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**TWO HUNDRED CONJECTURES AND ONE
HUNDRED AND FIFTY OPEN PROBLEMS
ON FERMAT PSEUDOPRIMES**

Education Publishing

2013

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(COLLECTED PAPERS)

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Columbus, Ohio 43212
USA
Tel. (614) 485-0721

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EAN: 9781599732572
ISBN: 978-1-59973-257-2

INTRODUCTION

Prime numbers have always fascinated mankind. For mathematicians, they are a kind of “black sheep” of the family of integers by their constant refusal to let themselves to be disciplined, ordered and understood. However, we have at hand a powerful tool, insufficiently investigated yet, which can help us in understanding them: Fermat pseudoprimes. It was a night of Easter, many years ago, when I rediscovered Fermat’s “little” theorem. Excited, I found the first few Fermat absolute pseudoprimes (561, 1105, 1729, 2465, 2821, 6601, 8911...) before I found out that these numbers are already known. Since then, the passion for study these numbers constantly accompanied me.

Exceptions to the above mentioned theorem, Fermat pseudoprimes seem to be more malleable than prime numbers, more willing to let themselves to be ordered than them, and their depth study will shed light on many properties of the primes, because it seems natural to look for the rule studying it’s exceptions, as a virologist search for a cure for a virus studying the organisms that have immunity to the virus.

In this book I gathered together 30 of my articles posted on VIXRA about Poulet numbers (Fermat pseudoprimes to base 2), Carmichael numbers (absolute Fermat pseudoprimes), Fermat pseudoprimes to base 3 and other relative Fermat pseudoprimes, also 30 sequences of such numbers posted by me on OEIS and 150 open problems regarding, of course, these numbers, problems listed in the last part, Part four, of this book.

I titled the book in this way to show how many new and exciting things one can say more about this class of numbers, but, though indeed these collected papers contain 200 conjectures about Fermat pseudoprimes, listed in the Part one of this book, these collected papers contain also many observations about the properties of Fermat pseudoprimes and generic formulas for many subclasses of such numbers. Also, among these 200 conjectures there are some conjectures concerning strictly prime numbers, more precisely few types of primes which have arisen in the study of Fermat pseudoprimes.

Probably that, from the conjectures stated in this book, some of them are false, some of them are naïve and some of them are unimportant, but I do believe that many of them have potential. It should also be mentioned that many of the observations from this book of collected papers are due to hazard (not to appeal to notions like inspiration): I was often seeking to illustrate a certain pattern when I found an entirely different one: this is because most of the calculations were made laborious, not using computer programs, so that on the path between “input” and “output” I could detect interesting relationships, other than those I was looking to find.

All articles in this book use well-known notions of number theory, with only two exceptions, namely one article in which I have defined a new concept, *id est* “a set of Smarandache-Coman divisors of order k of a composite integer n with m prime factors” and one article in which I have showed several application of this concept in the study of Fermat pseudoprimes.

Finally, one last observation: don’t blame us (the author and the publisher) for the heterogeneous aspect of the content of this book, which is the natural consequence of the fact that it is a collection of articles and is not designed as a whole from the beginning.

SUMMARY

Part one. Two hundred conjectures on Fermat pseudoprimes

Part two. Thirty articles on Fermat pseudoprimes

1. A formula for generating primes and a possible infinite series of Poulet numbers
2. A new class of Fermat pseudoprimes and few remarks about Cipolla pseudoprimes
3. Formulas that generate subsets of 3-Poulet numbers and few types of chains of primes
4. A conjecture about 2-Poulet numbers and a question about primes
5. A formula that generates a type of pairs of Poulet numbers
6. A method of finding subsequences of Poulet numbers
7. A possible infinite subset of Poulet numbers generated by a formula based on Wieferich primes
8. Four sequences of integers regarding balanced primes and Poulet numbers
9. Six polynomials in one and two variables that generate Poulet numbers
10. A conjecture about a large subset of Carmichael numbers related to concatenation
11. A conjecture about primes based on heuristic arguments involving Carmichael numbers
12. A conjecture regarding the relation between Carmichael numbers and the sum of their digits
13. A list of 13 sequences of Carmichael numbers based on the multiples of the number 30
14. A possible generic formula for Carmichael numbers
15. An interesting and unexpected property of Carmichael numbers and a question
16. Connections between the three prime factors of 3-Carmichael numbers
17. Formulas for generating primes involving emirps, Carmichael numbers and concatenation
18. Four conjectures regarding Fermat pseudoprimes and few known types of pairs of primes
19. Special properties of the first absolute Fermat pseudoprime, the number 561
20. Six conjectures and the generic formulas for two subsets of Poulet numbers
21. A pattern that relates Carmichael numbers to the number 66
22. A generic formula of 2-Poulet numbers and also a method to obtain sequences of n-Poulet numbers
23. Few interesting results regarding Poulet numbers and Egyptian fraction expansion
24. The Smarandache-Coman divisors of order k of a composite integer n with m prime factors
25. Seventeen sequences of Poulet numbers characterized by a certain set of Smarandache-Coman divisors
26. Few types of chains of primes arising in the study of pseudoprimes
27. Ten conjectures about certain types of pairs of primes arising in the study of 2-Poulet numbers
28. Two conjectures that relates any Poulet number by a type of triplets respectively of duplets of primes
29. A type of primes that seem to lead to sequences of infinite Poulet numbers in a recurrent formula

30. A very interesting formula of a subset of Poulet numbers involving consecutive powers of a power of two

Part three. Thirty sequences of Fermat pseudoprimes

1. Poulet numbers with two prime factors
2. Poulet numbers with three prime factors
3. Poulet numbers with three prime factors divisible by a smaller Poulet number
4. Poulet numbers of the form $(6^k + 1)(6^k n + 1)$, where k, n are integers different from 0
5. Poulet numbers of the form $(6^k - 1)((6^k - 2)^n + 1)$, where k, n are integers different from 0
6. Poulet numbers of the form $7200n^2 + 8820n + 2701$
7. Poulet numbers of the form $144n^2 + 222n + 85$
8. Poulet numbers of the form $8p^n + p^2$, where p is prime
9. Poulet numbers of the form $(n^2 + 2n)/3$
10. Poulet numbers that can be written as $2p^2 - p$, where p is also a Poulet number
11. Poulet numbers of the form $m^n^2 + (11m - 23)n + 19m - 49$
12. Poulet numbers that can be written as $(p^2 + 2p)/3$, where p is also a Poulet number
13. Poulet numbers that can be written as $p^2 n - p^n + p$, where p is also a Poulet number
14. Primes of the form $(24p + 1)/5$, where p is a Poulet number
15. The smallest m for which the n -th Carmichael number can be written as $p^{2(m+1)} - p^m$
16. Carmichael numbers of the form $(30k + 7)(60k + 13)(150k + 31)$
17. Carmichael numbers of the form $C = (30n - 7)(90n - 23)(300n - 79)$
18. Carmichael numbers of the form $C = (30n - 17)(90n - 53)(150n - 89)$
19. Carmichael numbers $C = (60k + 13)(180k + 37)(300k + 61)$
20. Carmichael numbers $C = (30n - 29)(60n - 59)(90n - 89)(180n - 179)$
21. Carmichael numbers $C = (330k + 7)(660k + 13)(990k + 19)(1980k + 37)$
22. Carmichael numbers of the form $C = (30n - p)(60n - (2p + 1))(90n - (3p + 2))$, where $p, 2p + 1, 3p + 2$ are all three primes
23. Carmichael numbers of the form $C = p(2p - 1)(3p - 2)(6p - 5)$, where p is prime
24. Carmichael numbers of the form $C = p(2p - 1)(n(2p - 2) + p)$, where p and $2p - 1$ are primes
25. Carmichael numbers of the form $n(2n - 1)(p^n - p + 1)(2p^n - 2p + 1)$, where p is odd
26. Carmichael numbers of the form $3n(9n + 2)(18n - 1)$, where n is odd
27. Carmichael numbers that have only prime divisors of the form $10k + 1$
28. Carmichael numbers divisible by a smaller Carmichael number
29. Carmichael numbers divisible by 1729
30. Fermat pseudoprimes n to base 3 of the form $n = (3^{4k+2} - 1)/8$

Part four. One hundred and fifty open problems regarding Fermat pseudoprimes

Part one. Two hundred conjectures on Fermat pseudoprimes

Conjecture 1: There are infinite many Poulet numbers of the form $(4^k - 1)/3$, where k is positive integer.

Conjecture 2: Any number of the form $(4^k - 1)/3$, where k is prime, $k \geq 5$, is a Poulet number.

Conjecture 3: The formula $(n^{(n^k + k + n - 1)} - 1)/(n^2 - 1)$ generates an infinity of Fermat pseudoprimes to base n for any integer n , $n > 1$.

Conjecture 4: Any 3-Poulet number which has not a prime factor of the form $30k + 23$ can be written as $p^*((n + 1)^*p - n^*p)^*((m + 1)^*p - m^*p)$ or as $p^*((n^*p - (n + 1)^*p)^*(m^*p - (m + 1)^*p)$.

Conjecture 5: Any Poulet number with two prime factors can be written as $P = (q - 30^*n)^*(r + 30^*n)$, where q and r are primes or are equal to 1 and n is positive integer, $n \geq 1$.

Conjecture 6: There is an infinity of Poulet numbers of the form $p^2 + 81^*p + 3^*p^*q$, where p is a prime of the form $30^*k + 13$ and q is a prime of the form $30^*k + 41$, where k is an integer, $k \geq 0$.

Conjecture 7: There is an infinity of Poulet numbers of the form $p^2 + 81^*p + 3^*p^*q$, where p is a prime of the form $30^*k + 41$ and q is a prime of the form $30^*k + 13$, where k is an integer, $k \geq 0$.

Conjecture 8: If the number $p^2 + 81^*p + 3^*p^*q$, where p is a prime of the form $30^*k + 13$ and q is a prime of the form $30^*k + 41$, is a Poulet number, then the number $p^2 + 81^*p + 3^*p^*q$, where p is a prime of the form $30^*k + 41$ and q is a prime of the form $30^*k + 13$ is a Poulet number too (k is an integer, $k \geq 0$).

Conjecture 9: For every Wieferich prime p there is an infinity of Poulet numbers which are equal to $n^*p - n + 1$, where n is integer, $n > 1$.

Conjecture 10: The numbers formed through deconcatanation of Carmichael numbers not divisible by 5 that ends in the digits that form a number of the form $6^*k - 1$ and the respective number are congruent to 2 (mod 6) or to 5 (mod 6).

Conjecture 11: There is no a Carmichael number with three prime divisors to can be written as $(6^*x + 1)^*(6^*y + 1)^*(6^*z - 1)$, they are all of the form $(6^*x + 1)^*(6^*y + 1)^*(6^*z + 1)$, $(6^*x - 1)^*(6^*y - 1)^*(6^*z - 1)$ or $(6^*x + 1)^*(6^*y - 1)^*(6^*z - 1)$.

Conjecture 12: For any Carmichael number C that has only prime factors of the form $6^*k + 1$ is true at least one of the following five relations:

- (1) C is a Harshad number;
- (2) If we note with $s(m)$ the sum of the digits of the integer m then C is divisible by $n^*s(C) - n + 1$, where n is integer;
- (3) C is divisible by $s((C + 1)/2)$;
- (4) C is divisible by $n^*s((C + 1)/2) - n + 1$, where n is integer;
- (5) $s(C) = s((C + 1)/2)$.

Conjecture 13: The number $(30*n + 7)*(60*n + 13)*(150*n + 31)$ is a Carmichael number if (but not only if) $30*n + 7$, $60*n + 13$ and $150*n + 31$ are all three prime numbers.

Conjecture 14: The number $(30*n - 29)*(60*n - 59)*(90*n - 89)*(180*n - 179)$ is a Carmichael number if (but not only if) $30*n - 29$, $60*n - 59$, $90*n - 89$ and $180*n - 179$ are all four prime numbers.

Conjecture 15: The number $(330*n + 7)*(660*n + 13)*(990*n + 19)*(1980*n + 37)$ is a Carmichael number if $330*n + 7$, $660*n + 13$, $990*n + 19$ and $1980*n + 37$ are all four prime numbers.

Conjecture 16: The number $(30*n - 7)*(90*n - 23)*(300*n - 79)$ is a Carmichael number if (but not only if) $30*n - 7$, $90*n - 23$ and $300*n - 79$ are all three prime numbers.

Conjecture 17: The number $(30*n + 13)*(90*n + 37)*(150*n + 61)$ is a Carmichael number if (but not only if) $30*n + 13$, $90*n + 37$ and $150*n + 61$ are all three prime numbers.

Conjecture 18: Any Carmichael number can be written as $(n^2*p^2 - q^2)/(n^2 - 1)$, where p and q are primes or power of primes or are equal to 1 and n is positive integer, $n > 1$.

Conjecture 19: For any Carmichael numbers with three prime factors, $C = d_1*d_2*d_3$, where $d_1 < d_2 < d_3$, is true one of the following two statements:

(1) d_2 can be written as $d_1*(n + 1) - n$ and d_3 can be written as $d_1*(m + 1) - m$;

(2) d_2 can be written as $d_1*n - (n + 1)$ and d_3 can be written as $d_1*m - (m + 1)$,

where m and n are natural numbers.

Conjecture 20: There is an infinity of reversible primes p with the property that the number obtained through concatenation of the digits of p with a number of n digits of 0, where n is equal to one less than the digits of p , and finally with the digit 1 is a prime.

Conjecture 21: If p and $2p + 1$ are both primes, then the number $n = p*(2p + 1) - 2^k*p$ is Fermat pseudoprime to base $p + 1$ for at least one natural value of k .

Conjecture 22: If p and $2p - 1$ are both primes, $p > 3$, then the number $n = p*(2p - 1) - 2^k*p$ is Fermat pseudoprime to base $p - 1$ for at least one natural value of k .

Conjecture 23: If p and q are primes, where $q = \sqrt{2*p - 1}$, then the number $p*q$ is Fermat pseudoprime to base $p + 1$.

Conjecture 24: If p is prime, $p > 3$, and k integer, $k > 1$, then the number $n = p*(k*p - k + 1)$ is Fermat pseudoprime to base $k*p - k$ and to base $k*p - k + 2$.

Conjecture 25: Any prime number p can be written as $p = (C*q - 1)/(q - 1)$, where C is a Carmichael number and q is a prime.

Conjecture 26: If the number $360*(a*b) + 1$, where a and b are primes, is equal to c^2 , where c is prime, then exists an infinite series of Carmichael numbers of the form $a*b*d$, where d is a natural number (obviously odd, but not necessarily prime).

Conjecture 27: The expression $n^E \bmod 544 = n$, where n is any natural number, is true if E is an Euler pseudoprime.

Conjecture 28: For any biggest prime factor of a Poulet number p_1 with two prime factors exists a series with infinite many Poulet numbers p_2 formed this way: $p_2 \bmod (p_1 - d) = d$, where d is the biggest prime factor of p_1 .

Conjecture 29: Any Poulet number p_2 divisible by d can be written as $(p_1 - d)*n + d$, where n is natural, if exists a smaller Poulet number p_1 with two prime factors divisible by d .

Conjecture 30: For any biggest prime factor of a Poulet number p_1 exists a series with infinite many Poulet numbers p_2 formed this way: $p_2 \bmod (p_1 - d) = d$, where d is the biggest prime factor of p_1 .

Conjecture 31: Any Poulet number P with three or more prime divisors has at least one prime divisor q for that can be written as $P = q*((r + 1)*q - r)$, where r is a natural number.

Conjecture 32: The only Poulet number divisible by a smaller Poulet number that can't be written as $p*((m + 1)*p - m)*((n + 1)*p - n)$, where m, n, p are natural numbers, are multiples of 5461 and can be written as $5461*(42*k - 13)$.

Conjecture 33: There are infinite many Poulet numbers of the form $7200*n^2 + 8820*n + 2701$.

Conjecture 34: There are infinite many Poulet numbers of the form $144*n^2 + 222*n + 85$.

Conjecture 35: If a Poulet number can be written as $8*p^n + p^2$, where n is an integer number and p one of it's prime factors, than can be written this way for any of it's prime factors.

Conjecture 36: For any m natural, $m > 1$, there exist a series with infinite many Fermat pseudoprimes to base 2, P , formed this way: $P = (n^m + m*n)/(m + 1)$.

Conjecture 37: There are infinite many Poulet numbers that can be written as $(n + 1)*p^2 - n*p$, where n is natural, $n > 0$, and p is another Poulet number.

Conjecture 38: For any Poulet number p there are infinite many Poulet numbers that can be written as $(n + 1)*p^2 - n*p$, where n is natural, $n > 0$.

Conjecture 39: For any Poulet number, p_1 , there exist infinite many Poulet numbers, p_2 , formed this way: $p_2 = (p_1^n + n*p_1)/(n + 1)$, where n natural, $n > 1$.

Conjecture 40: For any Carmichael number, C_1 , there exist infinite many Carmichael numbers, C_2 , formed this way: $C_2 = (C_1^n + n*C_1)/(n + 1)$, where n natural, $n > 1$.

Conjecture 41: There is no absolute Fermat pseudoprime m for which $n = (5*m - 1)/24$ is a natural number.

Conjecture 42: Any Carmichael number C divisible by p and $2p - 1$ (where p and $2p - 1$ are prime numbers) can be written as $C = p*(2p - 1)*(n*(2p - 2) + p)$.

Conjecture 43: For any odd number p we have an infinite number of Carmichael numbers of the form $n^{2n-1}(p^n - p + 1)(2p^n - 2p + 1)$.

Conjecture 45: A Carmichael number C_1 can be written as $C_1 = (C_2 + C_3)/2$, where C_2 and C_3 are also Carmichael numbers, only if both C_1 and C_3 are divisible by C_2 .

Conjecture 44: All Carmichael numbers C (not only with three prime divisors) of the form $10^n + 1$ that have only prime divisors of the form $10^k + 1$ can be written as $C = (30^a + 1)(30^b + 1)(30^c + 1)$, $C = (30^a + 11)(30^b + 11)(30^c + 11)$, or $C = (30^a + 1)(30^b + 11)(30^c + 11)$. In other words, there are no such numbers of the form $C = (30^a + 1)(30^b + 1)(30^c + 11)$.

Conjecture 46: If $m^{126} + n = 1729$, $m^{126} > n$, then exists a series with infinite many Carmichael terms of the form $C \bmod m^{234} = n$.

Conjecture 47: If $m^{234} + n = 1729$, $m^{234} > n$, then exists a series with infinite many Carmichael terms of the form $C \bmod m^{234} = n$.

Conjecture 48: If $m^{342} + n = 1729$, $m^{342} > n$, then exists a series with infinite many Carmichael terms of the form $C \bmod m^{342} = n$.

Conjecture 49: For any prime factor of a Carmichael number C_1 exists a series with infinite many Carmichael terms C_2 formed this way: $C_2 \bmod m^{18d} = n$, where $m^{18d} + n = C_1$, where d is the prime factor of C_1 and m, n are natural numbers, $m^{18d} < n$.

Conjecture 50: There are infinitely many Fermat pseudoprimes to base 3 of the form $(3^{4k+2} - 1)/8$, where k is a natural number.

Conjecture 51: Any Poulet numbers P which have the numbers $p = 23$ and $q = 67$ as prime factors can be written as $P = p^*q^*(n^*(q - 1) + p) = 3^*p^3(3^*n + 1) - p^2(15^*n + 2) + 6^*p^*n$, where n non-null positive integer.

Conjecture 52: Any Poulet numbers P which have the numbers $p = 30^*k + 23$ and $q = 90^*k + 67$, where k non-negative integer, as prime factors can be written as $P = 3^*p^3(3^*n + 1) - p^2(15^*n + 2) + 6^*p^*n$, where n non-null positive integer.

Conjecture 53: There is an infinity of Poulet numbers which have the numbers $p = 30^*k + 23$ and $q = 90^*k + 67$, where k non-negative integer, as prime factors (implicitly there is an infinity of pairs of primes of the form $[30^*k + 23, 90^*k + 67]$).

Conjecture 54: Any Poulet numbers P which have the numbers $p = 11$ and $q = 61$ as prime factors can be written as $P = p^*q^*(n^*(q - 1) + p) = 6^*p^3(6^*n + 1) - p^2(66^*n + 5) + 30^*p^*n$, where n non-null positive integer.

Conjecture 55: Any Poulet numbers P which have the numbers $p = 30^*k + 11$ and $q = 180^*k + 61$, where k non-negative integer, as prime factors can be written as $P = 6^*p^3(6^*n + 1) - p^2(66^*n + 5) + 30^*p^*n$, where n non-null positive integer.

Conjecture 56: There is an infinity of Poulet numbers which have the numbers $p = 30*k + 11$ and $q = 180*k + 61$, where k non-negative integer, as prime factors (implicitly there is an infinity of pairs of primes of the form $[30*k + 11, 180*k + 61]$).

Conjecture 57: The length of the period of the rational number which is the sum, from $n = 1$ to $n = \infty$, of the numbers $1/(C_n - 1)$, where $\{C_1, C_2, \dots, C_n\}$ is the ordered set of Carmichael numbers, is always multiple of 66.

Conjecture 58: Any 2-Poulet number P can be written at least in one way as $P = (q*2^a*3^b*5^c \pm 1)*2^n + 1$, where q is a prime, a square of prime or a semiprime, a, b, c are non-negative integers and n is non-null positive integer.

Conjecture 59: If r is equal to the positive rational number $1/(d_1 - 1) + 1/(d_2 - 1) + \dots + 1/(d_n - 1)$, where d_1, \dots, d_n are the prime factors of a Poulet number P , and m is equal to the last denominator obtained applying the Egyptian fraction expansion to r , then the number $m + 1$ is a prime or a power of prime for an infinity of Poulet numbers.

Conjecture 60: If r is equal to the positive rational number $1/(d_1 - 1) + 1/(d_2 - 1) + \dots + 1/(d_n - 1)$, where d_1, \dots, d_n are the prime factors of a Poulet number P , and r is represented by the irreducible fraction x/y , where x, y positive integers, then the number $y + 1$ is a prime or a power of prime for an infinity of Poulet numbers.

Conjecture 61: If d_1, \dots, d_n are the prime factors of a Poulet number P , then the number $\text{lcm}((d_1 - 1), (d_2 - 1), \dots, (d_n - 1))$ is a prime or a power of prime for an infinity of Poulet numbers.

Conjecture 62: There is an infinity of 2-Poulet numbers which have the set of Smarandache-Coman divisors of order 1 equal to $\{p, p\}$, where p is prime.

Conjecture 63: There is an infinity of 2-Poulet numbers which have the set of SC divisors of order 2 equal to $\{p, p + 20*k\}$, where p is prime and k is non-null integer.

Conjecture 64: There is an infinity of 2-Poulet numbers which have the set of SC divisors of order 2 equal to $\{a, b\}$, where $a + b + 1$ is prime.

Conjecture 65: There is an infinity of 2-Poulet numbers which have the set of SC divisors of order 2 equal to $\{a, b\}$, where $a + b - 1$ is prime.

Conjecture 66: There is an infinity of 2-Poulet numbers which have the set of SC divisors of order 2 equal to $\{a, b\}$, where $a + b - 1$ and $a + b + 1$ are twin primes.

Conjecture 67: There is an infinity of pairs of 2-Poulet numbers which have the set of SC divisors of order 2 equal to $\{a, b\}$, respectively to $\{c, d\}$, where $a + b = c + d$ and a, b, c, d are primes.

Conjecture 68: There is an infinity of pairs of 2-Poulet numbers which have the set of SC divisors of order 2 equal to $\{a, b\}$, respectively to $\{c, d\}$, where $a + b + 1 = c + d - 1$.

Conjecture 69: There is an infinity of 2-Poulet numbers which have the set of SC divisors of order 6 equal to $\{p, q\}$, where $\text{abs}\{p - q\} = 6*k$, where p and q are primes and k is non-null positive integer.

Conjecture 70: There is an infinity of 2-Poulet numbers which have the set of SC divisors of order 6 equal to $\{a, b\}$, where $\text{abs}\{a - b\} = p$ and p is prime.

Conjecture 71: There is an infinity of 2-Poulet numbers which have the set of SC divisors of order 6 equal to $\{p, q\}$, where one from the numbers p and q is prime and the other one is twice a prime.

Conjecture 72: There is an infinity of 3-Poulet numbers which have the set of SC divisors of order 1 equal to $\{a, b, c\}$, where $a + b + c$ is prime and a, b, c are primes.

Conjecture 73: There is an infinity of 3-Poulet numbers which have the set of SC divisors of order 1 equal to $\{a, b, c\}$, where $a + b + c - 1$ and $a + b + c + 1$ are twin primes.

Conjecture 74: There is an infinity of 3-Poulet numbers which have the set of SC divisors of order 1 equal to $\{n, n, n\}$.

Conjecture 75: There is an infinity of 3-Poulet numbers which have the set of SC divisors of order 2 equal to $\{5, p, q\}$, where p and q are primes and $q = p + 6*k$, where k is non-null positive integer.

Conjecture 76: There is an infinity of Poulet numbers divisible by 15 which have the set of SC divisors of order 1 equal to $\{2, 4, 7, n_1, \dots, n_i\}$, where n_1, \dots, n_i are non-null positive integers and $i > 0$.

Conjecture 77: There is an infinity of Poulet numbers divisible by 15 which have the set of SC divisors of order 1 equal to $\{2, 4, 23, n_1, \dots, n_i\}$, where n_1, \dots, n_i are non-null positive integers and $i > 0$.

Conjecture 78: There is an infinity of Poulet numbers which are multiples of any Poulet number divisible by 15 which has the set of SC divisors of order 1 equal to $\{2, 4, n_1, \dots, n_i\}$, where $n_1 = n_2 = \dots = n_i = 7$ and $i > 0$.

Conjecture 79: There is an infinity of pairs of primes, not necessarily consecutive, of the form $(p, p + 24)$.

Conjecture 80: There is, for any odd prime $q, q > 3$, an infinity of primes of the form $q + p*(p + 24) - 1$, where p and $p + 24$ are both primes.

Conjecture 81: There is, for any pair of twin primes $(p, q), (p, q) \neq (3, 5)$, an infinity of pairs of twin primes of the form $(p + m*n - 1, q + m*n - 1)$, where m and n are both primes and $n = m + 24$.

Conjecture 82: There is an infinity of 2-Poulet numbers $P, P = p_1*p_2$, for which $p_2 - p_1 + 1 = q_1*q_2$, where q_1 and q_2 are primes and $q_2 - q_1 = 24$.

Conjecture 83: There is an infinity of 2-Poulet numbers $P, P = p_1*p_2$, for which $p_2 - p_1 + 1 = q$, where q is prime or a square of prime.

Conjecture 84: There is an infinity of 2-Poulet numbers $P, P = p_1*p_2$, for which $p_2 - p_1 - 1 = q$, where q is prime or a square of prime.

Conjecture 85: There is an infinity of 2-Poulet numbers P , $P = p_1 * p_2$, for which $p_2 - p_1 - 1$ and $p_2 - p_1 + 1$ are both primes.

Conjecture 86: For any pair of twin primes (q_1, q_2) there exist at least a pair of primes (p_1, p_2) such that $q_1 = p_2 - p_1 - 1$ and $q_2 = p_2 - p_1 + 1$ are both primes and $P = p_1 * p_2$ is a 2-Poulet number.

Conjecture 87: There is an infinity of pairs of primes, not necessarily consecutive, of the form $(p, p + 84)$.

Conjecture 88: There is an infinity of 2-Poulet numbers P , $P = p_1 * p_2$, for which $p_2 - p_1 = 84$.

Conjecture 89: Any Carmichael number C divisible by 7 and 13 can be written as $C = 7 * 13 * (18 * k + 1)$ or as $C = 7 * 13 * (18 * k + 13)$, where k is natural.

Conjecture 90: Any Carmichael number C divisible by 19 and 37 can be written as $C = 19 * 37 * (18 * k + 1)$, where k is natural.

Conjecture 91: Any Carmichael number C can be written as a product of three numbers in one of two ways:

I. $C = n * (a * n - a + 1) * (b * n - b + 1)$, where n is the smallest prime divisor of C and a, b natural numbers;

II. $C = n * (a * n - a - 1) * (b * n - b - 1)$, where n is the smallest prime divisor of C and a, b natural numbers.

Conjecture 92: Any Poulet number P , except for a set of definable exceptions, can be written as $P = ((d - 1) * (d + m) / m) + 1$, where d is any prime factor of P (the first exception is the 15-th Poulet number, the number 4371).

Conjecture 93: For any Poulet number P , except for a set of definable exceptions, is true one of the following two relations: $(P/d) - 1$ is divisible by $d - 1$ or $d - 1$ is divisible by $(P/d) - 1$ for any d prime factor of P (the first exception is the 15-th Poulet number, the number 4371).

Conjecture 94: For any 2-Poulet number P , except for a set of definable exceptions, is true the following relation: $P = ((d - 1) * (d + m) / m) + 1$, where m is natural and d is any of the two prime factors of P (the first three exceptions are the 19-th, the 35-th and the 38-th Poulet numbers, *i.e.* the numbers $7957 = 73 * 109$, $18721 = 97 * 193$ and $23377 = 97 * 241$).

Conjecture 95: For any Poulet number P there exist n such that $P - 80 * n$ is prime.

Conjecture 96: For any Poulet number P there exist n such that $P - 150 * n$ is prime.

Conjecture 97: For any 3-Carmichael number $C = d_1 * d_2 * d_3$, except for a set of definable exceptions, is true that $d_3^2 > d_1^2 + d_2^2$ (the first exception is the Carmichael number $116682721 = 281 * 617 * 673$).

Conjecture 98: Any Carmichael number C , except for a set of definable exceptions, can be written as $C = p * q + p - q$, where p and q are primes.

Conjecture 99: Any Carmichael number C of the form $10^*k + 1$, not divisible by 3, can be written at least in one of the following two forms: $C = 1800^*n^2 + 840^*n + p$ or $C = 1800^*n^2 - 840^*n - q$, where p and q are primes (p of the form $30^*h + 1$ and q of the form $30^*h + 29$).

