H/\partial q$, where q, p and H are the (canonical) position, the momentum and the Hamiltonian, respectively. Then

$$d[q,p]/dt = d(qp - pq)/dt = \dot{q}p + q\dot{p} - \dot{p}q - p\dot{q} = (q\dot{p} - \dot{p}q) + (\dot{q}p - p\dot{q}) = (0) + (0) = 0$$

As

$$[x, p] = [\Sigma_k\ a\ (f_{kj} - f_{ik})\ x_{ik}\ x_{kj}] = [\Sigma_k\ a\ (f_{kj} - f_{ik})\ x_{ik}(0)\ exp(i2\pi f_{ik}t)\ x_{kj}(0)\ exp(i2\pi f_{kj}t)] \qquad (4)$$

where $f_{ij} \neq 0$ for $i \neq j$ but $f_{ij} = 0$ for $i = j$, and as $f_{ik} + f_{kj} = f_{ij}$, then $[x, p]$ is a matrix of the type: $g = [g_{ij}\ exp(i2\pi f_{ij}t)]$. As $dg/dt = [g_{ij}\ exp(i2\pi f_{ij}t)\ i2\pi f_{ij}]$, then $dg/dt = 0$ only if g is diagonal ($i = j$): $g = [g_{ii}]$, which corresponds with (1), (2) and (3).

Now, let be the equation [3] (pp. 207-210):

$dy/dt = B\ y$, where B is a constant (independent of t) matrix with distinct characteristic roots (the equation, $det(B - \lambda I) = 0$, has distinct values of λ)

Doing $y = T\ z$, where T is a non singular ($det\ T \neq 0$) constant matrix whose columns are the eigenvectors of B, we have

$T\ dz/dt = B\ T\ z$

$dz/dt = T^{-1}\ B\ T\ z$

If $T^{-1}\ B\ T = \Lambda$, where Λ is a diagonal matrix, then we have k equations

$dz_k/dt = \lambda_k\ z_k$

whose solutions are

$z_k(t) = exp(\lambda_k t)\ z_k(0)$

But, using only matrices, it is also

$Z(t) = [exp(\lambda_k t)]\ Z(0)$

$Y(0) = T\ Z(0),\ Z(0) = T^{-1}\ Y(0) = T^{-1}\ I = T^{-1}$, doing $Y(0) = I$

$Y(t) = T\ Z(t) = T\ [exp(\lambda_k t)]\ Z(0) = T\ [exp(\lambda_k t)]\ T^{-1}$

If for the matrix B, it can be obtained a diagonal matrix $[exp(\lambda_k t)]$, then, from the matrix (4), it can also be obtained the diagonal matrix (2) with the condition (3).

Note, however, that to make (1) applicable to any (elemental or fundamental) particle, free or bound, and not only for an electron bound to an atom, two methods were used: one due to Dirac and the other using the wave function.

In the first method [4] (pp. 96-98):

$$[x, p_x] = x\, p_x - p_x\, x = i\hbar \{x, p_x\} = i\hbar \qquad (5)$$

where $\{x, p_x\}$ is a (classical) Poisson's bracket (PB), used in classical mechanics. For r dimensions, the general PB is

$$\{u, v\} = \Sigma_r ((\partial u/\partial q_r)\, (\partial v/\partial p_r) - (\partial u/\partial p_r)\, (\partial v/\partial q_r))$$

where u and v are dynamic variables. For the x dimension alone, it would be

$$\{u_x, v_x\} = (\partial u_x/\partial x)\, (\partial v_x/\partial p_x) - (\partial u_x/\partial p_x)\, (\partial v_x/\partial x)$$

then

$$\{x, p_x\} = (\partial x/\partial x)\, (\partial p_x/\partial p_x) - (\partial x/\partial p_x)\, (\partial p_x/\partial x) = (1)\, (1) - (0)\, (0) = 1$$

In the second method, using a function f and a Heisenberg's observable Ω, instead of the Schrödinger's wave function Ψ, which does not exist, it is only a supposition [1]; we have [5] (pp. 56, 183):

$$(\partial/\partial x) \cdot f\, \Omega = (\partial/\partial x)\, (f\, \Omega) = (\partial f/\partial x)\, \Omega + f\, (\partial \Omega/\partial x) = ((\partial f/\partial x) + f\, (\partial/\partial x))\, \Omega$$

$$(\partial/\partial x) \cdot f = (\partial f/\partial x) + f\, (\partial/\partial x)$$

$$[\partial/\partial x, f] = (\partial/\partial x) \cdot f - f\, (\partial/\partial x) = (\partial f/\partial x) + f\, (\partial/\partial x) - f\, (\partial/\partial x) = \partial f/\partial x$$

for $f(x) = x$:

$$[\partial/\partial x, x] = \partial x/\partial x = 1$$

$$-i\hbar\, [\partial/\partial x, x] = -i\hbar$$

$$-i\hbar\, ((\partial/\partial x) \cdot x - x\, (\partial/\partial x)) = -i\hbar$$

$$-i\hbar\, (\partial/\partial x) \cdot x - x\, (-i\hbar\, \partial/\partial x) = -i\hbar$$

and substituting the operator $-i\hbar\, \partial/\partial x$ by p_x, it would be:

$$p_x \cdot x - x\, p_x = -i\hbar$$

$$x\, p_x - p_x \cdot x = i\hbar$$

that is

$$[x, p_x] = x\, p_x - p_x\, x = i\hbar \tag{6}$$

Note, for last, that from (1), it is obtained the Heisenberg's uncertainty relation [5] (pp. 276-277)

$$\Delta x\, \Delta p \geq \frac{\hbar}{2} \tag{7}$$

where Δx and Δp are the uncertainties in the position and in the momentum of the particle, respectively. (7) is applied to any particle (because of this it is considered a principle). Therefore, (1) has to be applicable to any particle.

In summary, the position-momentum commutator is a diagonal matrix.

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