Conjecture 100: For any 2-Poulet number $P = d_1^*d_2$, except for a set of definable exceptions, there exist the non-null positive integers m, n such that $P \bmod (n^*d_2 - n) = (m + 1)^*d_1 - m$.

Conjecture 101: The number $N = p^*(p + 2) + 1729$, where p and $p + 2$ are twin primes, is always equal to a product of powers of the numbers 2 and 3 and of prime factors of the form $30^*k + 1, 30^*k + 7, 30^*k + 13$ and $30^*k + 19$.

Conjecture 102: Any Poulet number P of the form $10^*k + 1, 10^*k + 5$ or $10^*k + 9$ (not of the form $10^*k + 3$ or $10^*k + 7$) can be written as $P = (a^2 - b^2)/8$, where a and b are non-null positive integers.

Conjecture 103: Let C be a Carmichael number and $s(C)$ be its digital root. If C is not a Harshad number and $s(C)$ is not equal to 1, then any prime factor d of C , $d \neq 3$, can be written as $d = n^*s(C) - n \pm 1$, where n is a non-null positive integer.

Conjecture 104: Let C be a Carmichael number and $s(C)$ be the sum of its digits. Any C except for a set of definable exceptions can be written as $C = n^*s(C) - n + 1$, where n is a non-null positive integer.

Conjecture 105: There exist an infinity of 3-Carmichael numbers $C = d_1^*d_2^*d_3$ for which the number $N = 2^*(d_1 + d_2 + d_3 - 6)^*(C - 2) - 1$ is also a Carmichael number.

Conjecture 106: There exist an infinity of n -Carmichael numbers $C = d_1^*d_2^* \dots^*d_n$ for which the number $N = 2^*(d_1 + d_2 + \dots + d_n - 2^*n)^*(C - 2) - 1$ is also a Carmichael number.

Conjecture 107: For any Carmichael number C , except for a set of definable exceptions, is true at least one of the two following statements:

- (1) $C - 1$ is divisible by $s(C) - 1$;
- (2) $C - 1$ is divisible by $s(C) + 1$, where $s(C)$ is the sum of digits of C .

Conjecture 108: Any Carmichael number C can be written as $C = (n^*(n + p))/8$, where n is non-null positive integer and p is prime or power of prime.

Conjecture 109: There is an infinity of Carmichael numbers of the form $(104^*(104 + p))/8$, where p is prime (for $p = 29$ and 113 are obtained Carmichael numbers 1729 and 2821).

Conjecture 110: There is an infinity of primes p of the form $p = 7^*q - 104$, where q is a prime of the form $6^*k + 1$.

Conjecture 111: There is an infinity of primes p of the form $p = q_1^*q_2 - 8^*(q_1 + 6)$, where q_1 and $q_1 + 6$ are primes of the form $6^*k + 1$ and q_2 is also prime of the form $6^*k + 1$.

Conjecture 112: Any Carmichael number C can be written as $C = (n^*(n + a))/(a + 1)$, where n and a are non-null positive integers.

Conjecture 113: For any Poulet number P_1 there is an infinity of primes p such that $P_1 + p*(p - 1) = P_2$, where P_2 is also a Poulet number.

Conjecture 114: For any Poulet number P_1 there is an infinity of primes p such that $P_1 + 2*p*(p - 1) = P_2$, where P_2 is also a Poulet number.

Conjecture 115: There is an infinity of Carmichael numbers C of the form $C = 2821 + 26520*n$, where n is non-negative integer (for $n = 0, 1$ and 7 are obtained the Carmichael numbers 2821, 29341 and 188461).

Conjecture 116: There is an infinity of Carmichael numbers C of the form $C = p^4 + 60*p$, where p is prime (for $p = 7$ and 13 are obtained the Carmichael numbers 2821 and 29341).

Conjecture 117: Any Carmichael number C can be written as $C = 2^n*p + q$, where p and q are primes and $\text{abs}\{p - q\} = 10*k$, where k is non-negative integer.

Conjecture 118: There is an infinity of Carmichael numbers C of the form $C = n*(n + 1) + p^2$, where n is non-null positive integer and p is prime (for n equal to 23 and 32 and p equal to 3 and 7 are obtained 561 and 1105).

Conjecture 119: There is an infinity of Carmichael numbers C which can be written as $C = 2^m + 7^n$, where m and n are non-null positive integers (for m equal to 9 and 6 and n equal to 2 and 4 are obtained 561 and 2465).

Conjecture 120: There is an infinity of Carmichael numbers C which can be written as $C = (p^2 + n)/(n + 1)$, where p is prime and n non-null positive integer.

Conjecture 121: There is an infinity of Carmichael numbers C which can be written as $C = (p^2*q^2 + n)/(n + 1)$, where p and q are primes and n non-null positive integer.

Conjecture 122: There is an infinity of Carmichael numbers C which can be written as $C = (\text{sqrt}(4*p - 1)/3) + 2$, where p is prime.

Conjecture 123: Any Carmichael number C can be written as $C = p*q + q - p$, where p and q are primes (appears that often $q - p$ is of the form $10*k$).

Conjecture 124: Any 3-Carmichael number C divisible by 5 can be written as $C = p*(q - 2^n)$, where p is one of its prime factors beside 5, q is a prime and n is non-null positive integer.

Conjecture 125: For any p odd prime, $p \neq 5$, there exist an infinity of numbers n such that $q = 5*p + 2^n$ is prime.

Conjecture 126: For any Poulet number P , P not divisible by 3 and 5, there exist non-null positive integers m, n such that the number $(P + 5^m)/(6^n)$ is prime or power of prime.

Conjecture 127: For any Poulet number P , P divisible by 5 and not divisible by 3, there exist a non-null positive integer m such that the number $(P + 5^m)/30$ is prime.

Conjecture 128: For any Poulet number P , P divisible by 3 and 5, there exist a non-null positive integer m such that the number $(P + 5^m)/10$ is prime.

Conjecture 129: For any n non-null positive integer, n not divisible by 5, except for a set of definable exceptions, there exist a non-null positive integer m such that the number $6^n - 5^m$ is prime or is equal to 1 (the first three exceptions are the numbers 134, 341 and 431).

Conjecture 130: For any Poulet numbers P_1 and P_2 there exist an infinity of primes p such that $P_1 + P_2 + p$ is also a Poulet number.

Conjecture 131: For any Poulet numbers P_1, P_2, \dots, P_n there exist an infinity of primes p such that $P_1 + P_2 + \dots + P_n + p$ is also a Poulet number.

Conjecture 132: There exist an infinity of Carmichael numbers C which can be written as $C = 1729 \cdot (2^m \cdot 2^n + 1)$.

Conjecture 133: There exist an infinity of Carmichael numbers C_1 for which exist a Carmichael number C_2 such that $C_2 = 2 \cdot C_1 - d$, where d is a prime factor of C_1 .

Conjecture 134: There exist an infinity of pairs of Carmichael numbers C_1 and C_2 such that $C_2^2 - C_1^2 = n^2$, where n is non-null positive integer.

Conjecture 135: There exist an infinity of distinct Carmichael numbers C_1, C_2 and C_3 which satisfy the formula $2 \cdot C_2 - C_1 = C_3$.

Conjecture 136: For any Poulet number $P = n + d$, where d is a prime factor of P , there exist an infinity of primes p that can be written as $p = (n + d^k)/d$, where k positive integer.

Conjecture 137: There exist an infinity of primes p which can be written as $p = (P + n)/(n + 1)$, for any Poulet number P , where n is non-null positive integer.

Conjecture 138: There exist an infinity of primes p which can be written as $p = \sqrt{(n \cdot P - n + 1)}$, for any Poulet number P , where n is non-null positive integer.

Conjecture 139: There exist an infinity of primes p which can be written as $p = (q^2 + n)/(n + 1)$, for any prime number q , where n is non-null positive integer.

Conjecture 140: There exist an infinity of primes p which can be written as $p = \sqrt{((q^2 + n)/(n + 1))}$, for any prime number q , where n is non-null positive integer.

Conjecture 141: There exist an infinity of 3-Poulet numbers $P = d_1 \cdot d_2 \cdot d_3$, where $d_1 < d_2 < d_3$, for which the numbers $2 \cdot d_1 + 1$, $3 \cdot d_2 + 2$ and $3 \cdot d_3 + 2$ are all three primes.

Conjecture 142: There exist, for any Poulet number $P = d_1 \cdot d_2 \cdot d_3$, where $d_1 < d_2 < d_3$, an infinity of positive integers n such that the numbers $n \cdot d_1 + n - 1$, $(n + 1) \cdot d_2 + n$ and $(n + 1) \cdot d_3 + n$ are all three primes.

Conjecture 143: There exist an infinity of 3-Poulet numbers $P = d_1 \cdot d_2 \cdot d_3$, where $d_1 < d_2 < d_3$, for which the number $p = 60 \cdot d_1 + 30 \cdot d_2 + d_3$ is prime or square of prime.

Conjecture 144: There exist an infinity of 4-Poulet numbers $P = d_1 \cdot d_2 \cdot d_3 \cdot d_4$, where $d_1 < d_2 < d_3 < d_4$, for which the number $p = 90 \cdot d_1 + 60 \cdot d_2 + 30 \cdot d_3 + d_4$ is prime or square of prime.

Conjecture 145: There exist an infinity of n-Poulet numbers $P = d_1 * d_2 * d_3 * \dots * d_n$, where $d_1 < d_2 < d_3 < \dots < d_n$, for which the number $p = (n - 1) * 30 * d_1 + \dots + 2 * 30 * d_2 + 30 * d_3 + d_4$ is prime or square of prime.

Conjecture 146: For any 3-Poulet number P one of the two following statements it's true:

- (1) $36 * (P - d)$ is divisible by $(d^2 - d)$;
- (2) $36 * (d^2 - d)$ is divisible by $(P - d)$, where d is any prime factor of P.

Conjecture 147: For any 2-Poulet number P, squarefree, one of the two following statements it's true:

- (1) $36 * (P - d)$ is divisible by $(d^2 - d)$;
- (2) $36 * (d^2 - d)$ is divisible by $(P - d)$, where d is any prime factor of P.

Conjecture 148: For any 2-Poulet number P, squarefree, the following statement it's true: $360 * (P - d_1) / (d_1^2 - d_1) = 360 * (d_2^2 - d_2) / (P - d_2) = m$, where d_1 and d_2 are the two prime factors and m is a natural number.

Conjecture 149: For any Fermat pseudoprime to base n with two prime factors, P, squarefree, the following statement it's true:

$360 * (P - d_1) / (d_1^2 - d_1) = 360 * (d_2^2 - d_2) / (P - d_2) = m$, where d_1 and d_2 are the two prime factors and m is a natural number.

Conjecture 150: For any 3-Poulet number P one of the two following statements it's true:

- (1) $12 * d_1 * (P - d_2)$ is divisible by $(d_2^2 - d_2)$;
- (2) $12 * d_1 * (d_3^2 - d_3)$ is divisible by $(P - d_3)$, where $d_1 < d_2 < d_3$ are the prime factors of P.

Conjecture 151: Any Carmichael number C divisible by Carmichael number 8911 can be written as $C = 8911 * (198 * n + 1)$.

Conjecture 152: There exist an infinity of Carmichael numbers of the form $C = ((p^2 + 1) / 2) * p$, where p is prime.

Conjecture 153: There exist an infinity of Carmichael numbers of the form $C = 7 * 11 * 13 * 41 * (30 * n + 1)$.

Conjecture 154: There exist an infinity of Carmichael numbers of the form $C = 952 * n + 561$.

Conjecture 155: There exist an infinity of Poulet numbers of the form $P = 4 * n^2 + 172 * n + 529$, where n is positive integer.

Conjecture 156: There exist an infinity of Poulet numbers of the form $P = 20 * n^2 + 364 * n + 177$, where n is positive integer.

Conjecture 157: There exist an infinity of Carmichael numbers of the form $C = p^2 + n^2 - 1$, where p is prime and n is non-null positive integer.

Conjecture 158: For any m non-null positive integer there exist a series with infinite many Poulet numbers P formed this way: $P = (60 * m * n - 29) * (60 * m * n - (60 * m + 29))$, where n non-null positive integer.

Conjecture 159: For any m natural number there exist a series with infinite many Poulet numbers P formed this way: $P = (40*n - 10*m - 9)*(40*n - 10*m - 129)$, where n is non-null positive integer.

Conjecture 160: There exist an infinity of triplets of Carmichael numbers (C_1, C_2, C_3) such that $C_1 = 2*C_2 + C_3 - 30$.

Conjecture 161: There exist an infinity of Carmichael numbers C of the form $C = (40*k + 1)*(60*k + 1)*(100*k + 1)$, where k is non-null positive integer.

Conjecture 162: There exist an infinity of Poulet numbers P of the form $P = (42*m + 1)*(42*n + 1)$.

Conjecture 163: There exist an infinity of Poulet numbers P of the form $P = n^2 + 81*n + 1599$.

Conjecture 164: There exist an infinity of Poulet numbers P of the form $P = n^2 + 81*n + 159$.

Conjecture 165: There exist an infinity of Poulet numbers P of the form $P = n^2 + 81*n + 1629$.

Conjecture 166: There exist an infinity of Poulet numbers P of the form $P = n^2 + 81*n + 479$.

Conjecture 167: There exist an infinity of Poulet numbers P of the form $P = n^2 + 81*n + 39$.

Conjecture 168: There exist an infinity of Poulet numbers P of the form $P = n^2 + 10*n + 10$.

Conjecture 169: There exist an infinity of Poulet number P_2 of the form $21*q - (P_1 + 1)$, where q is prime and P_1 is also a Poulet number.

Conjecture 170: For any Poulet number P_1 of the form $3*k + 1$ there exist another Poulet number P_2 of the form $3*h + 2$ such that $P_1 + P_2 + 1 = P_3$, where P_3 is also a Poulet number (obviously of the form $3*g + 1$).

Conjecture 171: For any Poulet number P_1 of the form $3*k + 2$ there exist another Poulet number P_2 of the form $3*h + 1$ such that $P_1 + P_2 + 1 = P_3$, where P_3 is also a Poulet number (obviously of the form $3*g + 1$).

Conjecture 172: There exist an infinity of Poulet number P_2 of the form $(P_1 + 30)*9 + 10$, where P_1 is also a Poulet number.

Conjecture 173: All Carmichael numbers which are divisible both with 37 and 73 are of the form $10*k + 1$.

Conjecture 174: All Carmichael numbers which are divisible both with 37 and 73 have a prime divisor of the form $30*h + 23$.

Conjecture 175: There exist an infinity of Carmichael numbers C such that $C*s(C) - 90 = p$, where $s(C)$ is the sum of the digits of C and p is prime or power of prime.

Conjecture 176: There exist an infinity of Carmichael numbers C of the form $n^k + 10^h$, where n is positive odd integer greater than or equal to 3 and k, h non-null positive integers.

Conjecture 177: There exist an infinity of Poulet numbers P of the form $P = (d^2 + d)/2$, where d is a prime factor of P .

Conjecture 178: There exist an infinity of Poulet numbers P_2 of the form $P_2 = 2*P_1 - 1$, where P_1 is also a Poulet number.

Conjecture 179: There exist an infinity of Carmichael numbers C of the form $C = p^4 + 60*p$, where p is prime.

Conjecture 180: There are no Carmichael numbers of the following forms: $30*k + 3$, $30*k + 17$, $30*k + 23$, $30*k + 27$.

Conjecture 181: There exist an infinity of Carmichael numbers for each of the following forms: $30*k + 1$, $30*k + 5$, $30*k + 7$, $30*k + 9$, $30*k + 11$, $30*k + 13$, $30*k + 15$, $30*k + 19$, $30*k + 21$, $30*k + 25$, $30*k + 29$.

Conjecture 182: There exist an infinity of Poulet numbers of the form $(2*p*(p + 100*k) + 3*p^2)/5$, where k is non-null positive integer and p and $p + 100$ are both primes.

Conjecture 183: Without a set of definable exceptions, is true the following statement for Poulet numbers: if a Poulet number can be written as $2^k*p^n + p^2$, where k, n are integers, $k \geq 0$, and p one of its prime factors, that can be written as $2^k*q^m + q^2$, where m is also integer, for any of its other prime factors q . Thus, if a Poulet number can't be written as $2^k*p^n + p^2$, where k, n are integers, $k \geq 0$, and p one of its prime factors, that can't be written as $2^k*q^m + q^2$, where m is also integer, for any of its other prime factors q .

Conjecture 184: For any Poulet number P is true that $P - 5$ is equal to a sum of distinct powers of the number two.

Conjecture 185: There is an infinity of Carmichael numbers in every recurrent sequence formed in the following way: $a(0) = 2*(C_m - 1)$, where C_m is the m -th Carmichael number, then $a(n + 1) = a(n) + C_{m+n+1}$, where C_{m+n+1} is the $(m+n+1)$ th Carmichael number.

Conjecture 186: There is an infinity of Poulet numbers P such that the number N obtained by concatenating to the left P with the number 28 is (equal to) a forth power of a prime.

Conjecture 187: There is an infinity of duplets of numbers (q, n) , where q is prime and n is non-null positive integer, such that the number $P = (q^4 - 1)/3^n + 1$ is a Poulet number.

Conjecture 188: Any Carmichael number C can be written as $C = p*q + q - p$, where p and q are primes.

Conjecture 189: Any Carmichael number C can be written as $C = p^2 + p + d$, where p is prime and d is a prime factor of C .

Conjecture 190: Any Carmichael number C can be written as $C = a^2 + b^2 + c^2 - 2$, where a, b, c are primes.

Conjecture 191: Any Carmichael number C can be written as $C = p*2^n + 2^n - p$, where p is prime and n is non-null positive integer.

Conjecture 192: There exist infinite many Poulet numbers of the form $P = 1029600 \cdot p + 1$, where p is prime.

Conjecture 193: There exist infinite many Poulet numbers of the form $P_2 = 9 \cdot (P_1 - 1) + q^2$, where q is prime and P_1 is also a Poulet number.

Conjecture 194: For any two Poulet numbers P_1 and P_2 that share at least one prime divisor, the diophantine equation $x \cdot (P_1 - 1) = y \cdot (P_2 - 1)$ has solutions $[x, y]$ on integers, other than the trivial one $[x = (P_2 - 1), y = (P_1 - 1)]$.

Conjecture 195: Any Poulet number P not divisible by 5 can be written at least in one way as $P = m + n + q$, where m, n, q are primes, not necessarily all three distinct, of the same form from the following four ones: $30 \cdot k + 1, 30 \cdot k + 3, 30 \cdot k + 7$ or $30 \cdot k + 9$.

Conjecture 196: Any Poulet number P not divisible by 5 can be written at least in one way as $P = 2 \cdot m + n$, where m and n are distinct primes of the same form from the following four ones: $30 \cdot k + 1, 30 \cdot k + 3, 30 \cdot k + 7$ or $30 \cdot k + 9$.

Conjecture 197: Any square of a prime of the form $p^2 = 10 \cdot k + 1$ can be written as $p^2 = m + n + q$, where m, n, q are primes, not necessarily all three distinct, of the form $10 \cdot k + 7$.

Conjecture 198: Any square of a prime of the form $p^2 = 10 \cdot k + 9$ can be written as $p^2 = m + n + q$, where m, n, q are primes, not necessarily all three distinct, of the form $10 \cdot k + 3$.

Conjecture 199: Any Poulet number P that has a prime divisor d of the form $60 \cdot k + 13$ is the starting term in the following recurrent formula which produce a sequence containing an infinity of Poulet numbers P_i : $P_i = ((P \cdot 2 - d) \cdot 2 - d) \cdot 2 - d \dots$.

Conjecture 200: There is an infinity of Poulet numbers P of the form $P = ((2^n)^k) \cdot ((2^n)^{k+1} + 2^n + 1) + 1$, where k is a non-null positive integer, for any n non-null positive integer.

Part two. Thirty articles on Fermat pseudoprimes

1. A formula for generating primes and a possible infinite series of Poulet numbers

Abstract. An amazingly easy to formulate but rich in consequences property of Fermat pseudoprimes to base 2 (Poulet numbers).

A formula for generating primes

I studied Fermat pseudoprimes for quite a while (I posted on OEIS few series and properties of Carmichael numbers and Poulet numbers) and I always believed that in the structure of pseudoprimes resides a key for obtaining primes. Here I expose such a formula that generates primes and products of few primes.

I first noticed that the first Poulet number, 341, can be written as $(2^{10} - 1)/3$ and after that I found other Poulet numbers that can be written as $(2^k - 1)/3$: 5461, 1398101, 22369621, 5726623061, 91625968981, respectively for $k = 14, 22, 26, 34, 38$ (I conjecture that there are infinite Poulet numbers of this form).

I then noticed that the third Poulet number, 645, can be written as $(2^{4*11^2} - 1)/3$ and after that I found other Poulet numbers that can be written as $(2^{k*q^2} - 1)/3$, where q is prime: 2465, 2821, 8321, respectively for $q = 43, 23, 79$ (I conjecture that there are infinite Poulet numbers of this form too).

From the first 23 Poulet numbers, 19 can be written as $(2^{k*q} - 1)/3$, where q is prime or square of prime!

So the formula to generate numbers q that are primes, squares of primes and products of few primes or squares of primes is simply $q = (3*P + 1)/2^k$, where P is a Poulet number and k is the biggest natural number for that q is an integer.

I list below few values of $N = 3*P + 1$, for 9 consecutive Poulet numbers with 12 digits taken randomly (I note generically with s the squarefree semiprimes and with r the products of 3 distinct prime factors):

for $P = 994738556701$ we get $N = 2^3*s$;
for $P = 994738580641$ we get $N = 2^2*746053935481$;
for $P = 994750702441$ we get $N = 2^2*r$;
for $P = 994767925201$ we get $N = 2^2*746075943901$;
for $P = 994788345601$ we get $N = 2^2*746091259201$;
for $P = 994818048445$ we get $N = 2^3*s$;
for $P = 994830588181$ we get $N = 2^6*46632683821$;
for $P = 994853432581$ we get $N = 2^4*29^2*53^2*281^2$;
for $P = 994868271001$ we get $N = 2^2*r$.

We obtained, from 9 consecutive values of P, four primes, two semiprimes and two products of 3 distinct primes. It can easily be seen the potential of this formula as a generator of primes. I didn't forget the product of 3 squares; here's something interesting; we got through this formula primes, squarefree products of primes, squares of primes and squares of products of primes, but we didn't find a product to contain primes to a bigger power than two or both primes and squares of primes together, therefore we conjecture that there are no such numbers q, where $q = (3 \cdot P + 1)/2^k$ (and P is a Poulet number and k is the biggest n natural for that q is an integer).

We know take the four primes randomly generated, *i.e.* 746053935481, 746075943901, 746091259201 and 46632683821, and we see that they have also the property to generate primes; if we put them in a recurrent formula (Cunningham's chain type), we obtain for $M = 3 \cdot t + 1$ the following values:

for $t = 746053935481$ we get $M = 2^2 \cdot 559540451611$;
 for $t = 746075943901$ we get $M = 2^3 \cdot 1381 \cdot 202591223$;
 for $t = 746091259201$ we get $M = 2^2 \cdot 47 \cdot 11905711583$;
 for $t = 46632683821$ we get $M = 2^3 \cdot 174872256433$.

We now take a prime newly generated, 559540451611. We have:

$$3 \cdot 559540451611 + 1 = 2 \cdot 839310677417.$$

I believe these results are encouraging in the study of recurrent sequences of the type $P_n = (3 \cdot P_{n-1} + 1)/2^k$, where k is the biggest natural number for that P_n is an integer and P_0 is a Fermat pseudoprime to base 2.

A possible infinite series of Poulet numbers

We saw above that Poulet numbers 341, 5461, 1398101, 22369621, 5726623061, 91625968981 can be written as $(4^k - 1)/3$ for $k = 5, 7, 11, 13, 17, 19$. We didn't obtain a Poulet number for any other value of k from 1 to 19 beside those. We calculate now $(4^k - 1)/3$ for $k = 23, 29, 31, 37, 41$ and we get respectively:

: $23456248059221 = 47 \cdot 178481 \cdot 2796203$;
 : $96076792050570581 = 59 \cdot 233 \cdot 1103 \cdot 2089 \cdot 3033169$;
 : $1537228672809129301 = 715827883 \cdot 2147483647$;
 : $6296488643826193618261 = 223 \cdot 1777 \cdot 25781083 \cdot 616318177$;
 : $1611901092819505566274901 = 83 \cdot 13367 \cdot 164511353 \cdot 8831418697$.

Unfortunately I have just Mr. Richard Pinch's tables to verify if a number is a Poulet number or not (tables that are just up to 10^{12}) and there is no such a simple test to verify this as it is the Korselt criterion at Carmichael numbers. But the premises that the numbers we calculated are Poulet numbers are good: they are squarefree products of few primes. I don't have enough data to conjecture that a number of the form $(4^k - 1)/3$ is a Poulet number *if and only if* k is prime, $k \geq 5$ (which would be a tremendously result, to put prime numbers in a bijection with a subset of Poulet numbers!), but I do make two conjectures:

Conjecture 1: There are infinite many Poulet numbers of the form $(4^k - 1)/3$, where k is positive integer.

Conjecture 2: Any number of the form $(4^k - 1)/3$, where k is prime, $k \geq 5$, is a Poulet number.

The second conjecture, if true, would be, as I know, the first generic formula for an infinite series of Poulet numbers (of type “for any possible value of this we obtain necessarily that”, cause formulas that generates Poulet numbers, but not only Poulet numbers I submitted myself a few to OEIS). Besides this, the conjecture has yet another major implication: from the first million natural numbers, about 80 thousand are primes and just about 250 are Poulet numbers, which lead to the conclusion that Poulet numbers are far more rare than prime numbers. The conjecture, if true, would show that, in fact, for the first about 7 consecutive prime numbers, we have 7 corresponding Poulet numbers spread in the first about 40 thousand Poulet numbers and, consequently, the set of prime numbers is so just a mean set beside the set of Poulet numbers!

2. A new class of Fermat pseudoprimes and few remarks about Cipolla pseudoprimes

Abstract. I wrote an article entitled “A formula for generating primes and a possible infinite series of Poulet numbers”; the sequence I was talking about not only that is, indeed, infinite, but is also already known as the sequence of Cipolla pseudoprimes to base 2. Starting from comparing Cipolla pseudoprimes and some of my notes I discovered a new class of pseudoprimes.

Introduction

The article I was talking about in Abstract was my second encounter with Cipolla pseudoprimes. I first submitted a sequence to OEIS (A217853) to define a subset of Fermat pseudoprimes to base 3, *i.e.* numbers of the form $(3^{(4^k + 2)} - 1)/8$. I just later saw the note of Mr. Bruno Berselli on this sequence, that for p prime, $p = 2^k + 1$, is obtained the generating formula for Cipolla pseudoprimes to base 3, namely $(9^p - 1)/8$, and I made the connection with my further article, in which I was talking about the numbers of the form $(4^p - 1)/3$, namely Cipolla pseudoprimes to base 2.

The formula $(3^{(4^k + 2)} - 1)/8$ generates Fermat pseudoprimes to base 3 not only for $k = (p - 1)/2$, where p prime (which gives the formula for Cipolla pseudoprimes to base 3), but for other values of k too.

The first few Cipolla pseudoprimes to base 3 are 91, 7381, 597871, 3922632451, 317733228541, 2084647712458321, 168856464709124011 (for more of them, see the sequence A210454 in OEIS).

The first few terms generated by the formula above are 91, 7381, 597871, 48427561, 3922632451, 317733228541, 25736391511831, 2084647712458321, 168856464709124011 (for more of them, see the sequence A217853 in OEIS).

It can be seen that the formula generates until the number 168856464709124011 three more Fermat pseudoprimes to base 3: 48427561, 3922632451 and 25736391511831.

It seemed logic to try to generalize the formula $(3^{(4*k + 2)} - 1)/8$, hoping that it can be obtained a class of pseudoprimes that would contain the set of Cipolla pseudoprimes, but instead of this I obtained something even more interesting, an entirely different class of Fermat pseudoprimes (containing pseudoprimes which are not in the Cipolla sequence and, *vice versa*, not containing pseudoprimes that are in Cipolla sequence).

A formula that generates Fermat pseudoprimes

Conjecture: The formula $(n^{(n*k + k + n - 1)} - 1)/(n^2 - 1)$ generates an infinity of Fermat pseudoprimes to base n for any integer $n, n > 1$.

Verifying the conjecture

For $n = 2$ the formula becomes $(2^{(3*k + 1)} - 1)/3$ and generates the following Fermat pseudoprimes to base 2, for $k = 3, 7, 11$: 341, 1398101, 5726623061.

For $n = 3$ the formula becomes $(3^{(4*k + 2)} - 1)/8$ and generates Fermat pseudoprimes to base 3 for 14 values of k from 1 to 20.

For $n = 4$ the formula becomes $(4^{(5*k + 3)} - 1)/15$ and generates the following Fermat pseudoprime to base 4, for $k = 1$: 4369.

For $n = 5$ the formula becomes $(5^{(6*k + 4)} - 1)/24$ and generates the following Fermat pseudoprime to base 5, for $k = 1$: 406901.

Unfortunately the first term of the sequence (corresponding to $k = 1$) for $n = 7$ is larger than 10^{10} and I do not have the possibility to extend the verifying, but seems there is enough data to justify the conjecture.

Conclusion

It can easily be seen that, for $n = 2$, the sequence of Cipolla pseudoprimes to base 2 contains until the pseudoprime 5726623061 two more pseudoprimes than the pseudoprimes I defined above (5461 and 22369621 – for the sequence of Cipolla pseudoprimes to base 2 see the sequence A210454 in OEIS) and I have shown above that Cipolla pseudoprimes to base 3 contains until the pseudoprime 168856464709124011 three less pseudoprimes than the pseudoprimes I defined above so it's no need for a further proof that neither one of the two classes is not a subset of the other.

Reference

Cipolla Pseudoprimes, Y. Hamahata and Y. Kokobun

3. Formulas that generate subsets of 3-Poulet numbers and few types of chains of primes

Abstract. A simple list of sequences of products of three numbers, many of them, if not all of them, having probably an infinity of terms that are Fermat pseudoprimes to base 2 with three prime factors.

Note: I named with “3-Poulet numbers” the Fermat pseudoprimes to base 2 with 3 prime factors, obviously by similarity with the name “3-Carmichael numbers” for absolute Fermat pseudoprimes. For a list with 3-Poulet numbers see the sequence A215672 in OEIS.

I.

Poulet numbers with three prime factors of the form $p*((n+1)^*p-n*p)*((m+1)^*p-m*p)$, where p prime, m, n natural:

$$\begin{aligned} 10585 &= 5*29*73 = 5*(5*7 - 6)*(5*18 - 17); \\ 13741 &= 7*13*151 = 7*(7*2 - 1)*(7*25 - 24); \\ 13981 &= 11*31*41 = 11*(11*3 - 2)*(11*4 - 3); \\ 29341 &= 13*37*61 = 13*(13*3 - 2)*(13*5 - 4); \\ 137149 &= 23*67*89 = 23*(23*3 - 2)*(23*4 - 3). \end{aligned}$$

II.

Poulet numbers with three prime factors of the form $p*((n*p - (n + 1)*p)*(m*p - (m + 1)*p)$, where p prime, m, n natural:

$$6601 = 7*23*41 = 7*(7*4 - 5)*(7*7 - 8).$$

Conjecture: Any 3-Poulet number which has not a prime factor of the form $30k + 23$ can be written as $p*((n + 1)^*p - n*p)*((m + 1)^*p - m*p)$ or as $p*((n*p - (n + 1)*p)*(m*p - (m + 1)*p)$.

III.

Poulet numbers with three prime factors of the form $p*(p + 2*n)*(p + 2^2*n - 2)$, where p prime, n natural:

$$\begin{aligned} 561 &= 3*11*17 \\ p &= 3; p + 2*4 = 11; p + 2^2*4 - 2 = 17, \text{ so } [p, n] = [3, 4]; \end{aligned}$$

$$\begin{aligned} 1105 &= 5*13*17 \\ p &= 5; p + 2*4 = 13; p + 2^2*4 - 2 = 17, \text{ so } [p, n] = [5, 4]. \end{aligned}$$

IV.

Poulet numbers with three prime factors of the form $p*(p + 2*n)*(p + 2^k*n)$, where p prime and n, k natural:

$$\begin{aligned} 1729 &= 7*13*19 \\ p &= 7; p + 2*3 = 13; p + 2^2*3 = 19, \text{ so } [p, n, k] = [7, 3, 2]; \end{aligned}$$

$$2465 = 5*17*29$$

$p = 5; p + 2^*6 = 17; p + 2^2*6 = 29$, so $[p, n, k] = [5, 6, 2]$;

$2821 = 7*13*31$

$p = 7; p + 2^*3 = 17; p + 2^3*3 = 31$, so $[p, n, k] = [5, 6, 3]$;

$29341 = 13*37*61$

$p = 13; p + 2^*12 = 37; p + 2^2*12 = 61$, so $[p, n, k] = [13, 12, 2]$.

V.

Poulet numbers with three prime factors of the form $(1 + 2^k*m)*(1 + 2^k*n)*(1 + 2^k*(m + n))$, where k, m, n natural:

$13981 = 11*31*41$

$1 + 2^1*5 = 11, 1 + 2^1*15 = 31, 1 + 2^1*(5 + 15) = 41$, so $[k, m, n] = [1, 5, 15]$;

$252601 = 41*61*101$

$1 + 2^2*10 = 41, 1 + 2^2*15 = 61, 1 + 2^2*(10 + 15) = 101$, so $[k, m, n] = [2, 10, 15]$.

VI.

Poulet numbers with three prime factors of the form $(1 + 2^k*m)*(1 + 2^k*n)*(1 + 2^k*(m + n + 2))$, where k, m, n natural:

$561 = 3*11*17$

$1 + 2^1*1 = 3, 1 + 2^1*5 = 11, 1 + 2^1*(1 + 5 + 2) = 17$, so $[k, m, n] = [1, 1, 5]$.

VII.

Poulet numbers with three prime factors of the form $p*(p + 2^n)*(p + 2^n + 2*(n + 1))$, where p prime, n natural:

$6601 = 7*23*41$

$p = 7; p + 2^*8 = 31; p + 2^*8 + 2^*9 = 41$, so $[p, n] = [7, 8]$.

VIII.

Poulet numbers with three prime factors of the form $3*(3 + 2^k)*(3 + q*2^h)$, where q prime and k, h natural:

$645 = 3*5*43$ so $[q, h, k] = [5, 1, 3]$;

$1905 = 3*5*127$ so $[q, h, k] = [31, 1, 2]$;

$8481 = 3*11*257$ so $[q, h, k] = [127, 3, 1]$.

Notes

The chains of primes of the form $[p, p + 2^n, \dots, p + 2^k*n]$ seems to be a very interesting object of study; such chains are, for instance, $[3, 5, 7, 11, 19]$ for $[p, n, k] = [3, 1, 4]$ and $[3, 13, 23, 43, 83, 163]$ for $[p, n, k] = [3, 5, 5]$.

Also it would be interesting to study the chains of primes formed starting from a prime p and adding 2^k*n , where n is an arbitrarily chosen natural number and k the smallest values for which $p + 2^k*n$ is prime. Such a chain is, for instance, $[7, 13, 19, 31, 103, 199, 1543, 3079]$ for $[p, n] = [7, 3]$ and $[k_1, k_2, k_3, k_4, k_5, k_6, k_7] = [1, 2, 3, 5, 6, 9, 10]$.

An interesting triplet of primes is $[p + 2^*m, p + 2^*n, p + 2^*(m + n)]$ where p is prime and m, n natural; such triplets are $[11, 13, 17]$ for $[p, m, n] = [7, 2, 3]$ or $[23, 43, 59]$ for $[p, m, n] = [7, 8, 18]$. Generalizing, the triplet would be $[p + 2^k*m, p + 2^k*n, p + 2^k*(m + n)]$; such a triplet is $[11, 19, 23]$ for $[p, k, m, n] = [7, 2, 1, 3]$.

4. A conjecture about 2-Poulet numbers and a question about primes

Abstract. To find generic formulas for Poulet numbers (beside, of course, the formula that defines them) was for long time one of my targets; I maybe found such a formula for Poulet numbers with two prime factors, involving the multiples of the number 30, that also is rising an interesting question about primes.

Conjecture:

Any Poulet number with two prime factors can be written as $P = (q - 30*n)*(r + 30*n)$, where q and r are primes or are equal to 1 and n is positive integer, $n \geq 1$.

Note: For a list of 2-Poulet numbers see the sequence A214305 that I submitted to OEIS.

Verifying the conjecture for the first few 2-Poulet numbers:

$$: P = 341 = 11*31 = (41 - 30*1)*(1 + 30*1) = (31 - 30*1)*(311 + 30*1);$$

$$: P = 1387 = 19*73 = (61 - 30*2)*(1327 + 30*2) = (79 - 30*2)*(13 + 30*2);$$

$$: P = 2047 = 23*89 = (31 - 30*1)*(2017 + 30*1) = (53 - 30*1)*(59 + 30*1) = (61 - 30*2)*(1987 + 30*2) = (83 - 30*2)*(29 + 30*2);$$

$$: P = 2701 = 37*73 = (31 - 30*1)*(2671 + 30*1) = (67 - 30*1)*(43 + 30*1) = (103 - 30*1)*(7 + 30*1) = (97 - 30*2)*(13 + 30*2) = (151 - 30*5)*(2551 + 30*5);$$

$$: P = 3277 = 29*113 = (59 - 30*1)*(83 + 30*1) = (89 - 30*2)*(53 + 30*2) = (211 - 30*7)*(3067 + 30*7) = (241 - 30*8)*(3037 + 30*8) = (421 - 30*14)*(2857 + 30*14) = (571 - 30*19)*(2707 + 30*19) = (601 - 30*20)*(2677 + 30*20) = (631 - 30*21)*(2647 + 30*21).$$

Note: It is remarkable in how many ways a 2-Poulet number can be written this way.

Note: The conjecture might probably be extended for all Poulet numbers not divisible by 3 or 5, not only with two prime factors.

Verifying the extended conjecture for first few Poulet numbers with more than two prime factors not divisible by 3 or 5:

$$: P = 1729 = 7*13*19 = (31 - 30*1)*(1699 + 30*1) = (43 - 30*1)*(103 + 30*1);$$

$$: P = 2821 = 7 \cdot 13 \cdot 31 = (31 - 30 \cdot 1) \cdot (2791 + 30 \cdot 1) = (37 - 30 \cdot 1) \cdot (373 + 30 \cdot 1) = (61 - 30 \cdot 1) \cdot (61 + 30 \cdot 1);$$

$$: P = 6601 = 7 \cdot 23 \cdot 41 = (31 - 30 \cdot 1) \cdot (6571 + 30 \cdot 1) = (53 - 30 \cdot 1) \cdot (257 + 30 \cdot 1) = (71 - 30 \cdot 1) \cdot (131 + 30 \cdot 1) = (191 - 30 \cdot 1) \cdot (11 + 30 \cdot 1).$$

Note: This conjecture is rising the following question: which pairs of primes (x, y) , at least one of them bigger than 30, have the property that can be written as $(p - 30 \cdot n, q + 30 \cdot n)$, where p and q are primes or are equal to 1 and n is positive integer, $n \geq 1$.

5. A formula that generates a type of pairs of Poulet numbers

Abstract. Starting from the observation that the number $13^2 + 81 \cdot 13 + 3 \cdot 13 \cdot 41$ is a Poulet number (2821), and the number $41^2 + 81 \cdot 41 + 3 \cdot 13 \cdot 41$ is a Poulet number too (6601), and following my interest for the number 30, I found a formula that generates such pairs of Poulet numbers like (2821, 6601).

Observation: The formula $p^2 + 81 \cdot p + 3 \cdot p \cdot q$, where p is a prime of the form $30 \cdot k + 13$ and q is a prime of the form $30 \cdot k + 41$ (case I), or, vice versa, p is a prime of the form $30 \cdot k + 41$ and q is a prime of the form $30 \cdot k + 13$ (case II), generates Poulet numbers.

Examples:

: for $(p, q) = (13, 41)$, we got 2821, a Poulet number;

: for $(p, q) = (41, 13)$, we got 6601, a Poulet number;

: for $(p, q) = (43, 71)$, we got 14491, a Poulet number;

: for $(p, q) = (71, 43)$, we got 19951, a Poulet number.

Conjecture 1: There is an infinity of Poulet numbers of the form $p^2 + 81 \cdot p + 3 \cdot p \cdot q$, where p is a prime of the form $30 \cdot k + 13$ and q is a prime of the form $30 \cdot k + 41$, where k is an integer, $k \geq 0$.

Conjecture 2: There is an infinity of Poulet numbers of the form $p^2 + 81 \cdot p + 3 \cdot p \cdot q$, where p is a prime of the form $30 \cdot k + 41$ and q is a prime of the form $30 \cdot k + 13$, where k is an integer, $k \geq 0$.

Conjecture 3: If the number $p^2 + 81 \cdot p + 3 \cdot p \cdot q$, where p is a prime of the form $30 \cdot k + 13$ and q is a prime of the form $30 \cdot k + 41$, is a Poulet number, then the number $p^2 + 81 \cdot p + 3 \cdot p \cdot q$, where p is a prime of the form $30 \cdot k + 41$ and q is a prime of the form $30 \cdot k + 13$ is a Poulet number too (k is an integer, $k \geq 0$).

Note: The differences between the two numbers that form such a pair might also have interesting properties; in the examples above, we have $6601 - 2821 = 3780$ and $19951 - 14491 = 5460$. Note that $5460 - 3780 = 1680 = 41^2 - 1$.

Note: There are many Poulet numbers that can be written as $p^2 + 81p + 3p^*q$, where p, q primes, but it's not satisfied the reciprocity from the formula above.

6. A method of finding subsequences of Poulet numbers

Abstract. I was studying the Fermat pseudoprimes in function of the remainder of the division by different numbers, when I noticed that the study of the remainders of the division by 28 seems to be very interesting. Starting from this, I discovered a method to easily find subsequences of Poulet numbers. I understand through “finding subsequences of Poulet numbers” finding such numbers that share a non-trivial property, *i.e.* not a sequence defined like: “Poulet numbers divisible by 7”.

Introduction

The way of finding such subsequences is simply to calculate the remainder of the division of a Poulet number P by the number $4*q$, where q is a prime which does not divide P ; surprisingly, few values of these remainders seems to occur more often than others.

Few subsequences of Poulet numbers

For $q = 7$, we found out that, from the first 40 Poulet numbers not divisible by 7, 14 numbers can be written as $P = 28*n + 1$, where n is obviously a natural number; these numbers are:
: 561, 645, 1905, 2465, 3277, 4033, 4369, 5461, 10585, 18705, 25761, 31417, 33153, 34945.

For $q = 11$, we found out that, from the first 40 Poulet numbers not divisible by 11, 6 numbers can be written as $P = 44*n + 1$; these numbers are:
: 2465, 6601, 15709, 15841, 30889, 31417.

Also for $q = 11$ and the first 40 Poulet numbers not divisible by 11, we found out that 6 numbers can be written as $P = 44*n + 5$; these numbers are:
: 1105, 2821, 4681, 5461, 8321, 18705.

For $q = 13$, we found out that, from the first 40 Poulet numbers not divisible by 13, 9 numbers can be written as $P = 52*n + 1$; these numbers are:
: 3277, 4369, 4681, 5461, 7957, 8321, 18721, 30889, 34945.

Also for $q = 13$ and the first 40 Poulet numbers not divisible by 13, we found out that 5 numbers can be written as $P = 52*n + 29$; these numbers are:
: 341, 4033, 10585, 23377, 33153.

For $q = 17$, we found out that, from the first 50 Poulet numbers not divisible by 17, 8 numbers can be written as $P = 68*n + 1$; these numbers are:
: 341, 1905, 7957, 15709, 31417, 31621, 49981, 52633.

Also for $q = 17$ and the first 50 Poulet numbers not divisible by 17, we found out that 4 numbers can be written as $P = 68*n + 45$; these numbers are:
: 10585, 16705, 49141, 60701.

For $q = 19$, we found out that, from the first 50 Poulet numbers not divisible by 19, 4 numbers can be written as $P = 76*n + 5$; these numbers are:
: 1905, 4033, 29341, 31621.

Also for $q = 19$ and the first 50 Poulet numbers not divisible by 19, we found out that 4 numbers can be written as $P = 76*n + 37$; these numbers are:
: 341, 645, 4369, 8321.

Also for $q = 19$ and the first 50 Poulet numbers not divisible by 19, we found out that 4 numbers can be written as $P = 76*n + 45$; these numbers are:
: 4681, 8481, 23377, 49141.

For $q = 23$, we found out that, from the first 40 Poulet numbers not divisible by 23, 4 numbers can be written as $P = 92*n + 1$; these numbers are:
: 645, 1105, 23001, 25761.

Also for $q = 23$ and the first 40 Poulet numbers not divisible by 23, we found out that 4 numbers can be written as $P = 92*n + 45$; these numbers are:
: 4369, 7957, 18721, 31417.

Note: Yet is interesting to study the quotients n obtained through the method above, *i.e.* the numbers $n = (P - r)/4*q$, where r is the remainder, *e.g.* the numbers $n = (561 - 1)/4*7 = 2^2*5$, $n = (33153 - 1)/4*7 = 2^5*37$, $n = (2465 - 1)/4*11 = 2^3*7$, $n = (2821 - 5)/4*11 = 2^6$ and so on.

7. A possible infinite subset of Poulet numbers generated by a formula based on Wieferich primes

Abstract. I was studying the Poulet numbers of the form $n*p - n + 1$, where p is prime, numbers which appear often related to Fermat pseudoprimes (see the sequence A217835 that I submitted to OEIS) when I discovered a possible infinite subset of Poulet numbers generated by a formula based on Wieferich primes (I pointed out 4 such Poulet numbers).

It is known the following relation between the Fermat pseudoprimes to base 2 (Poulet numbers) and the Wieferich primes: the squares of the two known Wieferich primes, respectively $1194649 = 1093^2$ and $12327121 = 3511^2$, are Poulet numbers. I discovered yet another relation between these two classes of numbers:

Conjecture 1: For every Wieferich prime p there is an infinity of Poulet numbers which are equal to $n*p - n + 1$, where n is integer, $n > 1$.

Note: Because there are just two Wieferich primes known (it's not even known if there are other Wieferich primes beside these two), we verify the conjecture for these two and few values of n (until $n < 31$).

: $1093 \cdot 3 - 2 = 3277$, a Poulet number;
 : $1093 \cdot 4 - 3 = 4369$, a Poulet number;
 : $1093 \cdot 5 - 4 = 5461$, a Poulet number;
 : $3511 \cdot 14 - 13 = 49141$, a Poulet number.

Observation 1: The formula $n \cdot p - n + 1$, where p is Wieferich prime and n is integer, $n > 1$, leads often to semiprimes of the form $q \cdot (m \cdot q - m + 1)$ or of the form $q \cdot (m \cdot q + m - 1)$:

: $1093 \cdot 11 - 10 = 5 \cdot 2621$ and $2621 = 5 \cdot 655 - 654$;
 : $3511 \cdot 4 - 3 = 19 \cdot 739$ and $739 = 19 \cdot 41 - 40$;
 : $3511 \cdot 9 - 8 = 7 \cdot 4593$ and $4593 = 7 \cdot 752 - 751$;
 : $3511 \cdot 10 - 9 = 11 \cdot 3191$ and $3191 = 11 \cdot 319 - 318$;
 : $3511 \cdot 12 - 11 = 73 \cdot 577$ and $577 = 73 \cdot 8 - 7$;
 : $3511 \cdot 14 - 13 = 157 \cdot 313$ and $313 = 157 \cdot 2 - 1$;
 : $3511 \cdot 21 - 20 = 11 \cdot 6701$ and $6701 = 11 \cdot 670 - 669$;
 : $3511 \cdot 24 - 23 = 61 \cdot 1381$ and $1381 = 61 \cdot 23 - 22$;
 : $3511 \cdot 28 - 27 = 29 \cdot 3389$ and $3389 = 29 \cdot 121 - 120$;

: $1093 \cdot 11 - 10 = 41 \cdot 293$ and $293 = 41 \cdot 7 + 6$;
 : $1093 \cdot 18 - 17 = 11 \cdot 1787$ and $1787 = 11 \cdot 149 + 148$;
 : $1093 \cdot 29 - 28 = 11 \cdot 2879$ and $2879 = 11 \cdot 240 + 239$;
 : $3511 \cdot 4 - 3 = 19 \cdot 739$ and $739 = 19 \cdot 37 + 36$;
 : $3511 \cdot 19 - 18 = 17 \cdot 3923$ and $3923 = 17 \cdot 218 + 217$;
 : $3511 \cdot 31 - 30 = 233 \cdot 467$ and $467 = 233 \cdot 2 + 1$;
 : $3511 \cdot 28 - 27 = 29 \cdot 3389$ and $3389 = 29 \cdot 113 + 112$.

Note: Every Poulet number obtained so far through the formula above (until $n < 31$) is semiprime, in other words a 2-Poulet number.

Note: The class of primes p that can be written in both ways, like $p = n \cdot q - n + 1$ and like $m \cdot q + m - 1$, where q is prime and m and n are integers larger than 1, seems to be interesting to study. Such primes p are, for instance, $739 = 19 \cdot 41 - 40 = 19 \cdot 37 + 36$ and $3389 = 29 \cdot 121 - 120 = 29 \cdot 113 + 112$. Maybe is not a coincidence that both pairs of primes (p, q) are of the form $(10 \cdot k + 9, 10 \cdot h + 9)$.

Observation 2: Most of the 2-Poulet numbers (for a list with Fermat pseudoprimes to base 2 with two prime factors see the sequence A214305 in OEIS) can be written as $d \cdot (d \cdot n - n + 1)$ or as $d \cdot (d \cdot n + n - 1)$, where d is obviously one of the two prime factors and n is integer, $n > 1$: for instance $341 = 11 \cdot 31 = 11 \cdot (11 \cdot 3 - 2)$ and $1387 = 19 \cdot 73 = 19 \cdot (19 \cdot 4 - 3)$. But not all 2-Poulet numbers can be written in one of these two ways: for instance $23377 = 97 \cdot 241$, the 18th 2-Poulet number, can't be written this way.

Observation 3: I also noticed that two semiprimes obtained from the Wieferich primes through the formula above can be written as $q \cdot (q \cdot 38 + 17)$:

: $14041 = 19 \cdot 739 = 19 \cdot (19 \cdot 38 + 17)$; $52651 = 37 \cdot 1423 = 37 \cdot (37 \cdot 38 + 17)$.

Note: That would be also interesting to study the pairs of primes $(p, 38*p + 17)$; such pairs of primes are, for instance, $(7, 283)$, $(19, 739)$, $(37, 1423)$, $(73, 2791)$, $(79, 3019)$, $(103, 3931)$.

8. Four sequences of integers regarding balanced primes and Poulet numbers

Abstract. A simple list of sequences of integers that reveal interesting properties of few subsets of balanced primes.

I.

Balanced primes B that can be written as $B = P \pm 24$, where P is a Fermat pseudoprime to base two (a Poulet number):

1747, 2677, 4657, 41017, 188437, 195997 (...).

Comments:

B that can be written as $P + 24$: 1747;

B that can be written as $P - 24$: 2677, 4657, 41017, 188437, 195997.

Note that all these balanced primes are of the form $10*k + 7$!

Note: For a list of Poulet numbers see the sequence A001567 in OEIS. For a list of balanced primes see the sequence A006562 in OEIS.

II.

*Balanced primes $B2$ that can be written as $B1 + 330*n - 6$, where $B1$ is also a balanced prime and n is non-negative integer:*

257, 977, 1367, 1511, 1747, 1907, 2417, 2677 (...).

Comments:

$B1$ corresponding to the least n for that $B2$ can be written this way and the least n : $(263, 0)$, $(653, 1)$, $(53, 4)$, $(1187, 1)$, $(1753, 0)$, $(593, 4)$, $(1103, 4)$, $(373, 7)$.

Note that 7 from the first 12 balanced primes of the form $10*k + 7$ can be written this way!

Note: Seems that the formula $p + 330*n$ produces many primes when p is a balanced prime of the form $10*k + 3$ or $10*k + 7$; for instance the number $257 + 330*n$ is prime for $n = 0, 1, 5, 6, 8, 10, 12, 13, 14, 17, 18, 20, 21, 22, 26, 28, 31, 35, 39, 40, 43, 45, 47, 48, 49, 52, 53, 54, 59, 62, 64, 66, 67, 68, 69, 70, 71, 74, 77, 78, 81, 83, 85, 88, 94, 95$, that means for 46 values of n from the first 99. I also noticed that the same formula produces many primes and squares of primes when p is a square of prime; for instance the number $361 + 330*n$ is prime or square of prime for $n = 0, 1, 2, 4, 5, 6, 7, 8, 9, 13, 16, 18, 20, 22, 23, 26, 28, 29, 33, 37, 42, 43, 46, 51, 53, 54, 58, 60, 64, 68, 69, 74, 75, 77, 79, 81, 83, 84, 85, 88, 90, 91, 93, 96, 97$, that means for the first 45 values of n from the first 99.

III.

*Balanced primes $B2$ that can be written as $B1 + 330*n + 6$, where $B1$ is also a balanced prime and n is non-negative integer:*

263, 593, 1753, 2903, 2963, 4013 (...).

Comments:

B1 corresponding to the least n for that B2 can be written this way and the least n: (257, 0), (257, 1), (1747, 0), (257, 8), (977, 6), (1367, 8).

Note that 5 from the first 14 balanced primes of the form $10*k + 3$ can be written this way!

IV.

*Balanced primes B2 that can be written as $B1 + 1980*n$, where B1 is also a balanced prime and n is positive integer:*

3733, 4013, 4657, 6863, 11411, 11807, 11933, 13463, 15193, 15767, 16097, 16787, 16987, 17483, 19463, 19477, 20107, 20123, 22447, 23333, 23893, 24413, 25621, 26177, 26393, 26693, 26723, 27067 (...).

Comments:

The corresponding (B1, n): (1753, 1), (53, 2), (2677, 1), (2903, 2), (1511, 5), (1907, 5), (4013, 4), (7523, 3), (3313, 6), (11807, 2), (257, 8), (947, 8), (5107, 6), (7583, 5), (7583, 6), (3637, 8), (2287, 9), (6263, 7), (12547, 5), (9473, 6), (6073, 9), (653, 12), (21661, 2), (2417, 12), (24413, 1), (10853, 8), (2963, 12), (3307, 12).

Comments:

B2 may sometimes be written this way for more than one set of values of B1 and n (for instance $11933 = 4013 + 4*1980 = 53 + 6*1980$); we refered through the corresponding (B1, n) to the least value of n.

Note that 32 from the first 171 balanced primes can be written as $B + 1980*n$, where B is a smaller balanced prime.

Conjecture: Any balanced prime B beside the first one, 5, generates an infinity of balanced primes of the form $B + 1980*n$ (e.g. the second balanced prime, 53, generates for $n = 2, 6, 14, 56$ the balanced primes 4013, 11933, 27773, 110933).

Conjecture: Any balanced prime B beside the first one, 5, generates through the formula $B - 1980*n$ an infinity of balanced primes in absolute value (e.g. $5807 - 6*1980 = - 6073$, where 5807 and 6073 are balanced primes).

9. Six polynomials in one and two variables that generate Poulet numbers

Abstract. Fermat pseudoprimes were for me, and they still are, a class of numbers as fascinating as that of prime numbers; over time I discovered few polynomials that generate Poulet numbers (but not only Poulet numbers). I submitted all of them on OEIS; in this paper I get them together.

- (1) *Poulet numbers of the form $7200*n^2 + 8820*n + 2701$.*

First 8 terms: 2701, 18721, 49141, 93961, 226801, 314821, 534061, 665281 (sequence A214016 in OEIS).

Note: The Poulet numbers above were obtained for the following values of n : 0, 1, 2, 3, 5, 6, 8, 9.

- (2) *Poulet numbers of the form $144*n^2 + 222*n + 85$.*

First 8 terms: 1105, 2047, 3277, 6601, 13747, 16705, 19951, 31417 (sequence A214017 in OEIS).

Note: The Poulet numbers above were obtained for the following values of n : 2, 3, 4, 6, 9, 10, 11, 14.

- (3) *Poulet numbers of the form $3*(2*n + 1)*(18*n + 11)*(36*n + 17)$.*

First 4 terms: 561, 62745, 656601, 11921001 (sequence A213071 in OEIS).

Note: The Poulet numbers above were obtained for the following values of n : 0, 2, 5, 14.

Note: All 4 terms from above are Carmichael numbers.

- (4) *Poulet numbers of the form $(6*m - 1)*((6*m - 2)*n + 1)$.*

First 11 terms: 341, 561, 645, 1105, 1905, 2047, 2465, 3277, 4369, 4371, 6601 (sequence A210993 in OEIS).

Notes:

For $m = 1$ the formula becomes $20*n + 5$ and generates all the Poulet numbers divisible by 5 from the sequence above (beside 645, all of them have another solutions beside $n = 1$).

For $m = 2$ the formula becomes $110*n + 11$ and generates the Poulet numbers: 341, 561 etc.

For $m = 3$ the formula becomes $272*n + 17$ and generates the Poulet numbers: 561, 1105, 2465, 4369 etc.

For $m = 4$ the formula becomes $506*n + 23$ and generates the Poulet numbers: 2047, 6601 etc.

For $n = 1$ the formula generates a perfect square.

For $n = 2$ the formula becomes $3*(6*m - 1)*(4*m - 1)$ and were found the following Poulet numbers: 561 etc.

For $n = 3$ the formula becomes $(6*m - 1)*(18*m - 5)$ and were found the following Poulet numbers: 341, 2465 etc.

For $n = 4$ the formula becomes $(6*m - 1)*(24*m - 7)$ and were found the following Poulet numbers: 1105, 2047, 3277, 6601 etc.

Note: The formula is equivalent to Poulet numbers of the form $p*(n*p - n + 1)$, where p is of the form $6*m - 1$. From the first 68 Poulet numbers just 26 of them (1387, 2701, 2821, 4033, 4681, 5461, 7957, 8911, 10261, 13741, 14491, 18721, 23377, 29341, 31609, 31621, 33153, 35333, 42799, 46657, 49141, 49981, 57421, 60787, 63973, 65281) can't be written as $p*(n*p - n + 1)$, where p is of the form $6*m - 1$.

- (5) *Poulet numbers of the form $(6*m + 1)*(6*m*n + 1)$.*

First 10 terms: 1105, 1387, 1729, 2701, 2821, 4033, 4681, 5461, 6601, 8911 (sequence A214607 in OEIS).

Notes:

For $m = 1$ the formula becomes $42*n + 7$.
 For $m = 2$ the formula becomes $156*n + 13$.
 For $m = 3$ the formula becomes $342*n + 19$.
 For $m = 4$ the formula becomes $600*n + 25$.

For $n = 1$ the formula generates a perfect square.
 For $n = 2$ the formula becomes $(6*m + 1)*(12*m + 1)$ and were found the following Poulet numbers: 2701, 8911 etc.
 For $n = 3$ the formula becomes $(6*m + 1)*(18*m + 1)$ and were found the following Poulet numbers: 2821, 4033, 5461 etc.
 For $n = 4$ the formula becomes $(6*m + 1)*(24*m + 1)$ and were found the following Poulet numbers: 1387, 83665 etc. (see the sequence A182123 in OEIS).

Note: The formula is equivalent to Poulet numbers of the form $p*(n*p - n + 1)$, where p is of the form $6*m + 1$. From the first 68 Poulet numbers just 7 of them (7957, 23377, 33153, 35333, 42799, 49981, 60787) can't be written as $p*(n*p - n + 1)$, where p is of the form $6*m \pm 1$.

(6) *Poulet numbers of the form $m*n^2 + (11*m - 23)*n + 19*m - 49$.*

First 10 terms: 341, 645, 1105, 1387, 2047, 2465, 2821, 3277, 4033, 5461 (sequence A215326 in OEIS).

Note: The solutions (m, n) for the Poulet numbers from the sequence above are: (3, 9); (3, 13); (4, 14); (4, 16); (9, 11) and (4, 20); (3, 27); (3, 29); (4, 26); (3, 35); (290, 0).

10. A conjecture about a large subset of Carmichael numbers related to concatenation

Abstract. Though the method of concatenation has its recognised place in number theory, is rarely leading to the determination of characteristics of an entire class of numbers, which is not defined only through concatenation. We present here a property related to concatenation that appears to be shared by a large subset of Carmichael numbers

Introduction: I was studying the primes of the form $12*k + 5$ (*i.e.* the primes 5, 17, 29, 41, 53, 89, 101, 113, 137, 149, 173 and so on) when I noticed that the primes obtained through the concatenation of two of them are easily to find, especially the ones that end in the digits 29: 4129, 6529, 8929, 11329, 13729, 14929 and so on. When I looked on a certain subset of Carmichael numbers I observed an interesting property that appear to be common to the numbers from this subset (Observation) then I saw that the property is in fact common to a much larger subset of Carmichael numbers (Conjecture).

Observation: The numbers obtained through *deconcatenation* (I understand through this word the operation which is the reverse of concatenation) of the digits of the Carmichael numbers that have 29 as the last two digits and the respective two digits appear to be congruent to 5 (mod 6) or to 2 (mod 6).

I checked this property to the first 21 Carmichael numbers of the form $100*k + 29$:

: for 1729	we have	$(17 - 5)/6 = 2$;
: for 23382529	we have	$(233825 - 5)/6 = 38970$;
: for 146843929	we have	$(1468439 - 5)/6 = 244739$;
: for 172947529	we have	$(1729475 - 5)/6 = 288245$;
: for 188516329	we have	$(1885163 - 5)/6 = 314193$;
: for 246446929	we have	$(2464469 - 5)/6 = 410744$;
: for 271481329	we have	$(2714813 - 5)/6 = 452468$;
: for 484662529	we have	$(4846625 - 5)/6 = 807770$;
: for 593234929	we have	$(5932349 - 5)/6 = 988724$;
: for 934784929	we have	$(9347849 - 5)/6 = 1557974$;
: for 958762729	we have	$(9587627 - 5)/6 = 1597937$;
: for 1055384929	we have	$(10553849 - 5)/6 = 1758974$;
: for 1688214529	we have	$(16882145 - 5)/6 = 2813690$;
: for 1858395529	we have	$(18583955 - 5)/6 = 3097325$;
: for 1942608529	we have	$(19426085 - 5)/6 = 3237680$;
: for 6218177329	we have	$(62181773 - 5)/6 = 10363628$;
: for 7044493729	we have	$(70444937 - 5)/6 = 11740822$;
: for 10128932929	we have	$(101289329 - 5)/6 = 101289329$;
: for 10387489729	we have	$(103874897 - 5)/6 = 17312482$;
: for 11477658529	we have	$(114776585 - 5)/6 = 19129430$.
: for 12299638429	we have	$(122996384 - 2)/6 = 20499397$.

Note: I expressed this property in the way above so we can see yet another interesting pattern: many of the integers obtained through this operation have the sum of the digits equal to 29: 244739, 288245, 452468, 807770, 2813690, 3097325, 3237680, 10363628, 19129430.

Note: It would be interesting to see what kind of numbers we obtain if we reverse the operations above: let be x a number with the sum of the digits equal to 29, $x*6 + 5 = y$ and z the number obtained through concatenation of y and 29:

- : for $x = 2999$, $y = 17999$ and $z = 1799929$ prime;
- : for $x = 9299$, $y = 55799$ and $z = 1553*3593$ semiprime;
- : for $x = 9929$, $y = 59579$ and $z = 373*15973$ semiprime;
- : for $x = 9992$, $y = 59957$ and $z = 5995729$ prime;
- : for $x = 3899$, $y = 23399$ and $z = 2339929$ prime;
- : for $x = 3989$, $y = 23939$ and $z = 2393929$ prime.

If we take x a number with the sum of the digits equal to another prime of the form $6*k - 1$ instead 29, *i.e.* 41, and repeat the same operations from above, we obtain:

- : for $x = 59999$, $y = 359999$ and $z = 35999941$ prime;
- : for $x = 99599$, $y = 597599$ and $z = 59759941$ semiprime;
- : for $x = 99959$, $y = 599759$ and $z = 59975941$ prime;
- : for $x = 99995$, $y = 599975$ and $z = 59997541$ prime.

Even more than that, if we take x a number with the sum of the digits equal to 41, but we calculate z as the concatenation of y not cu 41 but with 29, we obtain:

- : for $x = 59999$, $y = 359999$ and $z = 35999929$ semiprime;
- : for $x = 95999$, $y = 575999$ and $z = 57599929$ prime;
- : for $x = 99599$, $y = 597599$ and $z = 59759929$ prime;
- : for $x = 99959$, $y = 599759$ and $z = 59975929$ semiprime;
- : for $x = 99995$, $y = 599975$ and $z = 59997599$ semiprime.

We saw that, taking randomly 15 numbers with the property that sum of their digits is equal to a prime of the form $6*k - 1$ (in fact not entirely random, because 2999 and 59999 are the smaller primes for which the sum of the digits is equal to 29, respectively 41), we obtained 9 primes and 6 semiprimes, so this direction of study seems to be prolific.

It is also interesting to see which are the smaller numbers with the property that the sum of their digits equals a prime p of the form $6*k - 1$: these numbers are: 29 (for $p = 11$), 89 (for $p = 17$), 599 (for $p = 23$), 2999 (for $p = 29$), 59999 (for $p = 41$), 299999 (for $p = 47$), 899999 (for $p = 53$), 5999999 (for $p = 59$), 89999999 (for $p = 71$), 299999999 (for $p = 83$), 899999999 (for $p = 89$), 2999999999 (for $p = 101$) and so on. If we concatenate, for instance, the number $6*2999999999 + 5$ with these numbers we obtain 1799999999929, 1799999999989, 17999999999599 (which are all semiprimes) and so on.

Conjecture: The numbers formed through deconcatanation of Carmichael numbers not divisible by 5 that ends in the digits that form a number of the form $6*k - 1$ and the respective number are congruent to 2 (mod 6) or to 5 (mod 6).

I checked this property to the first few Carmichael numbers that ends in digits of this form (beside the cases that I already considered above):

- : for 2821, where $821 \equiv 5 \pmod{6}$, we have $2 \equiv 2 \pmod{6}$;
- : for 8911, where $11 \equiv 5 \pmod{6}$, we have $89 \equiv 5 \pmod{6}$; but also $911 \equiv 5 \pmod{6}$, and we have $8 \equiv 2 \pmod{6}$;
- : for 15841, where $41 \equiv 5 \pmod{6}$, we have $158 \equiv 2 \pmod{6}$;
- : for 29341, where $41 \equiv 5 \pmod{6}$, we have $293 \equiv 5 \pmod{6}$; but also $341 \equiv 5 \pmod{6}$, and we have $29 \equiv 5 \pmod{6}$ and also $9341 \equiv 5 \pmod{6}$, and we have $2 \equiv 2 \pmod{6}$;
- : for 41041, where $41 \equiv 5 \pmod{6}$, we have $410 \equiv 2 \pmod{6}$;
- : for 52633, where $2633 \equiv 5 \pmod{6}$, we have $5 \equiv 5 \pmod{6}$;
- : for 101101, where $101 \equiv 5 \pmod{6}$, we have $101 \equiv 5 \pmod{6}$;
- : for 115921, where $5921 \equiv 5 \pmod{6}$, we have $11 \equiv 5 \pmod{6}$;
- : for 126217, where $17 \equiv 5 \pmod{6}$, we have $1262 \equiv 2 \pmod{6}$;
- : for 172081, where $2081 \equiv 5 \pmod{6}$, we have $17 \equiv 2 \pmod{6}$;
- : for 188461, where $461 \equiv 5 \pmod{6}$, we have $188 \equiv 2 \pmod{6}$;
- : for 252601, where $52601 \equiv 5 \pmod{6}$, we have $2 \equiv 2 \pmod{6}$;
- : for 294409, where $4409 \equiv 5 \pmod{6}$, we have $29 \equiv 5 \pmod{6}$; but also $94409 \equiv 5 \pmod{6}$, and we have $2 \equiv 2 \pmod{6}$;
- : for 314821, where $821 \equiv 5 \pmod{6}$, we have $314 \equiv 2 \pmod{6}$;
- : for 334153, where $53 \equiv 5 \pmod{6}$, we have $3341 \equiv 5 \pmod{6}$;
- : for 410041, where $41 \equiv 5 \pmod{6}$, we have $4100 \equiv 2 \pmod{6}$;
- : for 488881, where $881 \equiv 5 \pmod{6}$, we have $488 \equiv 2 \pmod{6}$;
- : for 512461, where $461 \equiv 5 \pmod{6}$, we have $512 \equiv 2 \pmod{6}$;
- : for 530881, where $881 \equiv 5 \pmod{6}$, we have $530 \equiv 2 \pmod{6}$; but also $30881 \equiv 5 \pmod{6}$, and we have $5 \equiv 5 \pmod{6}$;
- : for 658801, where $8801 \equiv 5 \pmod{6}$, we have $65 \equiv 2 \pmod{6}$;
- : for 748657, where $8657 \equiv 5 \pmod{6}$, we have $74 \equiv 2 \pmod{6}$;
- : for 838201, where $8201 \equiv 5 \pmod{6}$, we have $83 \equiv 2 \pmod{6}$;
- : for 852841, where $41 \equiv 5 \pmod{6}$, we have $8528 \equiv 2 \pmod{6}$;
- : for 1082809, where $809 \equiv 5 \pmod{6}$, we have $1082 \equiv 2 \pmod{6}$;
- : for 1152271, where $71 \equiv 5 \pmod{6}$, we have $11522 \equiv 2 \pmod{6}$;
- : for 1193221, where $221 \equiv 5 \pmod{6}$, we have $1193 \equiv 5 \pmod{6}$; but also $93221 \equiv 5 \pmod{6}$, and we have $11 \equiv 5 \pmod{6}$;

: for 1461241, where $41 \equiv 5 \pmod{6}$, we have $14612 \equiv 2 \pmod{6}$; but also $1241 \equiv 5 \pmod{6}$, and we have $146 \equiv 2 \pmod{6}$ and $61241 \equiv 5 \pmod{6}$, and we have $14 \equiv 2 \pmod{6}$;
 : for 1615681, where $5681 \equiv 5 \pmod{6}$, we have $161 \equiv 5 \pmod{6}$;
 : for 1773289, where $89 \equiv 5 \pmod{6}$, we have $17732 \equiv 2 \pmod{6}$; but also $73289 \equiv 5 \pmod{6}$, and we have $17 \equiv 2 \pmod{6}$.

We take now few bigger Carmichael numbers:

: for 998324255809, where $809 \equiv 5 \pmod{6}$, we have $998324255 \equiv 5 \pmod{6}$; but also $255809 \equiv 5 \pmod{6}$, and we have $998324 \equiv 2 \pmod{6}$ and $24255809 \equiv 5 \pmod{6}$, and we have $9983 \equiv 5 \pmod{6}$ and $324255809 \equiv 5 \pmod{6}$, and we have $998 \equiv 2 \pmod{6}$;
 : for 998667686017, where $17 \equiv 5 \pmod{6}$, we have $9986676860 \equiv 2 \pmod{6}$; but also $6017 \equiv 5 \pmod{6}$, and we have $99866768 \equiv 2 \pmod{6}$ and $7686017 \equiv 5 \pmod{6}$, and we have $99866 \equiv 2 \pmod{6}$ and $67686017 \equiv 5 \pmod{6}$, and we have $9986 \equiv 2 \pmod{6}$ and $667686017 \equiv 5 \pmod{6}$, and we have $998 \equiv 2 \pmod{6}$;
 : for 999607982113, where $113 \equiv 5 \pmod{6}$, we have $999607982 \equiv 2 \pmod{6}$;
 : for 999629786233, where $233 \equiv 5 \pmod{6}$, we have $999629786 \equiv 2 \pmod{6}$; but also $6233 \equiv 5 \pmod{6}$, and we have $99962978 \equiv 2 \pmod{6}$ and $786233 \equiv 5 \pmod{6}$, and we have $999629 \equiv 5 \pmod{6}$ and $9786233 \equiv 5 \pmod{6}$, and we have $99962 \equiv 2 \pmod{6}$.

Note: From all the cases which appear until the Carmichael number 1773289 (we saw that for a single Carmichael number we can meet the conditions from hypothesis more than once), I only met one exception: for 162401, where $401 \equiv 5 \pmod{6}$, we have $162 \equiv 0 \pmod{6}$; I didn't change yet the statement from conjecture, waiting for at least one more counterexample to set a pattern.

Conclusion: The results obtained for Carmichael numbers may have theoretical value, but for a more practical value, for instance to be helpful in a PRP test, let's see if these results can be extended for the class of Fermat pseudoprimes to base 2:

: for 341, where $41 \equiv 5 \pmod{6}$, we have $3 \equiv 3 \pmod{6}$;
 : for 2047, where $47 \equiv 5 \pmod{6}$, we have $20 \equiv 2 \pmod{6}$;
 : for 2701, where $701 \equiv 5 \pmod{6}$, we have $2 \equiv 2 \pmod{6}$;
 : for 3277, where $77 \equiv 5 \pmod{6}$, we have $32 \equiv 2 \pmod{6}$;
 : for 4371, where $71 \equiv 5 \pmod{6}$, we have $43 \equiv 1 \pmod{6}$.

Unfortunately, from the first 5 cases that we considered it becomes clear that the conjecture can't be extended on Poulet numbers. A resembling pattern seems not to exist in the case of prime numbers also, so this is a feature strictly of absolute Fermat pseudoprimes.

11. A conjecture about primes based on heuristic arguments involving Carmichael numbers

Abstract. The number 30 is important to me because I always believed in the utility of classification of primes in primes of the form $30*k + 1$, $30*k + 7$, $30*k + 11$, $30*k + 13$, $30*k + 17$, $30*k + 19$, $30*k + 23$ and $30*k + 29$ (which may be interpreted as well as primes of the form $30*h - 29$, $30*h - 23$, $30*h - 19$, $30*h - 17$, $30*h - 13$, $30*h - 11$, $30*h - 7$ and $30*h - 1$). The following conjecture involves the multiples of the number 30 and is based on the study of Carmichael numbers.

Conjecture: For any three distinct primes p, q, r there exist a positive integer n so that the numbers $x = 30*n - p, y = 30*n - q$ and $z = 30*n - r$ are all three primes.

Comments

I already showed in the article “A list of 13 sequences of Carmichael numbers based on the multiples of the number 30”, posted on VIXRA, the importance of the multiples of 30 in the study of Carmichael numbers.

I shall list randomly a number of ways in which a Carmichael number with three prime factors can be written in function of the multiples of the number 30 (we note with C a Carmichael number):

$C = (30*n - p)*(60*n - q)*(90*n - r)$, where n is a positive integer and p, q, r are primes.
Examples:

$$\begin{aligned} C &= 8911 = 7*19*67 = (30 - 23)*(60 - 41)*(90 - 23); \\ C &= 15841 = 7*31*73 = (30 - 23)*(60 - 29)*(90 - 17); \\ C &= 29341 = 13*37*61 = (30 - 17)*(60 - 23)*(90 - 29). \end{aligned}$$

$C = (30*n - p)*(90*n - q)*(120*n - r)$, where n is a positive integer and p, q, r are primes.
Example:

$$C = 52633 = 7*73*103 = (30 - 23)*(90 - 17)*(120 - 17).$$

But the most appealing form is the following one: $C = (30*n - p)*(30*n - q)*(30*n - r)$, where n is a positive integer and p, q, r are primes.

Examples:

$$\begin{aligned} C &= 1729 = 7*13*19 = (30*1 - 23)(30*1 - 17)(30*1 - 11); \\ C &= 1729 = 7*13*19 = (30*9 - 263)(30*9 - 257)(30*9 - 251); \\ C &= 2821 = 7*13*31 = (30*9 - 263)(30*9 - 257)(30*9 - 239); \\ C &= 6601 = 7*23*41 = (30*6 - 173)(30*6 - 157)(30*6 - 139); \\ C &= 8911 = 7*19*67 = (30*3 - 83)(30*3 - 71)(30*3 - 23); \\ C &= 15841 = 7*31*73 = (30*3 - 83)(30*3 - 59)(30*3 - 17). \end{aligned}$$

In fact, my initial intention was to conjecture that any Carmichael number can be written in this form, in other words that for any three prime factors p, q, r of a 3-Carmichael number there exist a positive integer n so that the numbers $x = 30*n - p, y = 30*n - q$ and $z = 30*n - r$ are all three primes.

Note: The reason for which I chose 3 primes for the conjecture instead of 2 or 4 is that 3 is the minimum number of prime factors of a Carmichael number but also because I would relate this conjecture with the study of Fermat’s last theorem.

Note: The conjecture implies of course that for any pair of twin primes (p, q) there exist a pair of primes $(30*n - p, 30*n - q)$ so that there are infinitely many pairs of primes.

12. A conjecture regarding the relation between Carmichael numbers and the sum of their digits

Abstract. Though they are a fascinating class of numbers, there are very many properties of Carmichael numbers still unstudied enough. I have always thought there is a connection between these numbers and the sum of their digits (few of them are also Harshad numbers). I try here to highlight such a possible connection.

Conjecture: For any Carmichael number C that has only prime factors of the form $6k + 1$ is true at least one of the following five relations:

- (1) C is a Harshad number;
- (2) If we note with $s(m)$ the sum of the digits of the integer m then C is divisible by $n \cdot s(C) - n + 1$, where n is integer;
- (3) C is divisible by $s((C + 1)/2)$;
- (4) C is divisible by $n \cdot s((C + 1)/2) - n + 1$, where n is integer;
- (5) $s(C) = s((C + 1)/2)$.

I verified below the conjecture for the first 23 Carmichael numbers of this type: 1729, 2821, 8911, 15841, 29341, 46657, 52633, 63973, 115921, 126217, 172081, 188461, 294409, 314821, 334153, 399001, 488881, 512461, 530881, 670033, 748657, 838201, 997633.

: 1729 is divisible by 19, where $19 = s(1729)$; so 1729 satisfies relation (1); also $s((1729 + 1)/2) = s(865) = 19$ so 1729 satisfies the relations (3) and (5) either;

: 2821 is divisible by 13, where $13 = s(2821)$; so 2821 satisfies relation (1); also $s((2821 + 1)/2) = s(1411) = 7$ and 2821 is divisible by 7 so 2821 satisfies the relation (3) either;

: 8911 is divisible by 19, where $19 = s(8911)$; so 8911 satisfies relation (1); also $s((8911 + 1)/2) = s(4456) = 19$ so 8911 satisfies the relations (3) and (5) either;

: $s(15841) = s((15841 + 1)/2) = 19$ and 15841 is divisible by 73 which is equal to $4 \cdot 19 - 3$; so 15841 satisfies relations (2), (4) and (5);

: $s(29341) = s((29341 + 1)/2) = 19$ and 29341 is divisible by 37 which is equal to $2 \cdot 19 - 1$; so 29341 satisfies relations (2), (4) and (5);

: $s((46657 + 1)/2) = 19$ and 46657 is divisible by 37, which is equal to $2 \cdot 19 - 1$; so 46657 satisfies relation (4);

: $s(52633) = s((52633 + 1)/2) = 19$ and 52633 is divisible by 73 which is equal to $4 \cdot 19 - 3$; so 52633 satisfies relations (2), (4) and (5);

: $s(63973) = s((63973 + 1)/2) = s(31987) = 28$; so 52633 satisfies relation (5);

: $s(115921) = 19$ and 115921 is divisible by 37 which is equal to $2 \cdot 19 - 1$;

: 126217 is divisible by 19, where $19 = s(126217)$; so 126217 satisfies relation (1); also $s((126217 + 1)/2) = s(63109) = 19$ so 126217 satisfies relations (3) and (5) either;

: $s(172081) = s((172081 + 1)/2) = s(86041) = 19$; so 172081 satisfies relation (5);

: $s((188461 + 1)/2) = s(94231) = 19$ and 188461 is divisible by 19; so 188461 satisfies relation (3); also $s(188461) = 28$ and 188461 is divisible by 109 which is equal to $4 \cdot 28 - 3$ so satisfies relation (2) either;

: $s(294409) = 28$ and 294409 is divisible by 109 which is equal to $4 \cdot 28 - 3$; so 294409 satisfies relation (2); $s((294409 + 1)/2) = s(147205) = 19$ and 294409 is divisible by 37, 73 and 109 which are equal to $19 \cdot 2 - 1$, $19 \cdot 4 - 3$ and $19 \cdot 6 - 5$ so 294409 satisfies relation (4) either;

: $s(314821) = s((314821 + 1)/2) = s(157411) = 19$; so 314821 satisfies relation(5);

: 334153 is divisible by 19, where $19 = s(334153)$; so 334153 satisfies relation (1);

$s(399001) = 22$ and 399001 is divisible by 211 which is equal to $22 \cdot 10 - 9$; so 399001 satisfies relation(2);
 : 488881 is divisible by 37, where $37 = s(488881)$; so 488881 satisfies relation (1);
 : $s(512461) = s((512461 + 1)/2) = s(256231) = 19$; so 512461 satisfies relation(5);
 : $s(530881) = 22$ and 530881 is divisible by 421 which is equal to $22 \cdot 20 - 19$; so 530881 satisfies relation(2);
 : $s(670033) = s((670033 + 1)/2) = s(335017) = 19$; so 512461 satisfies relation(5);
 : $s(748657) = 37$ and 748657 is divisible by 433 which is equal to $37 \cdot 12 - 11$; so 748657 satisfies relation(2);
 : $s((838201 + 1)/2) = s(419101) = 16$ and 838201 is divisible by 61 and 151 which are equal to $16 \cdot 4 - 3$ and $16 \cdot 10 - 5$; so 748657 satisfies relation(4);
 : $s(997633) = s((997633 + 1)/2) = s(498817) = 37$; so 997633 satisfies relation(5).

Note: We observed a subset a Carmichael numbers: the numbers $399001 = 31 \cdot 61 \cdot 211$ and $530881 = 13 \cdot 97 \cdot 421$ have both the sum of their digits $s(C) = 22$ and $s((C + 1)/2) = 25$; also, C is divisible by $n \cdot s(C) - n + 1$, where n is their greatest prime factor.

Note: Many other Carmichael numbers have resembling properties, the ones that have only prime factors of the form $6 \cdot k - 1$ for instance, but I didn't find yet another category of Carmichel numbers that could be set in such a closed form.

Note: For many Carmichael number C that are also Harshad number is true that $s(C) = s((C + 1)/2)$.

Note: For the odd Harshad numbers H that I checked, the first one that satisfy the relation $s(H) = s((H + 1)/2)$ is the number 1387, the fifth Poulet number, which yet again connect this property with Fermat pseudoprimes.

Observation: I also noticed few relations based on the sum of the digits that are satisfied by a Poulet number P that has only two prime factors, both of the form $6 \cdot k + 1$:

- (1) $s(P) = s((P + 1)/2)$;
- (2) Both prime factors of P can be written as $n \cdot s((P + 1)/2) + 1$, where n is integer;
- (3) Both prime factors of P can be written as $n \cdot s((P + 1)/2) + n + 1$, where n is integer;
- (4) Both prime factors of P can be written as $n \cdot s((P + 1)/2) - n + 1$, where n is integer;
- (5) Both prime factors of P can be written as $n \cdot s(P) - n + 1$, where n is integer.

I considered the first 15 Poulet numbers of this type: 1387, 2071, 4033, 4681, 5461, 7957, 10261, 14491, 18721, 23377, 31609, 31621, 42799, 49141, 49981 (for a list of Poulet numbers with two prime factors see the sequence A214305 in OEIS).

$s(1387) = s((1387 + 1)/2) = s(694) = 19$, so 1387 satisfies relation (1);
 $s(2071) = s((2071 + 1)/2) = s(1036) = 10$, so 2071 satisfies relation (1);
 $s(4033) = s((4033 + 1)/2) = s(2017) = 10$, so 4033 satisfies relation (1);
 $s(4681) = 19$ and $s((4681 + 1)/2) = s(2341) = 10$ and 4681 is divisible with 31 which is equal to $3 \cdot 10 + 1$ also with 151 which is equal to $15 \cdot 10 + 1$, so 4681 satisfies relation (2);
 $s(5461) = 16$ and $s((5461 + 1)/2) = s(2731) = 13$ and 4681 is divisible with 43 which is equal to $3 \cdot 13 + 4$ also with 127 which is equal to $9 \cdot 13 + 10$, so 1387 satisfies relation (3);
 $s(7957) = s((7957 + 1)/2) = s(3979) = 28$, so 7957 satisfies relation (1);

: $s(10261) = 10$ and $s((10261 + 1)/2) = s(5131) = 10$ and 10261 is divisible with 31 which is equal to $3 \cdot 10 + 1$ also with 331 which is equal to $33 \cdot 10 + 1$, so 10261 satisfies relation (2);
 : $s(14491) = s((14491 + 1)/2) = s(7246) = 19$, so 14491 satisfies relation (1);
 : $s(18721) = s((18721 + 1)/2) = s(9361) = 19$, so 18721 satisfies relation (1);
 : $s(23377) = 22$ and $s((23377 + 1)/2) = s(11689) = 25$ and 23377 is divisible with 97 which is equal to $4 \cdot 25 - 3$ also with 241 which is equal to $10 \cdot 25 - 9$, so 23377 satisfies relation (4);
 : $s(31609) = s((31609 + 1)/2) = s(15805) = 19$, so 31609 satisfies relation (1);
 : $s(31621) = 13$ and $s((31621 + 1)/2) = s(15811) = 16$ and 31621 is divisible with 103 which is equal to $6 \cdot 16 + 7$ also with 307 which is equal to $18 \cdot 16 + 19$, so 31621 satisfies relation (3);
 : $s(42799) = 31$ and $s((42799 + 1)/2) = s(21400) = 7$ and 42799 is divisible with 127 which is equal to $18 \cdot 7 + 1$ also with 337 which is equal to $48 \cdot 7 + 1$, so 42799 satisfies relation (2);
 : $s(49141) = s((49141 + 1)/2) = s(24571) = 19$, so 49141 satisfies relation (1);
 : $s(49981) = 31$ and 49981 is divisible with 151 which is equal to $31 \cdot 5 - 4$ also with 331 which is equal to $31 \cdot 11 - 10$, so 49981 satisfies relation (1).

Conclusion: The relation between the Fermat pseudoprimes and the sum of their digits seems to be obvious even that there are probably better ways to express this relation (I actually only wanted to highlight few such possible ways). The property of a composite odd integer n to be divisible with $s((n + 1)/2)$ deserves further study, also the property of a Harshad odd number n to have $s(n) = s((n + 1)/2)$: we saw that the smallest such number with this property is a Fermat pseudoprime to base 2, the number 1387. It would also be interesting to see what numbers that are products of more than three prime factors of the form $6 \cdot k + 1$ and are not Carmichael numbers satisfy the relations from the conjecture.

13. A list of 13 sequences of Carmichael numbers based on the multiples of the number 30

Abstract. The applications of the multiples of the number 30 in the study of Fermat pseudoprimes was for a long time one of my favourite subject of study; in this paper I shall list 13 sequences that I discovered, many of them, if not all of them, having probably an infinity of terms that are Carmichael numbers. I posted many of them on OEIS, where I analyzed more of their attributes; here I'll just list them, enumerate their first few terms and present few conjectures.

- (1) Carmichael numbers of the form $C = (30 \cdot n + 7) \cdot (60 \cdot n + 13) \cdot (150 \cdot n + 31)$.

First 6 terms: 2821, 488881, 288120421, 492559141, 776176261, 1632785701 (sequence A182085 in OEIS).

Conjecture: The number $(30 \cdot n + 7) \cdot (60 \cdot n + 13) \cdot (150 \cdot n + 31)$ is a Carmichael number if (but not only if) $30 \cdot n + 7$, $60 \cdot n + 13$ and $150 \cdot n + 31$ are all three prime numbers.

- (2) Carmichael numbers of the form $C = (30 \cdot n - p) \cdot (60 \cdot n - (2 \cdot p + 1)) \cdot (90 \cdot n - (3 \cdot p + 2))$, where p , $2 \cdot p + 1$, $3 \cdot p + 2$ are all three prime numbers.

First 6 terms: 1729, 172081, 294409, 1773289, 4463641, 56052361 (sequence A182087 in OEIS).

Comment: These numbers can be reduced to only two possible forms: $C = (30*n - 23)*(60*n - 47)*(90*n - 71)$ or $C = (30*n - 29)*(60*n - 59)*(90*n - 89)$.

- (3) Carmichael numbers of the form $C = (30*n - 29)*(60*n - 59)*(90*n - 89)*(180*n - 179)$.

First 4 terms: 31146661, 2414829781, 192739365541, 197531244744661 (sequence A182088 in OEIS).

Conjecture: The number $(30*n - 29)*(60*n - 59)*(90*n - 89)*(180*n - 179)$ is a Carmichael number if (but not only if) $30*n - 29$, $60*n - 59$, $90*n - 89$ and $180*n - 179$ are all four prime numbers.

- (4) Carmichael numbers of the form $C = (330*n + 7)*(660*n + 13)*(990*n + 19)*(1980*n + 37)$.

First 2 terms: 63973, 461574735553 (sequence A182089 in OEIS).

Conjecture: The number $(330*n + 7)*(660*n + 13)*(990*n + 19)*(1980*n + 37)$ is a Carmichael number if $330*n + 7$, $660*n + 13$, $990*n + 19$ and $1980*n + 37$ are all four prime numbers.

- (5) Carmichael numbers of the form $C = (30*n - 7)*(90*n - 23)*(300*n - 79)$.

First 5 terms: 340561, 4335241, 153927961, 542497201, 1678569121 (sequence A182132 in OEIS).

Conjecture: The number $(30*n - 7)*(90*n - 23)*(300*n - 79)$ is a Carmichael number if (but not only if) $30*n - 7$, $90*n - 23$ and $300*n - 79$ are all three prime numbers.

- (6) Carmichael numbers of the form $C = (30*n - 17)*(90*n - 53)*(150*n - 89)$.

First 5 terms: 29341, 1152271, 34901461, 64377991, 775368901 (sequence A182133 in OEIS).

Conjecture: The number $(30*n - 17)*(90*n - 53)*(150*n - 89)$ is a Carmichael number if (but not only if) $30*n - 17$, $90*n - 53$ and $150*n - 89$ are all three prime numbers.

- (7) Carmichael numbers of the form $C = (60*n + 13)*(180*n + 37)*(300*n + 61)$.

First 5 terms: 29341, 34901461, 775368901, 1213619761, 4562359201 (sequence A182416 in OEIS).

Conjecture: The number $(60*n + 13)*(180*n + 37)*(300*n + 61)$ is a Carmichael number if (but not only if) $60*n + 13$, $180*n + 37$ and $300*n + 61$ are all three prime numbers.

(8) Carmichael numbers of the form $C = (90*n + 1)*(180*n + 1)*(270*n + 1)*(540*n + 1)$.

First 2 terms: 2414829781, 192739365541.

Comment:

For $n = n/15$ the formula becomes $(6*n + 1)*(12*n + 1)*(18*n + 1)*(36*n + 1)$.

(9) Carmichael numbers of the form $C = (p + 30)*(q + 60)*(p*q + 90)$, where p and q are primes.

First 2 terms: 488881, 1033669.

Comment: We obtained Carmichael numbers for $[p, q] = [7, 13]$ and $[p, q] = [7, 31]$.

(10) Carmichael numbers of the form $C = (30*p + 1)*(60*p + 1)*(90*p + 1)$, where p is prime.

First 4 terms: 56052361, 216821881, 798770161, 1976295241.

Comment: We obtained Carmichael numbers for the following values of p : 7, 11, 17, 23.

(11) Carmichael numbers of the form $C = 1710*3^m + 60*n + 451$.

First 3 terms: 2821, 6601, 15841.

Comment: We obtained Carmichael numbers for the following values of $[m, n]$: $[0, 11]$, $[1, 17]$, $[2, 0]$.

(12) Carmichael numbers of the form $C = 1710*m + 30*n + 1$.

First 7 terms: 2821, 6601, 8911, 15841, 29341, 41041, 75361.

Comment: We obtained Carmichael numbers for the following values of $[m, n]$: $[1, 37]$, $[3, 49]$, $[5, 12]$, $[9, 15]$, $[17, 9]$, $[24, 0]$, $[44, 4]$.

(13) Carmichael numbers of the form $C = 60*n + 2281$.

First 17 terms: 2821, 6601, 15841, 29341, 41041, 101101, 115921, 172081, 188461, 252601, 314821, 340561, 399001, 410041, 488881, 512461, 530881.

Comment: We obtained Carmichael numbers for the following values of n : 9, 72, 226, 451, 646, 1647, 1894, 2830, 3103, 4172, 5209, 5638, 6612, 6796, 8110, 8503, 8810.

Conjecture: All Carmichael numbers C of the form $10*k + 1$ that have digital root equal to 1, 4 or 7 can be written as $C = 60*n + 2281$.

14. A possible generic formula for Carmichael numbers

Abstract. To find generic formulas for Carmichael numbers (beside, of course, the formula that defines them) was for long time one of my targets; I already found such a formula, based on Korselt's criterion; I possibly found now another such a formula.

Conjecture: Any Carmichael number can be written as $(n^2 \cdot p^2 - q^2)/(n^2 - 1)$, where p and q are primes or power of primes or are equal to 1 and n is positive integer, $n > 1$.

The first Carmichael number, 561, can be written as $(4 \cdot p^2 - q^2)/3$ for $[p, q] = [29, 41], [41, 71], [7^2, 89], [421, 29^2]$; it can also be written as $(16 \cdot p^2 - q^2)/15$ for $[p, q] = [23, 7], [29, 71]$ etc.

The second Carmichael number, 1105, can be written as $(4 \cdot p^2 - q^2)/3$ for $[p, q] = [29, 7], [31, 23], [53, 89], [59, 103], [67, 11^2], [829, 1657]$; it can also be written as $(9 \cdot p^2 - q^2)/8$ for $[p, q] = [37, 59], [7^2, 113], [61, 157]$ etc.

The third Carmichael number, 1729, can be written as $(4 \cdot p^2 - q^2)/3$ for $[p, q] = [37, 17], [43, 47], [67, 113], [73, 127], [103, 193], [433, 863], [1297, 2593]$; it can also be written as $(9 \cdot p^2 - q^2)/8$ for $[p, q] = [43, 53], [53, 107], [67, 163], [167, 487], [1153, 3457]$; it can also be written as $(16 \cdot p^2 - q^2)/15$ for $[p, q] = [41, 31], [47, 97], [97, 353], [157, 657], [173, 673], [251, 991]$; it can also be written as $(25 \cdot p^2 - q^2)/24$ for $[p, q] = [41, 23], [61, 227], [151, 727], [347, 1723]$ etc. (seems that the famous Hardy–Ramanujan number can set a record for how many ways can be written this way).

Few subsets of Carmichael numbers:

A subset of Carmichael numbers C has the following property: $C = (4 \cdot p^2 - q^2)/3$, where q is the smaller prime that verify the relation $q > \sqrt{3 \cdot C/4}$, and p is prime or a power of prime; few such numbers are:

1105, 1729, 6601, 41041, 75361, 340561, for corresponding $[p, q] = [7, 29], [17, 37], [19, 71], [71, 179], [239, 7^2], [509, 11^4]$.

Another subset of Carmichael numbers C has the following property: $C = (n^2 \cdot p^2 - 1)/(n^2 - 1)$, where p is the smaller prime that verify the relation $p > \sqrt{3 \cdot C/4}$; few such numbers are: 2465, 8911, 10585, 15841, 162401, for corresponding $[n, p] = [2, 43], [3, 89], [3, 97], [2, 109], [2, 349]$.

Another subset of Carmichael numbers C (but this time only related to the formula above) has the following property: $C = (4 \cdot p^2 - 7153)/3$, where p is prime; such numbers are: 561, 488881, for corresponding $p = 47, 607$ (interesting that $607 - 47 = 560$ and 561 is the first Carmichael number).

Another subset of Carmichael numbers C (this time too only related to the formula above) has the following property: $C = (p \cdot q^2 - 1723^2)/(p - 1)$, where p and q are primes or power of primes; few such numbers are: 1105 for $[p, q] = [1249, 59]$, 1729 for $[p, q] = [5^2, 347]$, 2465 for $[p, q] = [7^2, 251]$.

Note: The formula based on Korselt's criterion that I was talking about in Abstract is: $C = p^k + n \cdot p^2 - n \cdot p$ (if $C > p^k$) or $C = p^k - n \cdot p^2 + n \cdot p$ (if $p^k > C$) for any p prime divisor of C and any k natural number. See the sequence A213812 that I submitted to OEIS.

15. An interesting and unexpected property of Carmichael numbers and a question

Abstract. I was researching a kind of generalized Cunningham chains that generate, instead of primes, Fermat pseudoprimes to some base when purely by chance I noticed a property of absolute Fermat pseudoprimes, equally interesting and unexpected. By a childish simple operation, a new class of numbers is obtained from Carmichael numbers.

Like anyone that learned in school that digits are just a way to designate a number and to operate with it, I always looked with reluctance on the arbitrary play with digits. I personally gave credit to the method of concatenation when I saw the relation between it and Fermat pseudoprimes (see my articles, *A conjecture about a large subset of Carmichael numbers related to concatenation* and *Formulas for generating primes involving emirps, Carmichael numbers and concatenation*, posted on viXra).

The property of Carmichael numbers that I discovered now proves the extreme versatility of these numbers: by a childish simple operation, insertion of the digit 0 among the digits of these numbers, we obtain an entirely new class of numbers.

Thus we have the following numbers obtained from Carmichael numbers through the operation that I mentioned:

: 5601 (from 561)

We can see that $n^{5601} \bmod 5601 = n^3$ for n from 2 to 17 (not for $n = 18$);

: 28021 (from 2821)

We can see that $n^{28021} \bmod 28021 = n^7$ for n from 2 to 4 (not for $n = 5$);

: 24065 (from 2465)

We can see that $n^{24065} \bmod 24065 = n^5$ for n from 2 to 7 (not for $n = 8$).

Note: For the number 1729, which is the known Hardy–Ramanujan number, we have $p = 10729$, $p = 17029$ and $p = 17209$ all three primes! (so, of course, $n^p \bmod p = n$ for any value of n).

Note: For the relative Fermat pseudoprimes, to base 2 and respectively to base 3, we don't obtain resembling results through this operation.

Observation: By adding the digit 0 to Carmichael numbers, operation which itself it's not at all special, it's equivalent to a simple formula, the multiplication of a Carmichael number with the number 10, we obtain: $n^{5610} \bmod 5610 = n^{10}$ for $n = 2$ (not for $n = 3$) and the same result for the numbers 1105 and 1729. Through multiplication of the first Carmichael number, 561, with the number 8, we obtain the number 4488 and also $n^{4488} \bmod 4488 = n^8$ for $n = 2$ (not for $n =$

3). Through multiplication of the first Poulet number, 341, with the number 10, we obtain the number 3410 and also $n^{3410} \bmod 3410 = n^{10}$ for $n = 2$ (not for $n = 3$). Through multiplication of the first Fermat pseudoprime to base three, 91, with the number 10 we don't obtain resembling results. Seems that this property, that $2^{(P*k)} \bmod (P*k) = 2^k$, it's a property of Poulet numbers P (it can't be extended for Fermat pseudoprimes to base 3) while the property that I showed above it's a property of Carmichael numbers (it can't be extended for relative Fermat pseudoprimes).

Comment: The numbers m that satisfy the relation $n^m \bmod m = n^k$, where $k > 1$, for any consecutive integer value of n from 2 to some larger integer, numbers obtained from Carmichael numbers through this operation or not, seems to deserve further study.

Question: Are there any numbers m to satisfy the relation $n^m \bmod m = n^k$, where $k > 1$, for any value of n ?

16. Connections between the three prime factors of 3-Carmichael numbers

Abstract. It was always obvious to me that, beside Korselt's criterion, that gives a relation between any prime factor of a Carmichael number and the number itself, there must be a relation between the prime factors themselves; here I present a conjecture on the Carmichael numbers with three prime factors expressing the larger two prime factors as a function of the smallest one and few particular cases of connections between all three prime factors.

Introduction:

In the sequence A213812 that I posted in OEIS I showed a formula, derived from Korselt's criterion, to express a Carmichael number as a function of any of its prime factors and an integer. In the sequence A215672 that I posted in OEIS I extended this formula for a Poulet number with three or more prime factors, expressing such a number as a function of at least one of its prime factors and an integer. This formula relates a Fermat pseudoprime to one (in the case of Poulet numbers) or to any (in the case of Carmichael numbers) of its prime factors, but says nothing about the relation between the prime factors themselves.

In the sequence A215672 I showed that most of Fermat pseudoprimes to base 2 with three prime factors (so, implicitly, most of Carmichael numbers with three prime factors) can be written in one of the following two ways:

- (1) $p*((n+1)^p - n)*((m+1)^p - m);$
- (2) $p*((n^p - (n+1))*(m^p - (m+1))),$

where p is the smallest of the three prime factors and n, m are natural numbers.

Exempli gratia for Poulet numbers from first category:

$$10585 = 5*29*73 = 5*(5*7 - 6)*(5*18 - 17).$$

Exempli gratia for Poulet numbers from second category:

$$6601 = 7*23*41 = 7*(7*4 - 5)*(7*7 - 8).$$

From the first 37 Poulet numbers with three prime factors, just three (30889, 88561 and 91001) can't be written in one of this two ways.

Conjecture: For any Carmichael numbers with three prime factors, $C = d_1 * d_2 * d_3$, where $d_1 < d_2 < d_3$, is true one of the following two statements:

- (1) d_2 can be written as $d_1 * (n + 1) - n$ and d_3 can be written as $d_1 * (m + 1) - m$;
 - (2) d_2 can be written as $d_1 * n - (n + 1)$ and d_3 can be written as $d_1 * m - (m + 1)$,
- where m and n are natural numbers.

As I showed, this conjecture holds for the first 13 Carmichael numbers with three prime factors checked. In this article I present few connections that express not the larger two prime factors as a function of the smallest one, as above, but connects all the three prime factors.

Observation: For most of the Carmichael numbers with three prime factors, $C = d_1 * d_2 * d_3$, where $d_1 < d_2 < d_3$, is true one of the following seventh statements:

- (1) d_3 can be written as $d_1 * (m + 1) - n$ and as well as $d_2 * (n + 1) - m$;
- (2) d_3 can be written as $d_1 * (m - 1) + n$ and as well as $d_2 * (n - 1) + m$;
- (3) d_3 can be written as $d_1 + (m + 1) * n$ and as well as $d_2 + m * n$;
- (4) d_3 can be written as $d_1 * m - 2 * n$ and as well as $d_2 * n + 2 * m$;
- (5) d_3 can be written as $d_1 * m + 2 * n$ and as well as $d_2 * n - 2 * m$;
- (6) d_3 can be written as $d_1 * m - 2 * n$ and as well as $d_2 * n + m$;
- (7) d_3 can be written as $d_1 * m + n$ and as well as $d_2 * n - 2 * m$,

where m and n are natural numbers.

Carmichael numbers which verify the first statement:

For $C = 561 = 3 * 11 * 17$ we have $[m, n] = [5, 1]$:
Indeed, $3 * (5 + 1) - 1 = 17$ and $11 * (1 + 1) - 5 = 17$.

For $C = 162401 = 17 * 41 * 233$ we have $[m, n] = [13, 5]$:
Indeed, $17 * (13 + 1) - 5 = 233$ and $41 * (5 + 1) - 13 = 233$.

For $C = 314821 = 13 * 61 * 397$ we have $[m, n] = [30, 6]$:
Indeed, $13 * (30 + 1) - 6 = 397$ and $61 * (6 + 1) - 30 = 397$.

Carmichael numbers which verify the second statement:

For $C = 1105 = 5 * 13 * 17$ we have $[m, n] = [4, 2]$:
Indeed, $5 * (4 - 1) + 2 = 17$ and $13 * (2 - 1) + 4 = 17$.

For $C = 2821 = 7 * 13 * 31$ we have $[m, n] = [5, 3]$:
Indeed, $7 * (5 - 1) + 3 = 31$ and $13 * (3 - 1) + 5 = 31$.

For $C = 8911 = 7 * 19 * 67$ we have $[m, n] = [10, 4]$:
Indeed, $7 * (10 - 1) + 4 = 67$ and $19 * (4 - 1) + 10 = 67$.

For $C = 10585 = 5 * 29 * 73$ we have $[m, n] = [15, 3]$:
Indeed, $5 * (15 - 1) + 3 = 73$ and $29 * (3 - 1) + 15 = 73$.

For $C = 15841 = 7 * 31 * 73$ we have $[m, n] = [11, 3]$:

Indeed, $7*(11 - 1) + 3 = 73$ and $31*(3 - 1) + 11 = 73$.

For $C = 115921 = 13*37*241$ we have $[m, n] = [19, 7]$:
Indeed, $13*(19 - 1) + 7 = 241$ and $37*(7 - 1) + 19 = 241$.

For $C = 314821 = 13*61*397$ we have $[m, n] = [31, 7]$:
Indeed, $13*(31 - 1) + 7 = 397$ and $61*(7 - 1) + 31 = 397$.

For $C = 334153 = 19*43*409$ we have $[m, n] = [22, 10]$:
Indeed, $19*(22 - 1) + 10 = 409$ and $43*(10 - 1) + 22 = 409$.

Carmichael numbers which verify the third statement:

For $C = 1729 = 7*13*19$ we have $[m, n] = [1, 6]$:
Indeed, $7 + 2*6 = 19$ and $13 + 6 = 19$.

For $C = 2465 = 5*17*29$ we have $[m, n] = [1, 12]$:
Indeed, $5 + 2*12 = 29$ and $17 + 12 = 29$.

For $C = 29341 = 13*37*61$ we have $[m, n] = [1, 24]$:
Indeed, $13 + 2*24 = 61$ and $37 + 24 = 61$.

For $C = 252601 = 41*61*101$ we have $[m, n] = [2, 32]$:
Indeed, $41 + 3*20 = 101$ and $61 + 2*20 = 101$.

For $C = 294409 = 37*73*109$ we have $[m, n] = [1, 36]$:
Indeed, $37 + 2*36 = 109$ and $73 + 36 = 109$.

For $C = 399001 = 31*61*211$ we have $[m, n] = [5, 36]$:
Indeed, $31 + 6*30 = 211$ and $61 + 5*30 = 211$.

For $C = 410041 = 41*73*137$ we have $[m, n] = [2, 32]$:
Indeed, $41 + 3*32 = 137$ and $73 + 2*32 = 137$.

For $C = 488881 = 37*73*181$ we have $[m, n] = [3, 36]$:
Indeed, $37 + 4*36 = 181$ and $73 + 3*36 = 181$.

For $C = 512461 = 31*61*271$ we have $[m, n] = [7, 30]$:
Indeed, $31 + 8*30 = 271$ and $61 + 7*30 = 271$.

For $C = 1152271 = 43*127*211$ we have $[m, n] = [1, 84]$:
Indeed, $43 + 2*84 = 211$ and $127 + 84 = 211$.

For $C = 1152271 = 43*127*211$ we have $[m, n] = [1, 84]$:
Indeed, $43 + 2*84 = 211$ and $127 + 84 = 211$.

For $C = 1857241 = 31*181*331$ we have $[m, n] = [1, 150]$:
Indeed, $31 + 2*150 = 331$ and $181 + 150 = 331$.

Carmichael numbers which verify the fourth statement:

For $C = 52633 = 7 \cdot 73 \cdot 103$ we have $[m, n] = [15, 1]$:
Indeed, $7 \cdot 15 - 2 \cdot 1 = 103$ and $73 \cdot 1 + 2 \cdot 15 = 103$.

For $C = 1461241 = 37 \cdot 73 \cdot 541$ we have $[m, n] = [15, 7]$:
Indeed, $37 \cdot 15 - 2 \cdot 7 = 541$ and $73 \cdot 7 + 2 \cdot 15 = 541$.

Carmichael numbers which verify the fifth statement:

For $C = 46657 = 13 \cdot 37 \cdot 97$ we have $[m, n] = [7, 3]$:
Indeed, $13 \cdot 7 + 2 \cdot 3 = 97$ and $37 \cdot 3 - 2 \cdot 7 = 97$.

Carmichael numbers which verify the sixth statement:

For $C = 1193221 = 31 \cdot 61 \cdot 631$ we have $[m, n] = [21, 10]$:
Indeed, $31 \cdot 21 - 2 \cdot 10 = 631$ and $61 \cdot 10 + 21 = 631$.

Carmichael numbers which verify the seventh statement:

For $C = 530881 = 13 \cdot 97 \cdot 421$ we have $[m, n] = [32, 5]$:
Indeed, $13 \cdot 32 + 5 = 421$ and $97 \cdot 5 - 2 \cdot 32 = 421$.

Note: From the first 31 Carmichael numbers with three prime factors checked, only four of them ($6601 = 7 \cdot 23 \cdot 41$, $1024651 = 19 \cdot 199 \cdot 271$, $1615681 = 23 \cdot 199 \cdot 353$ and $1909001 = 41 \cdot 101 \cdot 461$) don't satisfy any of the seventh statements.

Note: Obviously the prime factors of Chernick's Carmichael numbers satisfy the third statement.

Note: There are Carmichael numbers, like $314821 = 13 \cdot 61 \cdot 397$, that satisfy both the first and the second statement. The triplets of primes like $[p_1, p_2, p_3] = [13, 61, 397]$, for which $p_3 = p_1 \cdot (m + 1) - n = p_2 \cdot (n + 1) - m = p_1 \cdot m + n + 1 = p_2 \cdot n + m + 1$, deserve further study, also the question if and when the products $p_1 \cdot p_2 \cdot p_3$ are Carmichael numbers.

Note: The Carmichael number $252601 = 41 \cdot 61 \cdot 101$ can be written as $p \cdot (p \cdot n - m) \cdot (p \cdot (n + 1) - (m + 1))$, where p is prime and m, n natural numbers (because $61 = 41 \cdot 2 - 21$ and $101 = 41 \cdot 3 - 22$). Also the triplets of primes of the form $[p, p \cdot n - m, p \cdot (n + 1) - (m + 1)]$ deserve further study as well as the question if and when the products of the primes that form such a triplet are Carmichael numbers.

Note: For Carmichael numbers with three prime factors, see the sequence A087788 in OEIS.

17. Formulas for generating primes involving emirps, Carmichael numbers and concatenation

Abstract. Observations on generating primes or products of very few primes from reversible primes and Carmichael numbers using the method of concatenation.

I. On the numbers obtained through concatenation from emirps and Carmichael numbers using only the digits of the number itself and the digits of its square

Note: First we notice that, if p is a reversible prime and the number q is the number obtained through concatenation of the digits of p^2 with the digits of p , then the number q/p is often the product of very few primes (for a list of emirps see the sequence A006567 in OEIS).

Observation: If p is a reversible prime and the number q obtained through concatenation of the digits of p^2 with the digits of p has the sum of digits equal to 29, then the number q/p is often a prime or a semiprime.

16913/13 = 1301 is prime;
136937/37 = 3701 is prime;
624179/79 = 7901 is prime;
564001751/751 = 751001 is prime;
10180811009/1009 = 101*99901 is semiprime;
17450411321/1321 = 7*1887143 is semiprime.

Note that the first digits of the resulted primes are the same with the digits of p . The pairs of primes [13, 1301], [37, 3701], [79, 7901], [751, 751001] and so on deserve further study.

Conjecture: There is an infinity of reversible primes p with the property that the number obtained through concatenation of the digits of p with a number of n digits of 0, where n is equal to one less than the digits of p , and finally with the digit 1 is a prime.

Note: We also notice that, if C is a Carmichael number and the number s is the number obtained through concatenation of the digits of C^2 with the digits of C , then the number C/s is often the product of very few primes (for a list of Carmichael numbers see the sequence A002997 in OEIS):

Few examples:

314721561/561 = $7^2 \cdot 107^2$;
79405921/8911 = $59 \cdot 1510339$;
79580412821/2821 = 28210001;
435732016601/6601 = $2593 \cdot 25457$;
711501714101472184350561/84350561 = $3 \cdot 2811685366666667$.

Note the interesting value of C/s for $C = 2821$.

II. On the numbers obtained through concatenation from emirps and Carmichael numbers using the digits of the number itself, the digits of its square and the digits 0001

Observation: If C is a Carmichael number then the number obtained through the concatenation of the digits of C with the digits 0001 is often a product of very few primes.

Few examples:

$5610001 = 1129 \cdot 4969$; $17290001 = 1051 \cdot 16451$; 28210001 is prime; $66010001 = 2593 \cdot 25457$;
 $89110001 = 59 \cdot 1510339$; $105850001 = 911 \cdot 116191$; 158410001 is prime.

Note the values obtained for 2821 and 15841, both divisible with 31.

Observation: If C is a Carmichael number divisible by 31 then the number obtained through the concatenation of the digits of C with the digits 0001 is often a product of very few primes.

28210001 is prime; 158410001 is prime; 753610001 is prime; 1720810001 is prime;
 $21009010001 = 7 \cdot 3001287143$; 9912830875210001 is prime.

Observation: If C is a Carmichael number having 561 (a Carmichael number, also) as the last digits then the following numbers are often a product of very few primes:

: M , obtained through the concatenation of the digits of C with the digits 0001;

: N , obtained through the concatenation of the digits of C^2 with the digits 0001.

Few examples:

$C = 340561$; $C^2 = 115981794721$

$M = 3405610001$ is semiprime; $N = 1159817947210001$ is semiprime;

$C = 8134561$; $C^2 = 66171082662721$

$M = 81345610001$ is prime; $N = 661710826627210001$ is prime;

$C = 10024561$; $C^2 = 100491823242721$

$M = 100245610001$ is prime; $N = 1004918232427210001$ is semiprime;

$C = 10402561$; $C^2 = 108213275358721$

$M = 1104025610001$ is semiprime; $N = 1082132753587210001$ is semiprime;

$C = 45318561$; $C^2 = 2053771971110721$

$M = 453185610001$ is semiprime; $N = 20537719711107210001$ is semiprime.

$C = 84350561$; $C^2 = 7115017141014721$

$M = 843505610001$ is semiprime; $N = 71150171410147210001$ is semiprime.

Note: Probably the formulas could be extrapolated for Carmichael numbers having as the last digits not 561 but another Carmichael number but the results that we obtained, *exempli gratia*, with Carmichael numbers 1729 and 6601 were not encouraging.

III. *On the numbers obtained through concatenation from emirps using only the digits of the number itself*

Observation: We noticed that, through successive concatenation of the digits of a reversible prime with the digits of its reversal, is obtained an interesting sequence of primes.

Primes obtained through concatenation of the digits of the numbers p, q, p, q and p, where p is an emirp and q is its reversal (this formula also conducts to products of very few primes):

1331133113, 9779977997, 769967769967769, 15111151151111511511

Observation: There is an infinity of primes formed this way.

IV. *On the extension of few of these observations from the set of emirps to set of all primes*

Note: We observed three interesting series of primes.

- (1) Primes q of the form n/p, where p is prime and n is formed through concatenation this way: first digits of n are the digits of the square of p and last digits of n are the digits of p itself:

First few such primes:

31, 71, 1301, 1901, 3701, 6101, 6701, 7901, 103001, 109001, 181001 (...).

(the corresponding p: 3, 7, 13, 19, 37, 61, 67, 79, 103, 109, 181)

- (2) Primes formed through successive concatenation of the digits of the prime p with the digits of its reversal, not necessarily prime, q (this formula also conducts to products of very few primes).

First few such primes:

1331133113, 2992299229, 4334433443, 9779977997, 127721127721127 (...).

Note that, from the primes obtained this way, we can also obtain interesting primes from adding numbers of the form $18 \cdot 10^k$.

Few examples:

- : $1331133113 + 18 \cdot 10^8 = 3131133113$ prime(which is the concatenation of q, q, p, q, p);
- : $9779977997 + 18 \cdot 10^7 = 9959977997$ prime;
- : $9779977997 + 18 \cdot 10^9 = 27779977997$ prime;
- : $769967769967769 + 18 \cdot 10^9 = 769985769967769$ prime;
- : $769967769967769 + 18 \cdot 10^{11} = 771767769967769$ prime.

- (3) Primes q formed through concatenation of the digits of the squares of the primes p with the digits 0001.

First few such primes:

90001, 490001, 2890001, 8410001, 18490001, 22090001

(the corresponding p : 3, 7, 17, 29, 43, 47).

18. Four conjectures regarding Fermat pseudoprimes and few known types of pairs of primes

Abstract. There are already known some relations between Fermat pseudoprimes and the pairs of primes $[p, 2^*p - 1]$. We will here show few relations between Fermat pseudoprimes and the pairs of primes of the type $[p, 2^*p - 1]$, $[p, 2^*p + 1]$, $[p, \sqrt{2^*p - 1}]$, respectively $[p, k^*p - k + 1]$.

Introduction

Due to mathematician Farideh Firoozbakht, we have in OEIS few interesting observations about the relation between Fermat pseudoprimes and the pairs of primes $[p, 2^*p - 1]$. We will list only few of them (see the sequences A005935 - A005937):

: if p and $2^*p - 1$ are both primes, and $p > 3$, then $p^*(2^*p - 1)$ is pseudoprime to base 3;

: if p and $2p - 1$ are both primes, then $p^*(2^*p - 1)$ is pseudoprime to base 5 iff p is of the form $10^*k + 1$;

: if p and $2^*p - 1$ are both primes, then $p^*(2^*p - 1)$ is pseudoprime to base 6 iff p is of the form $12^*k + 1$.

Now that the relation between Fermat pseudoprimes and the pairs of primes $[p, 2^*p - 1]$ appears to be clear, we will make four conjectures regarding the relation between Fermat pseudoprimes and the pairs of primes of the type $[p, 2^*p + 1]$, $[p, 2^*p - 1]$, $[p, \sqrt{2^*p - 1}]$, respectively $[p, k^*p - k + 1]$.

Conjecture 1: If p and $2^*p + 1$ are both primes, then the number $n = p^*(2^*p + 1) - 2^*k^*p$ is Fermat pseudoprime to base $p + 1$ for at least one natural value of k .

Verifying the conjecture:

(for the first 8 such pairs of primes)

For $[p, 2^*p + 1] = [3, 7]$ we have, for $k = 1$, $n = 15$, which is, indeed, pseudoprime to base $p + 1 = 4$.

For $[p, 2^*p + 1] = [5, 11]$ we have, for $k = 2$, $n = 35$, which is, indeed, pseudoprime to base $p + 1 = 6$.

For $[p, 2^*p + 1] = [11, 23]$ we have, for $k = 5$, $n = 143$, which is, indeed, pseudoprime to base $p + 1 = 12$.

For $[p, 2^*p + 1] = [23, 47]$ we have, for $k = 6$, $n = 805$, which is, indeed, pseudoprime to base $p + 1 = 24$.

For $[p, 2^*p + 1] = [29, 59]$ we have, for $k = 3$, $n = 1537$, which is, indeed, pseudoprime to base $p + 1 = 30$.

For $[p, 2^*p + 1] = [41, 83]$ we have, for $k = 9$, $n = 2665$, which is, indeed, pseudoprime to base $p + 1 = 42$.

For $[p, 2^*p + 1] = [53, 107]$ we have, for $k = 4$, $n = 5247$, which is, indeed, pseudoprime to base $p + 1 = 54$.

For $[p, 2^*p + 1] = [83, 167]$ we have, for $k = 24$, $n = 9877$, which is, indeed, pseudoprime to base $p + 1 = 84$.

Note: For the list of Sophie Germain primes, see the sequence A005384 in OEIS.

Conjecture 2: If p and $2^*p - 1$ are both primes, $p > 3$, then the number $n = p^*(2^*p - 1) - 2^*k^*p$ is Fermat pseudoprime to base $p - 1$ for at least one natural value of k .

Verifying the conjecture:

(for the first 6 such pairs of primes)

For $[p, 2^*p - 1] = [7, 13]$ we have, for $k = 4$, $n = 21$, which is, indeed, pseudoprime to base $p - 1 = 6$.

For $[p, 2^*p - 1] = [19, 37]$ we have, for $k = 10$, $n = 323$, which is, indeed, pseudoprime to base $p - 1 = 18$.

For $[p, 2^*p - 1] = [31, 61]$ we have, for $k = 5$, $n = 1581$, which is, indeed, pseudoprime to base $p - 1 = 30$.

For $[p, 2^*p - 1] = [37, 73]$ we have, for $k = 2$, $n = 2553$, which is, indeed, pseudoprime to base $p - 1 = 36$.

For $[p, 2^*p - 1] = [79, 157]$ we have, for $k = 7$, $n = 11297$, which is, indeed, pseudoprime to base $p - 1 = 78$.

For $[p, 2^*p - 1] = [97, 193]$ we have, for $k = 8$, $n = 17169$, which is, indeed, pseudoprime to base $p - 1 = 96$.

Note: For the list of primes p for which $2^*p - 1$ is also prime, see the sequence A005382 in OEIS.

Conjecture 3: If p and q are primes, where $q = \sqrt{2^*p - 1}$, then the number p^*q is Fermat pseudoprime to base $p + 1$.

Verifying the conjecture:

(for the first 8 such pairs of primes)

For $[p, q] = [13, 5]$ we have $p^*q = 65$ which is, indeed, pseudoprime to base 14.

For $[p, q] = [61, 11]$ we have $p^*q = 671$ which is, indeed, pseudoprime to base 62.

For $[p, q] = [181, 19]$ we have $p^*q = 3439$ which is, indeed, pseudoprime to base 182.

For $[p, q] = [421, 29]$ we have $p^*q = 12209$ which is, indeed, pseudoprime to base 422.

For $[p, q] = [1741, 59]$ we have $p^*q = 102719$ which is, indeed, pseudoprime to base 1742.

For $[p, q] = [1861, 61]$ we have $p^*q = 113521$ which is, indeed, pseudoprime to base 1862.

For $[p, q] = [2521, 71]$ we have $p^*q = 178991$ which is, indeed, pseudoprime to base 2522.

For $[p, q] = [3121, 79]$ we have $p^*q = 246559$ which is, indeed, pseudoprime to base 3122.

Note: For the list of primes p for which $\sqrt{2^*p - 1}$ is also prime, see the sequence A067756 in OEIS.

Conjecture 4: If p is prime, $p > 3$, and k integer, $k > 1$, then the number $n = p^*(k*p - k + 1)$ is Fermat pseudoprime to base $k*p - k$ and to base $k*p - k + 2$.

Verifying the conjecture:

For the first 4 such pairs of primes, when $p = 5$:

For $[p, 2*p - 1] = [5, 9]$ we have $p^*(2*p - 1) = 45$ which is, indeed, pseudoprime to bases 8 and 10.

For $[p, 3*p - 2] = [5, 13]$ we have $p^*(3*p - 2) = 65$ which is, indeed, pseudoprime to bases 12 and 14.

For $[p, 4*p - 3] = [5, 17]$ we have $p^*(4*p - 3) = 85$ which is, indeed, pseudoprime to bases 16 and 18.

For $[p, 5*p - 4] = [5, 21]$ we have $p^*(5*p - 4) = 105$ which is, indeed, pseudoprime to bases 20 and 22.

For the first 4 such pairs of primes, when $p = 7$:

For $[p, 2*p - 1] = [7, 13]$ we have $p^*(2*p - 1) = 91$ which is, indeed, pseudoprime to bases 12 and 14.

For $[p, 3*p - 2] = [7, 19]$ we have $p^*(3*p - 2) = 133$ which is, indeed, pseudoprime to bases 18 and 20.

For $[p, 4*p - 3] = [7, 25]$ we have $p^*(4*p - 3) = 175$ which is, indeed, pseudoprime to bases 26 and 28.

For $[p, 5*p - 4] = [7, 31]$ we have $p^*(5*p - 4) = 217$ which is, indeed, pseudoprime to bases 30 and 32.

For the next 4 such pairs of primes, when $k = 3$:

For $[p, 3*p - 2] = [11, 31]$ we have $p^*(3*p - 2) = 341$ which is, indeed, pseudoprime to bases 30 and 32.

For $[p, 3*p - 2] = [13, 37]$ we have $p^*(3*p - 2) = 481$ which is, indeed, pseudoprime to bases 36 and 38.

For $[p, 3*p - 2] = [23, 67]$ we have $p^*(3*p - 2) = 1541$ which is, indeed, pseudoprime to bases 66 and 68.

For $[p, 3*p - 2] = [37, 109]$ we have $p^*(3*p - 2) = 4033$ which is, indeed, pseudoprime to bases 108 and 110.

Note: The formula $p^*(k*p - k + 1)$, where p is prime and k integer, seems to appear often related to Fermat pseudoprimes (see the sequence A217835 that I submitted to OEIS).

19. Special properties of the first absolute Fermat pseudoprime, the number 561

Abstract. Though is the first Carmichael number, the number 561 doesn't have the same fame as the third absolute Fermat pseudoprime, the Hardy–Ramanujan number, 1729. I try here to repair this injustice showing few special properties of the number 561.

I will just list (not in the order that I value them, because there is not such an order, I value them all equally as a result of my more or less inspired work, though they may or not “open a path”) the interesting properties that I found regarding the number 561, in relation with other Carmichael numbers, other Fermat pseudoprimes to base 2, with primes or other integers.

1. The number $2*(3 + 1)*(11 + 1)*(17 + 1) + 1$, where 3, 11 and 17 are the prime factors of the number 561, is equal to 1729. On the other side, the number $2*\text{lcm}((7 + 1), (13 + 1), (19 + 1)) + 1$, where 7, 13 and 19 are the prime factors of the number 1729, is equal to 561. We have so a function on the prime factors of 561 from which we obtain 1729 and a function on the prime factors of 1729 from which we obtain 561.

Note: The formula $N = 2(d_1 + 1)*...*(d_n + 1) + 1$, where d_1, d_2, \dots, d_n are the prime divisors of a Carmichael number, leads to interesting results (see the sequence A216646 in OEIS); the formula $M = 2*\text{lcm}((d_1 + 1), \dots, (d_n + 1)) + 1$ also leads to interesting results (see the sequence A216404 in OEIS). But we didn't obtained anymore through one of these two formulas a Carmichael number from another, so this bivalent reation might only exist between the numbers 561 and 1729.*

2. The number 561 can be expressed as $C = a*b + b - a$, where b is prime and a can be any prime factor of the number 1729: $561 = 7*71 + 71 - 7 = 13*41 + 41 - 13 = 19*29 + 29 - 19$ (even more than that, for those that consider that 1 is a prime number, so a prime factor of 1729, $561 = 1*281 + 281 - 1$).

Note: The formula $(a + 1)(b + 1)*(b - a + 1) + 1$ seems to lead to interesting results: for instance, $(19 + 1)*(29 + 1)*(29 - 19 + 1) = 6601$, also a Carmichael number and for the pairs $[a, b] = [7, 71]$ and $[a, b] = [13, 41]$ we obtain through this formula primes, which make us think that this formula deserves further study. Also the triplets $[a, b, a*b + b - a]$, where a, b and $a*b + b - a$ are all three primes might lead to interesting results.*

*Note: I can't, unfortunately, to state that 561 is the first integer that can be written in three (or even four, if we consider that 1 is prime) distinct ways as $a*b + b - a$, where a and b are primes, because there is a smaller number that has this property: $505 = 3*127 + 124 = 11*43 + 32 = 13*37 + 24 = 17*29 + 12$. I yet assert that Carmichael numbers (probably the Fermat pseudoprimes to base 2 also) and the squares of primes can be written in many ways as such.*

3. Another interesting formula inspired by the number 561: we have the expression $(2*3 + 3)*(2*11 + 3)*(2*17 + 3) - 4$, where 3, 11 and 17 are the prime factors of 561, equal to 8321, a Fermat pseudoprime to base 2.

*Note: If we apply this formula to the prime factors of another Carmichael number, $2821 = 7*13*31$, we obtain $32041 = 179^2$, an interesting result.*

4. We consider the triplets of primes of the form $[p, p + 560, p + 1728]$. The first triplet of such primes, $[59, 619, 1787]$, we notice that has the following property: $59 + 619 + 1787 = 2465$, a Carmichael number.

Note: For the next two such triplets, $[83, 643, 1811]$ and $[149, 709, 1877]$ we didn't obtain convincing results.

5. The number 561 is the first term in the sequence of Fermat pseudoprimes to base 2 of the form $3 \cdot (4 \cdot n - 1) \cdot (6 \cdot n - 2)$, where n is integer different from 0.

Note: See the sequence A210993 in OEIS.

6. The number 561 is the first term in the sequence of Fermat pseudoprimes to base 2 of the form $3 \cdot n \cdot (9 \cdot n + 2) \cdot (18 \cdot n - 1)$, where n is an odd number.

Note: See the sequence A213071 in OEIS.

7. The number 561 is the first term in the sequence of Fermat pseudoprimes to base 2 of the form $8 \cdot p \cdot n + p^2$, where p is prime and n is integer (for $n = 0$ we include in this sequence the squares of the only two Wieferich primes known).

Note: See the sequence A218483 in OEIS.

8. The number 561 is the first term in the sequence of Fermat pseudoprimes to base 2 of the form $5 \cdot p^2 - 4 \cdot p$, where p is prime.

Note: See the sequence A213812 in OEIS.

9. The numbers obtained through the method of concatenation from reversible primes and the number 561 are often primes.

Note: We obtain 11 primes from the first 20 reversible primes concatenated with the number 561; these primes are: 37561, 73561, 79561, 97561, 149561, 157561, 311561, 337561, 347561, 359561, 389561.

10. The numbers obtained through the method of concatenation from palindromic primes and the number 561 are often primes.

Note: We obtain 9 primes from the first 20 palindromic primes concatenated with the number 561; these primes are: 101561, 131561, 151561, 191561, 313561, 373561, 727561, 797561, 929561.

11. The numbers obtained through the method of concatenation from the powers of 2 and the number 561 are often primes or products of few primes.

Note: The numbers 4561, 16561, 32561, 256561 are primes.

12. Yet another relation between the numbers 561 and 1729: the numbers obtained through the method of concatenation from the prime factors of 1729 raised to the third power and the number 561 are primes.

Note: These are the numbers: 343561 (where $7^3 = 343$); 2197561 (where $13^3 = 2197$) and 6859561 (where $19^3 = 6859$).

13. The number $(561 \cdot n - 1)/(n - 1)$, where n is integer different from 1, is often integer; more than that, is often prime.

Note: We obtained the following primes (in the brackets is the corresponding value of n): 701(5), 673(6), 641(8), 631(9), 617(11), 601(14), 421(-3), 449(-4), 491(-7), 521(-13) etc. I assert that for a Carmichael number C the number $(C \cdot n - 1)/(n - 1)$, where n is integer different from 1, is often an integer (comparing to other integers beside C). In fact the primes appear so often that I will risk a conjecture.

Conjecture: Any prime number p can be written as $p = (C \cdot q - 1)/(q - 1)$, where C is a Carmichael number and q is a prime.

14. The number 561 is the first term in the sequence of Fermat pseudoprimes to base 2 of the form $(n \cdot 109^2 - n)/360$, where n is integer (561 is obtained for $n = 17$).

Note: Another term of this sequence, obtained for $n = 19897$, is the Carmichael number 656601.

Note: The number 1729 is the first term in the sequence of Fermat pseudoprimes to base 2 of the form $(n \cdot 181^2 - n)/360$, where n is integer (1729 is obtained for $n = 19$). The next terms of the sequence, obtained for $n = 31$, is the Carmichael number 2821.

Note: Because the numbers 561 and 1729 have both three prime factors, the sequences from above can be eventually translated into the property of the numbers of the form $360 \cdot (a \cdot b) + 1$, where a and b are primes, to generate squares of primes. Corresponding to the sequences above, for $[a, b] = [3, 11]$ we obtain 109^2 and for $[a, b] = [7, 13]$ we obtain 181^2 .

Conjecture: If the number $360 \cdot (a \cdot b) + 1$, where a and b are primes, is equal to c^2 , where c is prime, then exists an infinite series of Carmichael numbers of the form $a \cdot b \cdot d$, where d is a natural number (obviously odd, but not necessarily prime).

Note: The numbers of the form $360 \cdot (a \cdot b) + c$, where a, b and c are primes, seems to have also the property to generate primes. Indeed, if we take for instance $[a, c] = [3, 7]$, we obtain primes for $b = 5, 11, 17, 23, 29, 31, 43, 47, 59, 67$ etc. (note the chain of 5 consecutive primes of the form $6 \cdot k - 1$).

15. The number 561 is the first term of the sequence of Carmichael numbers that can be written as $2^m + n^2$, where m and n are integers (561 is obtained for $m = 5$ and $n = 23$).

Note: The next few terms of this sequence are: $1105 = \sqrt{2^4 + 33^2}$, $2465 = \sqrt{2^6 + 49^2}$ etc.

16. Some Carmichael numbers are also Harshad numbers but the most of them aren't. The number 561 has yet another interesting related property; if we note with $s(n)$ the iterated sum of

the digits of a number n that not goes until the digital root but stops to the last odd prime obtained before this, than 561 is divisible by $s((561 + 1)/2)$ equivalent to $s(281)$ equivalent to 11. Also other Carmichael numbers have this property: 1105 is divisible by $s(1105) = 13$ and 6601 is divisible by $s(6601) = 7$.

17. For the randomly chosen, but consecutive, 7 primes (129689, 1299709, 1299721, 1299743, 1299763, 1299791 and 1299811) we obtained 3 primes and 3 semiprimes when introduced them in the formula $2*561 + p^2 - 360$.

18. Another relation between 561 and Hardy Ramanujan number: $(62745 + 24) \bmod 1728 = 561$ (where 24 is, e.g., the sum of the digits of the Carmichael number 62745 or a constant and 1728 is, obviously, one less than Hardy–Ramanujan number).

19. Yet another relation between 561 and Hardy Ramanujan number: $561 \bmod 73 = 1729 \bmod 73 = 50$. The formula $73*n + 50$, from which we obtain 561 and 1729 for $n = 7$ and $n = 23$, leads to other interesting results for n of the form $7 + 16*k$: we obtain primes for $n = 39, 71, 119, 167$ etc.

20. A formula that generating primes: $561^2 - 561 - 1 = 314159$ is prime; $561^4 - 561^3 - 561^2 - 561 - 1 = 98872434077$ is prime. Also for other Carmichael number the formula $C^2 - C - 1$ conducts to: $1105^2 - 1105 - 1 = 1219919$ prime, $6601^2 - 6601 - 1 = 43566599$ prime (semiprimes were obtained for the numbers 1729, 2465, 2821 and so on). Yet the number $2465^4 - 2465^3 - 1 = 36905532355999$ is prime and the number $15841^4 - 15841^3 - 1 = 62965543898204639$ is prime.

21. The formula $N = d_1^2 + d_2^2 + d_3^2 - 560$, where d_1, d_2 and d_3 are the only prime factors of a Carmichael number, and they are all three of the form $6*k + 1$, seems to generate an interesting class of primes:

- : for $C = 1729 = 7*13*19$ we have $N = 19$ prime;
- : for $C = 2821 = 7*13*31$ we have $N = 619$ prime;
- : for $C = 8911 = 7*19*67$ we have $N = 4339$ prime;
- : for $C = 15841 = 7*31*73$ we have $N = 5779$ prime.

22. The number 544, obtained as the difference between the first two Carmichael numbers, 1105 and 561, has also a notable property: the relation $n^C \bmod 544 = n$ seems to be verified for a lot of natural numbers n and a lot of Carmichael numbers C , especially when C is also an Euler pseudoprime.

Conjecture: *The expression $n^E \bmod 544 = n$, where n is any natural number, is true if E is an Euler pseudoprime.*

23. The difference between the squares of the first two Carmichael numbers, 1105 and 561, has also the notable property that results in a square of an integer: $952^2 = 1105^2 - 561^2$.

Conclusion: I am aware of the excessive use of the word “interesting” in this article, but this was the purpose of it: to show how many “interesting” paths can be opened just studying the number 561, not to follow until the last consequences one of these paths. I didn’t succeed to show that the properties of the number 561 eclipses the ones of the number 1729 (very present in this article) but hopefully I succeeded to show that they are both a pair of extraordinary numbers (and that the number 561 deserves his place on the license plate of a taxi-cab).

20. Six conjectures and the generic formulas for two subsets of Poulet numbers

Abstract. I was following an interesting “track”, *i.e.* the pairs of primes $[p, q]$ that apparently can form strictly Carmichael numbers of the form $p \cdot q \cdot (n \cdot (q - 1) + p)$, like for instance $[23, 67]$ and $[41, 241]$, when I observed that also all the Poulet numbers P which have the numbers $p = 30 \cdot k + 23$ and $q = 90 \cdot k + 67$ respectively $p = 30 \cdot k + 11$ and $q = 180 \cdot k + 61$ as prime factors can be written as $P = p \cdot q \cdot (n \cdot (q - 1) + p)$ and I made few conjectures.

I. The generic formula for Poulet numbers which have two prime factors of the form $30 \cdot k + 23$ and $90 \cdot k + 67$

Conjecture 1:

Any Poulet numbers P which have the numbers $p = 23$ and $q = 67$ as prime factors can be written as $P = p \cdot q \cdot (n \cdot (q - 1) + p) = 3 \cdot p^3 \cdot (3 \cdot n + 1) - p^2 \cdot (15 \cdot n + 2) + 6 \cdot p \cdot n$, where n non-null positive integer (we took $q = 3 \cdot p - 2$).

Verifying the conjecture (for the first few such Poulet numbers):

- : For $n = 1$ we have the Poulet number $P = 137149 = 23 \cdot 67 \cdot 89$;
- : For $n = 3$ we have the Poulet number (also Carmichael number) $P = 340561 = 13 \cdot 17 \cdot 23 \cdot 67$;
- : For $n = 9$ we have the Poulet number $P = 950797 = 23 \cdot 67 \cdot 617$;
- : For $n = 10$ we have the Poulet number $P = 1052503 = 23 \cdot 67 \cdot 683$;
- : For $n = 13$ we have the Poulet number $P = 1357621 = 23 \cdot 67 \cdot 881$.

Comment:

This formula is important for determining sequences of Poulet numbers; in their case there is not an instrument for obtaining such formulas as there is the Korselt’s criterion in the case of Carmichael numbers. See also the sequence A182515 that I submitted to OEIS.

Note:

The formula $P = p \cdot q \cdot (n \cdot (q - 1) + p)$ is not a pattern for any Poulet numbers which have two prime factors of the form p and $q = 3 \cdot p - 2$; for instance, for $[p, q] = [7, 19]$ and Carmichael numbers 1729 and 63973 the formula doesn’t apply.

Conjecture 2:

Any Poulet numbers P which have the numbers $p = 30 \cdot k + 23$ and $q = 90 \cdot k + 67$, where k non-negative integer, as prime factors can be written as $P = 3 \cdot p^3 \cdot (3 \cdot n + 1) - p^2 \cdot (15 \cdot n + 2) + 6 \cdot p \cdot n$, where n non-null positive integer.

Note:

As it can be seen, the formula from above it is not anymore derived from and equivalent to the formula $P = p \cdot q \cdot (n \cdot (q - 1) + p)$, equivalence that exists only in the case of the Conjecture 1.

Verifying the conjecture for $p = 53$ and $q = 157$ (for the first few such Poulet numbers):

- : For $n = 3$ we have the Poulet number (also Carmichael number) $P = 4335241 = 53 \cdot 157 \cdot 521$;
- : For $n = 10$ we have the Poulet number $P = 13421773 = 53 \cdot 157 \cdot 1613$;
- : For $n = 13$ we have the Poulet number (also Carmichael number) $P = 17316001 = 53 \cdot 157 \cdot 2081$.

Verifying the conjecture for $p = 113$ and $q = 337$ (for the first few such Poulet numbers):

- : For $n = 1$ we have the Poulet number (also Carmichael number) $P = 17098369 = 113 \cdot 337 \cdot 449$;
- : For $n = 7$ we have the Poulet number (also Carmichael number) $P = 93869665 = 5 \cdot 17 \cdot 29 \cdot 113 \cdot 337$;
- : For $n = 13$ we have the Poulet number $P = 170640961 = 113 \cdot 337 \cdot 4481$.

Note:

It is notable how easily we found Poulet numbers with this formula, for at least three values of n from $n = 1$ to $n = 13$, for any of the three pairs of primes considered: $[23, 67]$, $[53, 157]$, $[113, 337]$.

Conjecture 3:

There is an infinity of Poulet numbers which have the numbers $p = 30 \cdot k + 23$ and $q = 90 \cdot k + 67$, where k non-negative integer, as prime factors (implicitly there is an infinity of pairs of primes of the form $[30 \cdot k + 23, 90 \cdot k + 67]$).

II. The generic formula for Poulet numbers which have two prime factors of the form $30 \cdot k + 11$ and $180 \cdot k + 61$

Conjecture 4:

Any Poulet numbers P which have the numbers $p = 11$ and $q = 61$ as prime factors can be written as $P = p \cdot q \cdot (n \cdot (q - 1) + p) = 6 \cdot p^3 \cdot (6 \cdot n + 1) - p^2 \cdot (66 \cdot n + 5) + 30 \cdot p \cdot n$, where n non-null positive integer (we took $q = 6 \cdot p - 5$).

Verifying the conjecture (for the first such Poulet number):

- : For $n = 21$ we have the Poulet number (also Carmichael number) $P = 852841 = 11 \cdot 31 \cdot 41 \cdot 61$.

Note:

The formula $P = p \cdot q \cdot (n \cdot (q - 1) + p)$ is not a pattern for any Poulet numbers which have two prime factors of the form p and $q = 6 \cdot p - 5$; for instance, for $[p, q] = [7, 37]$ and Carmichael number $63973 = 7 \cdot 13 \cdot 19 \cdot 37$ the formula doesn't apply.

Conjecture 5:

Any Poulet numbers P which have the numbers $p = 30 \cdot k + 11$ and $q = 180 \cdot k + 61$, where k non-negative integer, as prime factors can be written as $P = 6 \cdot p^3 \cdot (6 \cdot n + 1) - p^2 \cdot (66 \cdot n + 5) + 30 \cdot p \cdot n$, where n non-null positive integer.

Verifying the conjecture for $p = 41$ and $q = 241$ (for the first few such Poulet numbers):

- : For $n = 2$ we have the Poulet number (also Carmichael number) $P = 5148001 = 41 \cdot 241 \cdot 521$;
- : For $n = 3$ we have the Poulet number (also Carmichael number) $P = 7519441 = 41 \cdot 241 \cdot 761$;
- : For $n = 4$ we have the Poulet number (also Carmichael number) $P = 9890881 = 7 \cdot 11 \cdot 13 \cdot 41 \cdot 241$;
- : For $n = 5$ we have the Poulet number (also Carmichael number) $P = 12262321 = 17 \cdot 41 \cdot 73 \cdot 241$.

Conjecture 6:

There is an infinity of Poulet numbers which have the numbers $p = 30 \cdot k + 11$ and $q = 180 \cdot k + 61$, where k non-negative integer, as prime factors (implicitly there is an infinity of pairs of primes of the form $[30 \cdot k + 11, 180 \cdot k + 61]$).

21. A pattern that relates Carmichael numbers to the number 66

Abstract. The length of the period of the rational number which is the sum, from $n = 1$ to $n = \infty$, of the numbers $1/(C_n - 1)$, where $\{C_1, C_2, \dots, C_n\}$ is the ordered set of Carmichael numbers, i.e. $\{561, 1105, 1729, 2465, \dots\}$, seems to be always multiple of 66. This property doesn't apply always when C_1, C_2, \dots, C_n are not consecutive, so this pattern could be a way to determinate if between two known Carmichael numbers there exist other unknown Carmichael numbers.

Conjecture:

The length of the period of the rational number which is the sum, from $n = 1$ to $n = \infty$, of the numbers $1/(C_n - 1)$, where $\{C_1, C_2, \dots, C_n\}$ is the ordered set of Carmichael numbers, is always multiple of 66.

Verifying the conjecture (for $n \leq 12$):

- : the sum $1/560 + 1/1104$ is equal to a rational number with the length of the period 66;
- : the sum $1/560 + 1/1104 + 1/1728$ is equal to a rational number with the length of the period 66;
- : the sum $1/560 + 1/1104 + 1/1728 + 1/2464$ is equal to a rational number with the length of the period 66;
- : the sum $1/560 + \dots + 1/2464 + 1/2820$ is equal to a rational number with the length of the period $1518 = 66 \cdot 23$;
- : the sum $1/560 + \dots + 1/2820 + 1/6600$ is equal to a rational number with the length of the period $1518 = 66 \cdot 23$;
- : the sum $1/560 + \dots + 1/6600 + 1/8910$ is equal to a rational number with the length of the period $4554 = 66 \cdot 69$;

- : the sum $1/560 + \dots + 1/8910 + 1/10584$ is equal to a rational number with the length of the period $31878 = 66 \cdot 483$;
- : the sum $1/560 + \dots + 1/10584 + 1/15840$ is equal to a rational number with the length of the period $31878 = 66 \cdot 483$;
- : the sum $1/560 + \dots + 1/15840 + 1/29340$ is equal to a rational number with the length of the period $286902 = 66 \cdot 4347$;
- : the sum $1/560 + \dots + 1/29340 + 1/41040$ is equal to a rational number with the length of the period $286902 = 66 \cdot 4347$;
- : the sum $1/560 + \dots + 1/41040 + 1/46656$ is equal to a rational number with the length of the period $286902 = 66 \cdot 4347$.

Note:

This is a characteristic only of absolute Fermat pseudoprimes; in the case of relative Fermat pseudoprimes, Poulet numbers for instance, this pattern doesn't apply: for the first two Poulet numbers, 341 and 561, we have the sum $1/340 + 1/560$ equal to a rational number with the length of the period 48.

Note:

This is a characteristic only of the sum of ordered Carmichael numbers, for instance, for the first and the third Carmichael numbers, 561 and 1729, we have the sum $1/560 + 1/1728$ equal to a rational number with the length of the period 6.

Comment:

This property could be a way to determine if between two known Carmichael numbers there exist other unknown Carmichael numbers.

22. A generic formula of 2-Poulet numbers and also a method to obtain sequences of n-Poulet numbers

Abstract. In this paper I present a formula based on 2-Poulet numbers which seems to conduct always to a prime, a square of prime or a semiprime, a conjecture that this formula is generic for 2-Poulet numbers, and, in case that the conjecture doesn't hold, I present another utility for this formula, namely to generate sequences of n-Poulet numbers.

Conjecture:

Any 2-Poulet number P can be written at least in one way as $P = (q \cdot 2^a \cdot 3^b \cdot 5^c \pm 1) \cdot 2^n + 1$, where q is a prime, a square of prime or a semiprime, a, b, c are non-negative integers and n is non-null positive integer.

In other words, there always exist a number $q = ((P - 1)/2^n \pm 1)/(2^a * 3^b * 5^c)$, where P is a 2-Poulet number, a, b, c are non-negative integers and n is non-null positive integer, such that q is a prime, a square of prime or a semiprime.

Note: In this paper I will consider the number 1 to be a prime (not to repeat the formulation: q is prime, square of prime, semiprime or is equal to number 1).

Verifying the conjecture:

[for the first ten 2-Poulet numbers, only for a restrictive version of the conjecture, considering just the formula $P = (q * 2^a * 3^b * 5^c + 1) * 2^n + 1$]

For $P = 341$, we have:

$$q = ((341 - 1)/2^1 - 1)/(2^0 * 3^0 * 5^0) = 13^2;$$

$$q = ((341 - 1)/2^2 - 1)/(2^2 * 3^1 * 5^0) = 7.$$

For $P = 1387$, we have:

$$q = ((1387 - 1)/2^1 - 1)/(2^2 * 3^0 * 5^0) = 173.$$

For $P = 2047$, we have:

$$q = ((2047 - 1)/2^1 - 1)/(2^1 * 3^0 * 5^0) = 7 * 73.$$

For $P = 2701$, we have:

$$q = ((2701 - 1)/2^1 - 1)/(2^0 * 3^0 * 5^0) = 19 * 71;$$

$$q = ((2701 - 1)/2^2 - 1)/(2^1 * 3^0 * 5^0) = 337.$$

For $P = 3277$, we have:

$$q = ((3277 - 1)/2^1 - 1)/(2^0 * 3^0 * 5^0) = 1637;$$

$$q = ((3277 - 1)/2^2 - 1)/(2^1 * 3^0 * 5^0) = 409.$$

For $P = 4033$, we have:

$$q = ((4033 - 1)/2^1 - 1)/(2^0 * 3^0 * 5^1) = 13 * 31;$$

$$q = ((4033 - 1)/2^2 - 1)/(2^0 * 3^0 * 5^0) = 19 * 53;$$

$$q = ((4033 - 1)/2^3 - 1)/(2^0 * 3^0 * 5^0) = 503;$$

$$q = ((4033 - 1)/2^4 - 1)/(2^0 * 3^0 * 5^0) = 251;$$

$$q = ((4033 - 1)/2^5 - 1)/(2^0 * 3^0 * 5^3) = 1;$$

$$q = ((4033 - 1)/2^6 - 1)/(2^1 * 3^0 * 5^0) = 31.$$

For $P = 4369$, we have:

$$q = ((4369 - 1)/2^1 - 1)/(2^0 * 3^0 * 5^0) = 37 * 59;$$

$$q = ((4369 - 1)/2^2 - 1)/(2^0 * 3^0 * 5^0) = 1091;$$

$$q = ((4369 - 1)/2^3 - 1)/(2^0 * 3^0 * 5^1) = 109;$$

$$((4369 - 1)/2^4 - 1)/(2^0 * 3^0 * 5^0) = 251;$$

For $P = 4681$, we have:

$$q = ((4681 - 1)/2^1 - 1)/(2^0 * 3^0 * 5^0) = 2339;$$

$$q = ((4681 - 1)/2^2 - 1)/(2^0 * 3^0 * 5^0) = 7 * 167;$$

$$q = ((4681 - 1)/2^3 - 1)/(2^3 * 3^0 * 5^0) = 73;$$

For $P = 5461$, we have:

$$q = ((2701 - 1)/2^1 - 1)/(2^0 * 3^0 * 5^0) = 2729;$$

$$q = ((2701 - 1)/2^2 - 1)/(2^2 * 3^0 * 5^0) = 11 * 31.$$

For $P = 7957$, we have:

$$q = ((7957 - 1)/2^1 - 1)/(2^0 * 3^0 * 5^1) = 37 * 43;$$

$$q = ((7957 - 1)/2^2 - 1)/(2^2 * 3^0 * 5^0) = 7 * 71.$$

Verifying the conjecture:

(for seven greater consecutive 2-Poulet numbers)

For $P = 27657600833$, we have:

$$\begin{aligned}q &= ((27657600833 - 1)/2^4 - 1)/(2^0*3^1*5^0) = 653*882389; \\q &= ((27657600833 - 1)/2^1 + 1)/(2^0*3^1*5^0) = 22433*205483; \\q &= ((27657600833 - 1)/2^2 + 1)/(2^0*3^0*5^0) = 6914400209. \\q &= ((27657600833 - 1)/2^4 + 1)/(2^0*3^0*5^0) = 6911*250123. \\q &= ((27657600833 - 1)/2^6 + 1)/(2^1*3^0*5^0) = 8093*26699.\end{aligned}$$

For $P = 27667059281$, we have:

$$\begin{aligned}q &= ((27667059281 - 1)/2^1 - 1)/(2^0*3^0*5^0) = 103*134306113; \\q &= ((27667059281 - 1)/2^4 + 1)/(2^1*3^0*5^0) = 864595603.\end{aligned}$$

For $P = 27675991081$, we have:

$$\begin{aligned}q &= ((27675991081 - 1)/2^1 - 1)/(2^0*3^0*5^0) = 10169*680401; \\q &= ((27675991081 - 1)/2^2 - 1)/(2^2*3^0*5^0) = 1109*779869.\end{aligned}$$

For $P = 27681232903$, we have:

$$\begin{aligned}q &= ((27681232903 - 1)/2^1 - 1)/(2^1*3^0*5^2) = 276812329; \\q &= ((27681232903 - 1)/2^2 + 1)/(2^4*3^0*5^0) = 67*807083.\end{aligned}$$

For $P = 27685810639$, we have:

$$q = ((27685810639 - 1)/2^1 + 1)/(2^3*3^0*5^1) = 4740721.$$

For $P = 27686175193$, we have:

$$\begin{aligned}q &= ((27686175193 - 1)/2^1 - 1)/(2^0*3^0*5^1) = 20208887; \\q &= ((27686175193 - 1)/2^1 + 1)/(2^0*3^0*5^0) = 2837*4879481; \\q &= ((27686175193 - 1)/2^3 + 1)/(2^2*3^0*5^2) = 113*306263.\end{aligned}$$

For $P = 27702689701$, we have:

$$\begin{aligned}q &= ((27702689701 - 1)/2^2 - 1)/(2^3*3^0*5^0) = 11*78700823; \\q &= ((27702689701 - 1)/2^1 + 1)/(2^0*3^0*5^0) = 15971*867281; \\q &= ((27702689701 - 1)/2^2 + 1)/(2^1*3^0*5^0) = 199*17401187.\end{aligned}$$

Comment:

If the Conjecture doesn't hold, it may be considered a more premissive version: Any 2-Poulet number P can be written at least in one way as $P = (q*2^a*3^b*5^c \pm 1)*2^n + 1$, where q is a prime, a square of prime or a semiprime and a, b, c, n are non-negative integers.

In this case we have, for instance for $P = 27686175193$, $q = ((27686175193 - 1)/2^0 - 1)/(2^0*3^0*5^0) = 27686175191$ which is prime.

Comment:

If the Conjecture doesn't hold, it has anyhow at least one utility: it's a method for finding sequences of Poulet numbers (not only 2-Poulet numbers).

Taking, for instance, $q = 223*r$, where r is prime, we have the sequence of Poulet numbers P defined as $P = (223*r + 1)*2^n + 1$, with the first three terms $\{41041, 10261, 52633\}$, obtained for the following values of (r, n) : $\{23, 1), (23, 3), (59, 2)\}$.

Taking, for instance, $q = 29*r$, where r is prime, we have the sequence of Poulet numbers P defined as $P = (29*r + 1)*3^n + 1$, with the first term 2701, obtained for the following value of (r, n) : $(31, 1)$.

Taking, for instance, $q = 37*r$, where r is prime, we have the sequence of Poulet numbers P defined as $P = (37*r + 1)*5^n + 1$, with the first term 561, obtained for the following value of (r, n) : $(3, 5)$.

Taking, for instance, $q = 13^2$, we have the sequence of Poulet numbers P defined as $P = (13^2 + 1)*2^n + 1$, with the first term 341, obtained for the following value of n : 1.

23. Few interesting results regarding Poulet numbers and Egyptian fraction expansion

Abstract. Considering r being equal to the positive rational number $1/(d_1 - 1) + 1/(d_2 - 1) + \dots + 1/(d_n - 1)$, where d_1, \dots, d_n are the prime factors of a Poulet number, the Egyptian fraction expansion applied to r leads to interesting results.

Note:

An Egyptian fraction is a sum of distinct unit fractions, such as $1/a + 1/b + 1/c + \dots + 1/m$, where the denominators a, b, c, \dots, m are positive, distinct, integers. Every positive rational number can be represented by an Egyptian fraction.

The Egyptian fraction expansion is an algorithm due to Fibonacci for computing Egyptian fractions: the number x/y , where x, y are positive, distinct, integers, is written as follows:

$x/y = 1/\text{ceiling}(y/x) + ((-1) \bmod x)/y * \text{ceiling}(y/x)$, where the function $\text{ceiling}(z)$ represents the smaller integer equal to or greater than z .

This algorithm is repeated to the second term of the summation above and so on until is obtained an Egyptian fraction.

Conjecture 1:

If r is equal to the positive rational number $1/(d_1 - 1) + 1/(d_2 - 1) + \dots + 1/(d_n - 1)$, where d_1, \dots, d_n are the prime factors of a Poulet number P , and m is equal to the last denominator obtained applying the Egyptian fraction expansion to r , then the number $m + 1$ is a prime or a power of prime for an infinity of Poulet numbers.

Examples:

- : For $P = 341 = 11 * 31$, we have $r = 1/10 + 1/30 = 2/15 = 1/8 + 1/120$; the number $m + 1 = 120 + 1 = 121 = 11^2$, a square of prime.
- : For $P = 561 = 3 * 11 * 17$, we have $r = 1/2 + 1/10 + 1/16 = 53/80 = 1/2 + 1/7 + 1/51 + 1/28560$; the number $m + 1 = 28560 + 1 = 28561 = 13^4$, a power of prime.
- : For $P = 645 = 3 * 5 * 43$, we have $r = 1/2 + 1/4 + 1/42 = 65/84 = 1/2 + 1/4 + 1/42$; the number $m + 1 = 42 + 1 = 43$, a prime number.
- : For $P = 1105 = 5 * 13 * 17$, we have $r = 1/4 + 1/12 + 1/16 = 19/48 = 1/3 + 1/16$; the number $m + 1 = 16 + 1 = 17$, a prime number.
- : For $P = 1387 = 19 * 73$, we have $r = 1/18 + 1/72 = 5/72 = 1/15 + 1/360$; the number $m + 1 = 360 + 1 = 361 = 19^2$, a square of prime.
- : For $P = 1729 = 7 * 13 * 19$, we have $r = 1/6 + 1/12 + 1/18 = 11/36 = 1/4 + 1/18$; the number $m + 1 = 18 + 1 = 19$, a prime number.
- : For $P = 1905 = 3 * 5 * 127$, we have $r = 1/2 + 1/4 + 1/126 = 191/252 = 1/2 + 1/4 + 1/126$; the number $m + 1 = 126 + 1 = 127$, a prime number.
- : For $P = 6601 = 7 * 23 * 41$, we have $r = 1/6 + 1/22 + 1/40 = 313/1320 = 1/5 + 1/27 + 1/11880$; the number $m + 1 = 11880 + 1 = 11881 = 109^2$, a square of prime.

- : For $P = 8911 = 7 \cdot 19 \cdot 67$, we have $r = 1/6 + 1/18 + 1/66 = 47/198 = 1/5 + 1/27 + 1/2970$; the number $m + 1 = 2970 + 1 = 2971$, a prime number.
- : For $P = 52633 = 7 \cdot 73 \cdot 103$, we have $r = 1/6 + 1/72 + 1/102 = 233/1224 = 1/6 + 1/43 + 1/2289 + 1/8031644 + 1/80634123646776$; the number $m + 1 = 80634123646776 + 1 = 80634123646777$, a prime number.

Note:

For the first ten Carmichael numbers C divisible by 7 and 19 (we don't have a comprehensive list of Poulet numbers indexed together with their prime factors) we always obtain for the number $m + 1$ a prime or a square of prime; we have the following values for $(C, m + 1)$: (1729, 19), (8911, 2971), (63973, 2^2), (126217, 19^2), (188461, 433), (748657, 433), (825265, 1009), (997633, 577), (1050985, 23), (1773289, 1321).

Conjecture 2:

If r is equal to the positive rational number $1/(d_1 - 1) + 1/(d_2 - 1) + \dots + 1/(d_n - 1)$, where d_1, \dots, d_n are the prime factors of a Poulet number P , and r is represented by the irreducible fraction x/y , where x, y positive integers, then the number $y + 1$ is a prime or a power of prime for an infinity of Poulet numbers.

Examples:

(as it can be seen above)

- : For $P = 341$, we have $r = x/y = 2/15$; the number $y + 1 = 15 + 1 = 16 = 2^4$, a power of prime.
- : For $P = 561$, we have $r = x/y = 53/80$; the number $y + 1 = 80 + 1 = 81 = 3^4$, a power of prime.
- : For $P = 1105$, we have $r = x/y = 19/48$; the number $y + 1 = 48 + 1 = 49$, a square of prime.
- : For $P = 1387$, we have $r = x/y = 5/72$; the number $y + 1 = 72 + 1 = 73$, a prime number.
- : For $P = 1729$, we have $r = x/y = 11/36$; the number $y + 1 = 36 + 1 = 37$, a prime number.
- : For $P = 6601$, we have $r = x/y = 313/1320$; the number $y + 1 = 1320 + 1 = 1321$, a prime number.
- : For $P = 8911$, we have $r = x/y = 47/198$; the number $y + 1 = 198 + 1 = 199$, a prime number.

Note:

As it can be seen above, the number y is sometimes equal to $\text{lcm}((d_1 - 1), (d_2 - 1), \dots, (d_n - 1))$, which is, for instance, the case of the Poulet number $1387 = 19 \cdot 73$, where $y = 72 = \text{lcm}(18, 72)$, but this is not always true: this is, for instance, the case of Poulet number 341, where $y = 15$ and $\text{lcm}(10, 30) = 30$.

Conjecture 3:

If d_1, \dots, d_n are the prime factors of a Poulet number P , then the number $\text{lcm}((d_1 - 1), (d_2 - 1), \dots, (d_n - 1))$ is a prime or a power of prime for an infinity of Poulet numbers.

24. The Smarandache-Coman divisors of order k of a composite integer n with m prime factors

Abstract. We will define in this paper the Smarandache-Coman divisors of order k of a composite integer n with m prime factors, a notion that seems to have promising applications, at a first glance at least in the study of absolute and relative Fermat pseudoprimes, Carmichael numbers and Poulet numbers.

Definition 1:

We call *the set of Smarandache-Coman divisors of order 1 of a composite positive integer n with m prime factors*, $n = d_1 * d_2 * \dots * d_m$, where the least prime factor of n , d_1 , is greater than or equal to 2, the set of numbers defined in the following way:

$SCD_1(n) = \{S(d_1 - 1), S(d_2 - 1), \dots, S(d_m - 1)\}$, where S is the Smarandache function.

Examples:

1. The set of SC divisors of order 1 of the number 6 is $\{S(2 - 1), S(3 - 1)\} = \{S(1), S(2)\} = \{1, 2\}$, because $6 = 2 * 3$;
2. $SCD_1(429) = \{S(3 - 1), S(11 - 1), S(13 - 1)\} = \{S(2), S(10), S(12)\} = \{2, 5, 4\}$, because $429 = 3 * 11 * 13$.

Definition 2:

We call *the set of Smarandache-Coman divisors of order 2 of a composite positive integer n with m prime factors*, $n = d_1 * d_2 * \dots * d_m$, where the least prime factor of n , d_1 , is greater than or equal to 3, the set of numbers defined in the following way:

$SCD_2(n) = \{S(d_1 - 2), S(d_2 - 2), \dots, S(d_m - 2)\}$, where S is the Smarandache function.

Examples:

1. The set of SC divisors of order 2 of the number 21 is $\{S(3 - 2), S(7 - 2)\} = \{S(1), S(5)\} = \{1, 5\}$, because $21 = 3 * 7$;
2. $SCD_2(2429) = \{S(7 - 2), S(347 - 2)\} = \{S(5), S(345)\} = \{5, 23\}$, because $2429 = 7 * 347$.

Definition 3:

We call *the set of Smarandache-Coman divisors of order k of a composite positive integer n with m prime factors*, $n = d_1 * d_2 * \dots * d_m$, where the least prime factor of n , d_1 , is greater than or equal to $k + 1$, the set of numbers defined in the following way:

$SCD_k(n) = \{S(d_1 - k), S(d_2 - k), \dots, S(d_m - k)\}$, where S is the Smarandache function.

Examples:

1. The set of SC divisors of order 5 of the number 539 is $\{S(7 - 5), S(11 - 5)\} = \{S(2), S(6)\} = \{2, 3\}$, because $539 = 7^2 * 11$;
2. $SCD_6(221) = \{S(13 - 6), S(17 - 6)\} = \{S(7), S(11)\} = \{7, 11\}$, because $221 = 13 * 17$.

Comment:

We obviously defined the sets of numbers above because we believe that they can have interesting applications, in fact we believe that they can even make us re-think and re-consider the Smarandache function as an instrument to operate in the world of number

theory: while at the beginning its value was considered to consist essentially in that to be a criterion for primality, afterwards the Smarandache function crossed a normal process of substantiation, so it was constrained to evolve in a relatively closed (even large) circle of equalities, inequalities, conjectures and theorems concerning, most of them, more or less related concepts. We strongly believe that some of the most important applications of the Smarandache function are still undiscovered. We were inspired in defining the Smarandache-Coman divisors by the passion for Fermat pseudoprimes, especially for Carmichael numbers and Poulet numbers, by the Korselt's criterion, one of the very few (and the most important from them) instruments that allow us to comprehend Carmichael numbers, and by the encouraging results we easily obtained, even from the first attempts to relate these two types of numbers, Fermat pseudoprimes and Smarandache numbers.

Smarandache-Coman divisors of order 1 of the 2-Poulet numbers:

(See the sequence A214305 in OEIS, posted by us, for a list with Poulet numbers with two prime factors)

$$\begin{aligned}
 \text{SCD}_1(341) &= \{S(11 - 1), S(31 - 1)\} = \{S(10), S(30)\} = \{5, 5\}; \\
 \text{SCD}_1(1387) &= \{S(19 - 1), S(73 - 1)\} = \{S(18), S(72)\} = \{6, 6\}; \\
 \text{SCD}_1(2047) &= \{S(23 - 1), S(89 - 1)\} = \{S(22), S(88)\} = \{11, 11\}; \\
 \text{SCD}_1(2701) &= \{S(37 - 1), S(73 - 1)\} = \{S(36), S(72)\} = \{6, 6\}; \\
 \text{SCD}_1(3277) &= \{S(29 - 1), S(113 - 1)\} = \{S(28), S(112)\} = \{7, 7\}; \\
 \text{SCD}_1(4033) &= \{S(37 - 1), S(109 - 1)\} = \{S(36), S(108)\} = \{6, 9\}; \\
 \text{SCD}_1(4369) &= \{S(17 - 1), S(257 - 1)\} = \{S(16), S(256)\} = \{6, 10\}; \\
 \text{SCD}_1(4681) &= \{S(31 - 1), S(151 - 1)\} = \{S(30), S(150)\} = \{5, 10\}; \\
 \text{SCD}_1(5461) &= \{S(43 - 1), S(127 - 1)\} = \{S(42), S(126)\} = \{7, 7\}; \\
 \text{SCD}_1(7957) &= \{S(73 - 1), S(109 - 1)\} = \{S(72), S(108)\} = \{6, 9\}; \\
 \text{SCD}_1(8321) &= \{S(53 - 1), S(157 - 1)\} = \{S(52), S(156)\} = \{13, 13\}.
 \end{aligned}$$

Comment:

It is notable how easily are obtained interesting results: from the first 11 terms of the 2-Poulet numbers sequence checked there are already foreseen few patterns.

Open problems:

1. Is for the majority of the 2-Poulet numbers the case that the two Smarandache-Coman divisors of order 1 are equal, as for the seven from the eleven numbers checked above?
2. Is there an infinity of 2-Poulet numbers for which the set of SCD of order 1 is equal to $\{6, 6\}$, the case of Poulet numbers 1387 and 2701, or with $\{6, 9\}$, the case of Poulet numbers 4033 and 7957?

Smarandache-Coman divisors of order 2 of the 2-Poulet numbers:

$$\begin{aligned}
 \text{SCD}_2(341) &= \{S(11 - 2), S(31 - 2)\} = \{S(9), S(29)\} = \{6, 29\}; \\
 \text{SCD}_2(1387) &= \{S(19 - 2), S(73 - 2)\} = \{S(17), S(71)\} = \{17, 71\}; \\
 \text{SCD}_2(2047) &= \{S(23 - 2), S(89 - 2)\} = \{S(21), S(87)\} = \{7, 29\}; \\
 \text{SCD}_2(2701) &= \{S(37 - 2), S(73 - 2)\} = \{S(35), S(71)\} = \{7, 71\}; \\
 \text{SCD}_2(3277) &= \{S(29 - 2), S(113 - 2)\} = \{S(27), S(111)\} = \{9, 37\}; \\
 \text{SCD}_2(4033) &= \{S(37 - 2), S(109 - 2)\} = \{S(35), S(107)\} = \{7, 107\}; \\
 \text{SCD}_2(4369) &= \{S(17 - 2), S(257 - 2)\} = \{S(15), S(255)\} = \{5, 17\}; \\
 \text{SCD}_2(4681) &= \{S(31 - 2), S(151 - 2)\} = \{S(29), S(149)\} = \{29, 149\};
 \end{aligned}$$

$$\begin{aligned} \text{SCD}_2(5461) &= \{S(43 - 2), S(127 - 2)\} = \{S(41), S(125)\} = \{41, 15\}; \\ \text{SCD}_2(7957) &= \{S(73 - 2), S(109 - 2)\} = \{S(71), S(107)\} = \{71, 107\}; \\ \text{SCD}_2(8321) &= \{S(53 - 2), S(157 - 2)\} = \{S(52), S(156)\} = \{17, 31\}. \end{aligned}$$

Comment:

In the case of SCD of order 2 of the 2-Poulet numbers there are too foreseen few patterns.

Open problems:

1. Is for the majority of the 2-Poulet numbers the case that the two Smarandache-Coman divisors of order 2 are both primes, as for the eight from the eleven numbers checked above?
2. Is there an infinity of 2-Poulet numbers for which the set of SCD of order 2 is equal to $\{p, p + 20 \cdot k\}$, where p prime and k positive integer, the case of Poulet numbers 4033 and 4681?

Smarandache-Coman divisors of order 1 of the 3-Poulet numbers:

(See the sequence A215672 in OEIS, posted by us, for a list with Poulet numbers with two prime factors)

$$\begin{aligned} \text{SCD}_1(561) &= \text{SCD}_1(3 \cdot 11 \cdot 17) = \{S(2), S(10), S(16)\} &= \{2, 5, 6\}; \\ \text{SCD}_1(645) &= \text{SCD}_1(3 \cdot 5 \cdot 43) = \{S(2), S(4), S(42)\} &= \{2, 4, 7\}; \\ \text{SCD}_1(1105) &= \text{SCD}_1(5 \cdot 13 \cdot 17) = \{S(4), S(12), S(16)\} &= \{4, 4, 6\}; \\ \text{SCD}_1(1729) &= \text{SCD}_1(7 \cdot 13 \cdot 19) = \{S(6), S(12), S(18)\} &= \{3, 4, 6\}; \\ \text{SCD}_1(1905) &= \text{SCD}_1(3 \cdot 5 \cdot 127) = \{S(2), S(4), S(126)\} &= \{2, 4, 7\}; \\ \text{SCD}_1(2465) &= \text{SCD}_1(5 \cdot 17 \cdot 29) = \{S(4), S(16), S(28)\} &= \{4, 6, 7\}; \\ \text{SCD}_1(2821) &= \text{SCD}_1(7 \cdot 13 \cdot 31) = \{S(6), S(12), S(30)\} &= \{3, 4, 5\}; \\ \text{SCD}_1(4371) &= \text{SCD}_1(3 \cdot 31 \cdot 47) = \{S(2), S(30), S(46)\} &= \{2, 5, 23\}; \\ \text{SCD}_1(6601) &= \text{SCD}_1(7 \cdot 23 \cdot 41) = \{S(6), S(22), S(40)\} &= \{3, 11, 5\}; \\ \text{SCD}_1(8481) &= \text{SCD}_1(3 \cdot 11 \cdot 257) = \{S(2), S(10), S(256)\} &= \{2, 5, 10\}; \\ \text{SCD}_1(8911) &= \text{SCD}_1(7 \cdot 19 \cdot 67) = \{S(6), S(18), S(66)\} &= \{3, 19, 67\}. \end{aligned}$$

Open problems:

1. Is there an infinity of 3-Poulet numbers for which the set of SCD of order 1 is equal to $\{2, 4, 7\}$, the case of Poulet numbers 645 and 1905?
2. Is there an infinity of 3-Poulet numbers for which the sum of SCD of order 1 is equal to 13, the case of Poulet numbers 561 ($2 + 5 + 6 = 13$), 645 ($2 + 4 + 7 = 13$), 1729 ($3 + 4 + 6 = 13$), 1905 ($2 + 4 + 7 = 13$) or is equal to 17, the case of Poulet numbers 2465 ($4 + 6 + 7 = 17$) and 8481 ($2 + 5 + 10 = 17$)?
3. Is there an infinity of Poulet numbers for which the sum of SCD of order 1 is prime, which is the case of the eight from the eleven numbers checked above? What about the sum of SCD of order 1 plus 1, the case of Poulet numbers 2821 ($3 + 4 + 5 + 1 = 13$) and 4371 ($2 + 5 + 23 + 1 = 31$) or the sum of SCD of order 1 minus 1, the case of Poulet numbers 1105 ($4 + 4 + 6 - 1 = 13$), 2821 ($3 + 4 + 5 - 1 = 11$) and 4371 ($2 + 5 + 23 - 1 = 29$)?

25. Seventeen sequences of Poulet numbers characterized by a certain set of Smarandache-Coman divisors

Abstract. In a previous article I defined the Smarandache-Coman divisors of order k of a composite integer n with m prime factors and I sketched some possible applications of this concept in the study of Fermat pseudoprimes. In this paper I make few conjectures about few possible infinite sequences of Poulet numbers, characterized by a certain set of Smarandache-Coman divisors.

Conjecture 1:

There is an infinity of 2-Poulet numbers which have the set of SC divisors of order 1 equal to $\{p, p\}$, where p is prime.

The sequence of this 2-Poulet numbers is: 341, 2047, 3277, 5461, 8321, 13747, 14491, 19951, 31417, ... (see the lists below).

Conjecture 2:

There is an infinity of 2-Poulet numbers which have the set of SC divisors of order 2 equal to $\{p, p + 20 \cdot k\}$, where p is prime and k is non-null integer.

The sequence of this 2-Poulet numbers is: 4033, 4681, 10261, 15709, 23377, 31609, ... (see the lists below).

Conjecture 3:

There is an infinity of 2-Poulet numbers which have the set of SC divisors of order 2 equal to $\{a, b\}$, where $a + b + 1$ is prime.

The sequence of this 2-Poulet numbers is: 1387, 2047, 2701, 3277, 4369, 4681, 8321, 13747, 14491, 18721, 31417, 31609, ... (see the lists below).

Note: This is the case of twelve from the first twenty 2-Poulet numbers.

Conjecture 4:

There is an infinity of 2-Poulet numbers which have the set of SC divisors of order 2 equal to $\{a, b\}$, where $a + b - 1$ is prime.

The sequence of this 2-Poulet numbers is: 4033, 8321, 10261, 13747, 14491, 15709, 19951, 23377, 31417, ... (see the lists below).

Conjecture 5:

There is an infinity of 2-Poulet numbers which have the set of SC divisors of order 2 equal to $\{a, b\}$, where $a + b - 1$ and $a + b + 1$ are twin primes.

The sequence of this 2-Poulet numbers is: 13747, 14491, 23377, 31417, ... (see the lists below).

Conjecture 6:

There is an infinity of pairs of 2-Poulet numbers which have the set of SC divisors of order 2 equal to $\{a, b\}$, respectively to $\{c, d\}$, where $a + b = c + d$ and a, b, c, d are primes.

Such pair of 2-Poulet numbers is: (4681, 7957), because $29 + 149 = 71 + 107 = 178$.

Conjecture 7:

There is an infinity of pairs of 2-Poulet numbers which have the set of SC divisors of order 2 equal to $\{a, b\}$, respectively to $\{c, d\}$, where $a + b + 1 = c + d - 1$.

Such pairs of 2-Poulet numbers are:

(3277, 8321), because $9 + 37 + 1 = 17 + 31 - 1 = 47$;

(19951, 5461), because $23 + 31 + 1 = 41 + 15 - 1 = 55$.

Conjecture 8:

There is an infinity of 2-Poulet numbers which have the set of SC divisors of order 6 equal to $\{p, q\}$, where $\text{abs}\{p - q\} = 6 \cdot k$, where p and q are primes and k is non-null positive integer.

The sequence of this 2-Poulet numbers is:

1387, 2047, 2701, 3277, 4033, 4369, 7957, 13747, 14491, 15709, 23377, 31417, 31609, ... (see the lists below).

Note: This is the case of thirteen from the first twenty 2-Poulet numbers.

Conjecture 9:

There is an infinity of 2-Poulet numbers which have the set of SC divisors of order 6 equal to $\{a, b\}$, where $\text{abs}\{a - b\} = p$ and p is prime.

The sequence of this 2-Poulet numbers is: 341, 4681, 10261, ... (see the lists below).

Conjecture 10:

There is an infinity of 2-Poulet numbers which have the set of SC divisors of order 6 equal to $\{p, q\}$, where one from the numbers p and q is prime and the other one is twice a prime.

The sequence of this 2-Poulet numbers is: 341, 4681, 5461, 10261, ... (see the lists below).

Conjecture 11:

There is an infinity of 3-Poulet numbers which have the set of SC divisors of order 1 equal to $\{a, b, c\}$, where $a + b + c$ is prime and a, b, c are primes.

The sequence of this 2-Poulet numbers is: 561, 645, 1729, 1905, 2465, 6601, 8481, 8911, 10585, 12801, 13741, ... (see the lists below).

Note: This is the case of eleven from the first twenty 2-Poulet numbers.

Conjecture 12:

There is an infinity of 3-Poulet numbers which have the set of SC divisors of order 1 equal to $\{a, b, c\}$, where $a + b + c - 1$ and $a + b + c + 1$ are twin primes.

The sequence of this 3-Poulet numbers is: 2821, 4371, 16705, 25761, 30121, ... (see the lists below)

Conjecture 13:

There is an infinity of 3-Poulet numbers which have the set of SC divisors of order 1 equal to $\{n, n, n\}$.

Such 3-Poulet number is 13981.

Conjecture 14:

There is an infinity of 3-Poulet numbers which have the set of SC divisors of order 2 equal to $\{5, p, q\}$, where p and q are primes and $q = p + 6*k$, where k is non-null positive integer.

Such 3-Poulet numbers are:

1729, because $SCD_2(1729) = \{5, 11, 17\}$ and $17 = 11 + 6*1$;

2821, because $SCD_2(2821) = \{5, 11, 29\}$ and $29 = 11 + 6*3$;

6601, because $SCD_2(6601) = \{5, 7, 13\}$ and $13 = 7 + 6*1$;

13741, because $SCD_2(13741) = \{5, 11, 149\}$ and $149 = 11 + 6*23$;

15841, because $SCD_2(15841) = \{5, 29, 71\}$ and $71 = 29 + 6*7$;

30121, because $SCD_2(30121) = \{5, 11, 329\}$ and $329 = 11 + 6*53$.

Conjecture 15:

There is an infinity of Poulet numbers divisible by 15 which have the set of SC divisors of order 1 equal to $\{2, 4, 7, n_1, \dots, n_i\}$, where n_1, \dots, n_i are non-null positive integers and $i > 0$.

The sequence of this 3-Poulet numbers is: 18705, 55245, 72855, 215265, 831405, 1246785, ... (see the lists below)

Conjecture 16:

There is an infinity of Poulet numbers divisible by 15 which have the set of SC divisors of order 1 equal to $\{2, 4, 23, n_1, \dots, n_i\}$, where n_1, \dots, n_i are non-null positive integers and $i > 0$.

The sequence of this 3-Poulet numbers is: 62745, 451905, ... (see the lists below)

Conjecture 17:

There is an infinity of Poulet numbers which are multiples of any Poulet number divisible by 15 which has the set of SC divisors of order 1 equal to $\{2, 4, n_1, \dots, n_i\}$, where $n_1 = n_2 = \dots = n_i = 7$ and $i > 0$.

Examples:

The Poulet number $645 = 3 \cdot 5 \cdot 43$, having $SCD_1(645) = \{2, 4, 7\}$, has the multiples the Poulet numbers 18705, 72885, which have $SCD_1 = \{2, 4, 7, 7\}$.

The Poulet number $1905 = 3 \cdot 5 \cdot 127$, having $SCD_1(1905) = \{2, 4, 7\}$, has the multiples 55245, 215265 which have $SCD_1 = \{2, 4, 7, 7\}$.

(see the sequence A215150 in OEIS for a list of Poulet numbers divisible by smaller Poulet numbers)

List of SC divisors of order 1 of the first twenty 2-Poulet numbers:

(see the sequence A214305 that I submitted to OEIS for a list of 2-Poulet numbers)

$$\begin{aligned}
 SCD_1(341) &= \{S(11-1), S(31-1)\} = \{S(10), S(30)\} = \{5, 5\}; \\
 SCD_1(1387) &= \{S(19-1), S(73-1)\} = \{S(18), S(72)\} = \{6, 6\}; \\
 SCD_1(2047) &= \{S(23-1), S(89-1)\} = \{S(22), S(88)\} = \{11, 11\}; \\
 SCD_1(2701) &= \{S(37-1), S(73-1)\} = \{S(36), S(72)\} = \{6, 6\}; \\
 SCD_1(3277) &= \{S(29-1), S(113-1)\} = \{S(28), S(112)\} = \{7, 7\}; \\
 SCD_1(4033) &= \{S(37-1), S(109-1)\} = \{S(36), S(108)\} = \{6, 9\}; \\
 SCD_1(4369) &= \{S(17-1), S(257-1)\} = \{S(16), S(256)\} = \{6, 10\}; \\
 SCD_1(4681) &= \{S(31-1), S(151-1)\} = \{S(30), S(150)\} = \{5, 10\}; \\
 SCD_1(5461) &= \{S(43-1), S(127-1)\} = \{S(42), S(126)\} = \{7, 7\}; \\
 SCD_1(7957) &= \{S(73-1), S(109-1)\} = \{S(72), S(108)\} = \{6, 9\}; \\
 SCD_1(8321) &= \{S(53-1), S(157-1)\} = \{S(52), S(156)\} = \{13, 13\}; \\
 SCD_1(10261) &= \{S(31-1), S(331-1)\} = \{S(30), S(330)\} = \{5, 11\}; \\
 SCD_1(13747) &= \{S(59-1), S(233-1)\} = \{S(58), S(232)\} = \{29, 29\}; \\
 SCD_1(14491) &= \{S(43-1), S(337-1)\} = \{S(42), S(336)\} = \{7, 7\}; \\
 SCD_1(15709) &= \{S(23-1), S(683-1)\} = \{S(22), S(682)\} = \{11, 31\}; \\
 SCD_1(18721) &= \{S(97-1), S(193-1)\} = \{S(96), S(192)\} = \{8, 8\}; \\
 SCD_1(19951) &= \{S(71-1), S(281-1)\} = \{S(70), S(280)\} = \{7, 7\}; \\
 SCD_1(23377) &= \{S(97-1), S(241-1)\} = \{S(96), S(240)\} = \{8, 6\}; \\
 SCD_1(31417) &= \{S(89-1), S(353-1)\} = \{S(88), S(352)\} = \{11, 11\}; \\
 SCD_1(31609) &= \{S(73-1), S(433-1)\} = \{S(72), S(432)\} = \{6, 9\}.
 \end{aligned}$$

List of SC divisors of order 2 of the first twenty 2-Poulet numbers:

(see the sequence A214305 that I submitted to OEIS for a list of 2-Poulet numbers)

$$\begin{aligned}
 SCD_2(341) &= \{S(11-2), S(31-2)\} = \{S(9), S(29)\} = \{6, 29\}; \\
 SCD_2(1387) &= \{S(19-2), S(73-2)\} = \{S(17), S(71)\} = \{17, 71\}; \\
 SCD_2(2047) &= \{S(23-2), S(89-2)\} = \{S(21), S(87)\} = \{7, 29\}; \\
 SCD_2(2701) &= \{S(37-2), S(73-2)\} = \{S(35), S(71)\} = \{7, 71\}; \\
 SCD_2(3277) &= \{S(29-2), S(113-2)\} = \{S(27), S(111)\} = \{9, 37\}; \\
 SCD_2(4033) &= \{S(37-2), S(109-2)\} = \{S(35), S(107)\} = \{7, 107\}; \\
 SCD_2(4369) &= \{S(17-2), S(257-2)\} = \{S(15), S(255)\} = \{5, 17\}; \\
 SCD_2(4681) &= \{S(31-2), S(151-2)\} = \{S(29), S(149)\} = \{29, 149\}; \\
 SCD_2(5461) &= \{S(43-2), S(127-2)\} = \{S(41), S(125)\} = \{41, 15\}; \\
 SCD_2(7957) &= \{S(73-2), S(109-2)\} = \{S(71), S(107)\} = \{71, 107\}; \\
 SCD_2(8321) &= \{S(53-2), S(157-2)\} = \{S(51), S(155)\} = \{17, 31\}; \\
 SCD_2(10261) &= \{S(31-2), S(331-2)\} = \{S(29), S(329)\} = \{29, 47\};
 \end{aligned}$$

$$\begin{aligned}
SCD_2(13747) &= \{S(59 - 2), S(233 - 2)\} = \{S(57), S(231)\} = \{19, 11\}; \\
SCD_2(14491) &= \{S(43 - 2), S(337 - 2)\} = \{S(41), S(335)\} = \{41, 67\}; \\
SCD_2(15709) &= \{S(23 - 2), S(683 - 2)\} = \{S(21), S(681)\} = \{7, 227\}; \\
SCD_2(18721) &= \{S(97 - 2), S(193 - 2)\} = \{S(95), S(191)\} = \{19, 191\}; \\
SCD_2(19951) &= \{S(71 - 2), S(281 - 2)\} = \{S(69), S(279)\} = \{23, 31\}; \\
SCD_2(23377) &= \{S(97 - 2), S(241 - 2)\} = \{S(95), S(239)\} = \{19, 239\}; \\
SCD_2(31417) &= \{S(89 - 2), S(353 - 2)\} = \{S(87), S(351)\} = \{29, 13\}; \\
SCD_2(31609) &= \{S(73 - 2), S(433 - 2)\} = \{S(71), S(431)\} = \{71, 431\}.
\end{aligned}$$

List of SC divisors of order 6 of the first twenty 2-Poulet numbers:

(see the sequence A214305 that I submitted to OEIS for a list of 2-Poulet numbers)

$$\begin{aligned}
SCD_6(341) &= \{S(11 - 6), S(31 - 6)\} = \{S(5), S(25)\} = \{5, 10\}; \\
SCD_6(1387) &= \{S(19 - 6), S(73 - 6)\} = \{S(13), S(67)\} = \{13, 67\}; \\
SCD_6(2047) &= \{S(23 - 6), S(89 - 6)\} = \{S(17), S(83)\} = \{17, 83\}; \\
SCD_6(2701) &= \{S(37 - 6), S(73 - 6)\} = \{S(31), S(67)\} = \{31, 67\}; \\
SCD_6(3277) &= \{S(29 - 6), S(113 - 6)\} = \{S(23), S(107)\} = \{23, 107\}; \\
SCD_6(4033) &= \{S(37 - 6), S(109 - 6)\} = \{S(31), S(103)\} = \{31, 103\}; \\
SCD_6(4369) &= \{S(17 - 6), S(257 - 6)\} = \{S(11), S(251)\} = \{11, 251\}; \\
SCD_6(4681) &= \{S(31 - 6), S(151 - 6)\} = \{S(25), S(145)\} = \{10, 29\}; \\
SCD_6(5461) &= \{S(43 - 6), S(127 - 6)\} = \{S(37), S(121)\} = \{37, 22\}; \\
SCD_6(7957) &= \{S(73 - 6), S(109 - 6)\} = \{S(67), S(103)\} = \{67, 103\}; \\
SCD_6(8321) &= \{S(53 - 6), S(157 - 6)\} = \{S(47), S(151)\} = \{47, 151\}; \\
SCD_6(10261) &= \{S(31 - 6), S(331 - 6)\} = \{S(25), S(325)\} = \{10, 13\}; \\
SCD_6(13747) &= \{S(59 - 6), S(233 - 6)\} = \{S(53), S(227)\} = \{53, 227\}; \\
SCD_6(14491) &= \{S(43 - 6), S(337 - 6)\} = \{S(37), S(331)\} = \{37, 331\}; \\
SCD_6(15709) &= \{S(23 - 6), S(683 - 6)\} = \{S(17), S(677)\} = \{17, 677\}; \\
SCD_6(18721) &= \{S(97 - 6), S(193 - 6)\} = \{S(91), S(187)\} = \{13, 17\}; \\
SCD_6(19951) &= \{S(71 - 6), S(281 - 6)\} = \{S(65), S(275)\} = \{13, 11\}; \\
SCD_6(23377) &= \{S(97 - 6), S(241 - 6)\} = \{S(91), S(235)\} = \{13, 47\}; \\
SCD_6(31417) &= \{S(89 - 6), S(353 - 6)\} = \{S(83), S(347)\} = \{83, 347\}; \\
SCD_6(31609) &= \{S(73 - 6), S(433 - 6)\} = \{S(67), S(427)\} = \{67, 61\}.
\end{aligned}$$

List of SC divisors of order 1 of the first twenty 3-Poulet numbers:

(see the sequence A215672 that I submitted to OEIS for a list of 3-Poulet numbers)

$$\begin{aligned}
SCD_1(561) &= SCD_1(3*11*17) = \{S(2), S(10), S(16)\} = \{2, 5, 6\}; \\
SCD_1(645) &= SCD_1(3*5*43) = \{S(2), S(4), S(42)\} = \{2, 4, 7\}; \\
SCD_1(1105) &= SCD_1(5*13*17) = \{S(4), S(12), S(16)\} = \{4, 4, 6\}; \\
SCD_1(1729) &= SCD_1(7*13*19) = \{S(6), S(12), S(18)\} = \{3, 4, 6\}; \\
SCD_1(1905) &= SCD_1(3*5*127) = \{S(2), S(4), S(126)\} = \{2, 4, 7\}; \\
SCD_1(2465) &= SCD_1(5*17*29) = \{S(4), S(16), S(28)\} = \{4, 6, 7\}; \\
SCD_1(2821) &= SCD_1(7*13*31) = \{S(6), S(12), S(30)\} = \{3, 4, 5\}; \\
SCD_1(4371) &= SCD_1(3*31*47) = \{S(2), S(30), S(46)\} = \{2, 5, 23\}; \\
SCD_1(6601) &= SCD_1(7*23*41) = \{S(6), S(22), S(40)\} = \{3, 11, 5\}; \\
SCD_1(8481) &= SCD_1(3*11*257) = \{S(2), S(10), S(256)\} = \{2, 5, 10\}; \\
SCD_1(8911) &= SCD_1(7*19*67) = \{S(6), S(18), S(66)\} = \{3, 19, 67\}; \\
SCD_1(10585) &= SCD_1(5*29*73) = \{S(4), S(28), S(72)\} = \{4, 7, 6\};
\end{aligned}$$

$$\begin{aligned}
\text{SCD}_1(12801) &= \text{SCD}_1(3*17*251) = \{S(2), S(16), S(250)\} = \{2, 6, 15\}; \\
\text{SCD}_1(13741) &= \text{SCD}_1(7*13*151) = \{S(6), S(12), S(150)\} = \{3, 4, 10\}; \\
\text{SCD}_1(13981) &= \text{SCD}_1(11*31*41) = \{S(10), S(30), S(40)\} = \{5, 5, 5\}; \\
\text{SCD}_1(15841) &= \text{SCD}_1(7*31*73) = \{S(6), S(30), S(72)\} = \{3, 5, 6\}; \\
\text{SCD}_1(16705) &= \text{SCD}_1(5*13*257) = \{S(4), S(12), S(256)\} = \{4, 4, 10\}; \\
\text{SCD}_1(25761) &= \text{SCD}_1(3*31*277) = \{S(2), S(30), S(276)\} = \{2, 5, 23\}; \\
\text{SCD}_1(29341) &= \text{SCD}_1(13*37*61) = \{S(12), S(36), S(60)\} = \{4, 6, 5\}; \\
\text{SCD}_1(30121) &= \text{SCD}_1(7*13*331) = \{S(6), S(12), S(330)\} = \{3, 4, 11\}.
\end{aligned}$$

List of SC divisors of order 2 of the first twenty 3-Poulet numbers:

(see the sequence A215672 that I submitted to OEIS for a list of 3-Poulet numbers)

$$\begin{aligned}
\text{SCD}_2(561) &= \text{SCD}_1(3*11*17) = \{S(1), S(9), S(15)\} = \{1, 6, 5\}; \\
\text{SCD}_2(645) &= \text{SCD}_1(3*5*43) = \{S(1), S(3), S(41)\} = \{1, 3, 41\}; \\
\text{SCD}_2(1105) &= \text{SCD}_1(5*13*17) = \{S(3), S(11), S(15)\} = \{3, 11, 5\}; \\
\text{SCD}_2(1729) &= \text{SCD}_1(7*13*19) = \{S(5), S(11), S(17)\} = \{5, 11, 17\}; \\
\text{SCD}_2(1905) &= \text{SCD}_1(3*5*127) = \{S(1), S(3), S(125)\} = \{1, 3, 15\}; \\
\text{SCD}_2(2465) &= \text{SCD}_1(5*17*29) = \{S(3), S(15), S(27)\} = \{3, 5, 9\}; \\
\text{SCD}_2(2821) &= \text{SCD}_1(7*13*31) = \{S(5), S(11), S(29)\} = \{5, 11, 29\}; \\
\text{SCD}_2(4371) &= \text{SCD}_1(3*31*47) = \{S(1), S(29), S(45)\} = \{1, 29, 6\}; \\
\text{SCD}_2(6601) &= \text{SCD}_1(7*23*41) = \{S(5), S(21), S(29)\} = \{5, 7, 13\}; \\
\text{SCD}_2(8481) &= \text{SCD}_1(3*11*257) = \{S(1), S(9), S(255)\} = \{1, 6, 17\}; \\
\text{SCD}_2(8911) &= \text{SCD}_1(7*19*67) = \{S(5), S(17), S(65)\} = \{5, 17, 13\}; \\
\text{SCD}_2(10585) &= \text{SCD}_1(5*29*73) = \{S(3), S(27), S(71)\} = \{3, 9, 71\}; \\
\text{SCD}_2(12801) &= \text{SCD}_1(3*17*251) = \{S(1), S(15), S(249)\} = \{1, 5, 83\}; \\
\text{SCD}_2(13741) &= \text{SCD}_1(7*13*151) = \{S(5), S(11), S(149)\} = \{5, 11, 149\}; \\
\text{SCD}_2(13981) &= \text{SCD}_1(11*31*41) = \{S(9), S(29), S(39)\} = \{6, 29, 13\}; \\
\text{SCD}_2(15841) &= \text{SCD}_1(7*31*73) = \{S(5), S(29), S(71)\} = \{5, 29, 71\}; \\
\text{SCD}_2(16705) &= \text{SCD}_1(5*13*257) = \{S(3), S(111), S(255)\} = \{3, 11, 17\}; \\
\text{SCD}_2(25761) &= \text{SCD}_1(3*31*277) = \{S(1), S(29), S(275)\} = \{1, 29, 11\}; \\
\text{SCD}_2(29341) &= \text{SCD}_1(13*37*61) = \{S(11), S(35), S(59)\} = \{11, 7, 59\}; \\
\text{SCD}_2(30121) &= \text{SCD}_1(7*13*331) = \{S(5), S(11), S(329)\} = \{5, 11, 329\}.
\end{aligned}$$

List of SC divisors of order 1 of the first ten Poulet numbers divisible by 3 and 5:

(see the sequence A216364 that I submitted to OEIS for a list of Poulet numbers divisible by 15)

$$\begin{aligned}
\text{SCD}_1(645) &= \text{SCD}_1(3*5*43) = \{2, 4, 7\}; \\
\text{SCD}_1(1905) &= \text{SCD}_1(3*5*127) = \{2, 4, 7\}; \\
\text{SCD}_1(18705) &= \text{SCD}_1(3*5*29*43) = \{2, 4, 7, 7\}; \\
\text{SCD}_1(55245) &= \text{SCD}_1(3*5*29*127) = \{2, 4, 7, 7\}; \\
\text{SCD}_1(62745) &= \text{SCD}_1(3*5*47*89) = \{2, 4, 23, 11\}; \\
\text{SCD}_1(72855) &= \text{SCD}_1(3*5*43*113) = \{2, 4, 7, 7\}; \\
\text{SCD}_1(215265) &= \text{SCD}_1(3*5*113*127) = \{2, 4, 7, 7\}; \\
\text{SCD}_1(451905) &= \text{SCD}_1(3*5*47*641) = \{2, 4, 23, 8\}; \\
\text{SCD}_1(831405) &= \text{SCD}_1(3*5*43*1289) = \{2, 4, 7, 23\}; \\
\text{SCD}_1(1246785) &= \text{SCD}_1(3*5*43*1933) = \{2, 4, 7, 23\}.
\end{aligned}$$

26. Few types of chains of primes arising in the study of pseudoprimes

Abstract. While studying Fermat pseudoprimes I met few interesting generic forms of numbers that have the property to generate chains of primes and pseudoprimes. I list in this paper few such types of chains.

I. Recurrent chains

I.1.

Chains of primes of the form $P_0, P_1 = P_0 * n - n + 1, P_2 = P_1 * n - n + 1, \dots, P_k = P_{k-1} * n - n + 1$, where P_0, P_1, \dots, P_k are primes and n is a positive integer, $n > 1$.

Note: For $n = 2$, we obtain the Cunningham chain of the second kind, *i.e.* $P_{i+1} = 2 * P_i - 1$.

For instance:

A chain of primes of length 6, for $n = 3$, should have the form: $p, 3 * p - 2, 9 * p - 8, 27 * p - 26, 81 * p - 80, 243 * p - 242$. It can be seen that p must be of the form $10 * k + 1$ for a chain of length bigger than 3. Such a chain, of length 5, is: 61, 181, 541, 1621, 4861.

A chain of primes of length 5, for $n = 4$, should have the form: $p, 4 * p - 3, 16 * p - 15, 64 * p - 63, 256 * p - 255$. It can be seen that p can't be of the form $10k + 7$. Such a chain, of length 4, is: 23, 89, 353, 1409.

Notes:

The formula $P_{i+1} = P_i * n - n + 1$ can lead to the formation of chains of Fermat pseudoprimes, for instance for Carmichael numbers $[P_0, P_1] = [1729, 46657]$ and $n = 27$.

The formula $P_{i+1} = P_i * (P_i * n - n + 1)$ can also lead to the formation of chains of Fermat pseudoprimes; for instance, for $n = 2$, $P_{i+1} = P_i * (2 * P_i - 1)$ leads to the formation of the following Poulet numbers $[P_0, P_1]$: [645, 831405], [1729, 5977153] etc.; for $n = 3$, $P_{i+1} = P_i * (3 * P_i - 2)$ leads to the formation of the following Poulet numbers $[P_0, P_1]$: [341, 348161], [645, 1246785] etc. (see the sequence A215343 in OEIS).

I.2.

Chains of primes of the form $P_0, P_1 = P_0 * n - n - 1, P_2 = P_1 * n - n - 1, \dots, P_k = P_{k-1} * n - n - 1$, where P_0, P_1, \dots, P_k are primes and n is a positive integer, $n > 1$.

For instance:

A chain of primes of length 6, for $n = 2$, should have the form: $p, 2 * p - 3, 4 * p - 9, 8 * p - 21, 16 * p - 45, 32 * p - 93$. It can be seen that p must be of the form $10k + 3$ for a chain of length bigger than 3. Such a chain, of length 4, is: 113, 223, 443, 883.

A chain of primes of length 5, for $n = 3$, should have the form: $p, 3 * p - 4, 9 * p - 16, 27 * p - 52, 81 * p - 160$. It can be seen that p must be of the form $10 * k + 7$ for a chain of length bigger than 3. Such a chain, of length 4, is: 7, 17, 47, 137.

Note: I met this type of numbers in the study of Fermat pseudoprimes to base 2 with three prime factors (see the sequence A215672 in OEIS). Most of them can be written as $p^*(p^n - n + 1)^*(p^m - m + 1)$ or as $p^*(p^n - n - 1)^*(p^m - m - 1)$.

I.3.

Chains of primes of the form $P_0, P_1 = P_0*n + n - 1, P_2 = P_1*n + n - 1, \dots, P_k = P_{k-1}*n + n - 1$, where P_0, P_1, \dots, P_k are primes and n is a positive integer, $n > 1$.

Note: For $n = 2$, we obtain the Cunningham chain of the first kind, i.e. $P_{i+1} = 2*P_i + 1$.

For instance:

A chain of primes of length 6, for $n = 3$, should have the form: $p, 3*p + 2, 9*p + 8, 27*p + 26, 81*p + 80, 243*p + 242$. It can be seen that p must be of the form $10*k + 9$ for a chain of length bigger than 3. Such a chain, of length 4, is: 29, 89, 269, 809.

I.4.

Chains of primes of the form $P_0, P_1 = P_0*n + n + 1, P_2 = P_1*n + n + 1, \dots, P_k = P_{k-1}*n + n + 1$, where P_0, P_1, \dots, P_k are primes and n is a positive integer, $n > 1$.

For instance:

A chain of primes of length 6, for $n = 2$, should have the form: $p, 2*p + 3, 4*p + 9, 8*p + 21, 16*p + 45, 32*p + 93$. It can be seen that p must be of the form $10*k + 7$ for a chain of length bigger than 3. Such a chain, of length 6, is: 47, 97, 197, 397, 797, 1597.

I.5.

Chains of primes of the form $P_0, P_1 = P_0*n - d*n + d, P_2 = P_1*n - d*n + d, \dots, P_k = P_{k-1}*n - d*n + d$, where P_0, P_1, \dots, P_k are primes, d is also a prime number and n is a positive integer, $n > 1$.

For instance:

A chain of this type of primes of length 6, for $n = 2$ and $d = 7$, should have the form: $p, 2*p - 7, 4*p - 21, 8*p - 49, 16*p - 105, 32*p - 217$. It can be seen that p must be of the form $30*k + 7$ for a chain of length bigger than 3.

A chain of this type of primes of length 6, for $n = 2$ and $d = 13$, should have the form: $p, 2*p - 13, 4*p - 39, 8*p - 91, 16*p - 195, 32*p - 403$. It can be seen that p must be of the form $30*k + 13$ for a chain of length bigger than 3. Such a chain, of length 4, is 163, 313, 613, 1213.

Note: I met this type of numbers in the study of Fermat pseudoprimes to base 2 with two prime factors (see the sequence A214305 in OEIS); for instance, for $n = 3$ and $d = 73$, $P_{i+1} = 3*P_i - 2*73$ leads to the formation of the following Poulet numbers $[P_0, P_1]$: [2701, 7957] etc.; for $n = 4$ and $d = 73$, $P_{i+1} = 4*P_i - 3*73$ leads to the formation of the following Poulet numbers $[P_0, P_1]$: [2701, 10585] etc.

II. Non-recurrent chains

II.1.

Chains of primes of the form $30*a^n - (a*p + a - 1)$, where p and $a*p + a - 1$ are primes and n has successive values of integers.

For instance:

For $p = 11$, $a = 2$, n from -1 to 3 we have, in absolute value, the following chain of primes of length 5: 83, 23, 37, 97, 157.

For $p = 23$, $a = 2$, n from -3 to 2 we have, in absolute value, the following chain of primes of length 6: 227, 167, 107, 47, 13, 73.

For $p = 7$, $a = 3$, n from -1 to 2 we have, in absolute value, the following chain of primes of length 4: 113, 23, 67, 157.

Note: I met this type of numbers in the study of Carmichael numbers of the form $C = ((30*a^n - (a^p + a - 1)) * ((30*b^n - (b^p + b - 1)) * ((30*c^n - (c^p + c - 1))))$, where p , $a^p + a - 1$, $b^p + b - 1$ and $c^p + c - 1$ are all primes. Many Carmichael numbers can be written in this form (see the sequence A182416 in OEIS).

II.2.

Chains of primes of the form $30*a^n + (a^p + a - 1)$, where p and $a^p + a - 1$ are primes and n has successive values of integers.

For instance:

For $p = 19$, $a = 3$, n from -1 to 2 we have, in absolute value, the following chain of primes of length 4: 31, 59, 149, 239.

Note: I met this type of numbers in the study of Carmichael numbers of the form $C = ((30*a^n + (a^p - a + 1)) * ((30*b^n + (b^p - b + 1)) * ((30*c^n + (c^p - c + 1))))$, where p , $a^p - a + 1$, $b^p - b + 1$ and $c^p - c + 1$ are all primes. Many Carmichael numbers can be written in this form (see the sequence A182416 in OEIS).

II.3.

Chains of primes of the form $2^p*n - 2^n + p$, where p and $2^p - 1$ are primes and n has successive values of integers.

For instance:

For $p = 7$, n from -5 to 3 we have, in absolute value, the following chain of primes of length 9: 53, 41, 29, 17, 5, 7, 19, 31, 43.

Note: I met this type of numbers in the study of Carmichael numbers of the form $C = p*(2^p - 1)*(2^p*n - 2^n + p)$. I conjecture that all Carmichael numbers divisible with p and $2^p - 1$, where p and $2^p - 1$ are primes, can be written in this form (see the sequence A182207 in OEIS).

27. Ten conjectures about certain types of pairs of primes arising in the study of 2-Poulet numbers

Abstract. There are many interesting, yet not studied enough, properties of Poulet numbers. In particular, the study of the 2-Poulet numbers appears to be most seductive because in their structure are found together three of the most important concepts in number theory: those of primes, semiprimes and pseudoprimes. In this paper we make few conjectures about primes or pairs of primes, including twin primes, that could be associated to the pairs of primes represented by the two prime factors of a 2-Poulet number.

Conjecture 1:

There is an infinity of pairs of primes of the form $(p, p + 24)$.

Note: It is not necessarily for the two primes to be consecutive, as is stipulated in de Polignac's Conjecture.

Conjecture 2:

There is, for any odd prime $q, q > 3$, an infinity of primes of the form $q + p*(p + 24) - 1$, where p and $p + 24$ are both primes.

Examples:

For $q = 5$, we have:

$$: 5 + 17*41 - 1 = 701, \text{ prime};$$

$$: 5 + 19*43 - 1 = 821, \text{ prime}.$$

For $q = 7$, we have:

$$: 7 + 5*29 - 1 = 151, \text{ prime};$$

$$: 7 + 19*43 - 1 = 823, \text{ prime}.$$

Conjecture 3:

There is, for any pair of twin primes $(p, q), (p, q) \neq (3, 5)$, an infinity of pairs of twin primes of the form $(p + m*n - 1, q + m*n - 1)$, where m and n are both primes and $n = m + 24$.

Examples:

For $(p, q) = (5, 7)$ we have:

$$: 5 + 19*43 - 1 = 821 \text{ and } 7 + 19*43 - 1 = 823, \text{ primes}.$$

For $(p, q) = (11, 13)$ we have:

$$: 11 + 7*31 - 1 = 227 \text{ and } 13 + 7*31 - 1 = 229, \text{ primes}.$$

Conjecture 4:

There is an infinity of 2-Poulet numbers $P, P = p_1*p_2$, for which $p_2 - p_1 + 1 = q_1*q_2$, where q_1 and q_2 are primes and $q_2 - q_1 = 24$.

Note: In other words, we can associate to an infinity of pairs of primes (p_1, p_2) , where p_1 and p_2 are the two prime factors of a 2-Poulet number, another pair of primes (q_1, q_2) .

Examples:

For $P = 129889 = 193 \cdot 673$ we have:

$$673 - 193 = 13 \cdot 37 \text{ and } 37 - 13 = 24.$$

For $P = 130561 = 137 \cdot 953$ we have:

$$953 - 137 = 19 \cdot 43 \text{ and } 43 - 19 = 24.$$

Conjecture 5:

There is an infinity of 2-Poulet numbers P , $P = p_1 \cdot p_2$, for which $p_2 - p_1 + 1 = q$, where q is prime or a square of prime.

Note: In other words, we can associate to an infinity of pairs of primes (p_1, p_2) , where p_1 and p_2 are the two prime factors of a 2-Poulet number, a prime or a square of prime q .

Examples:

For $P = 2047 = 23 \cdot 89$ we have: $89 - 23 + 1 = 67$, prime.

For $P = 2701 = 37 \cdot 73$ we have: $73 - 37 + 1 = 37$, prime.

For $P = 4033 = 37 \cdot 109$ we have: $109 - 37 + 1 = 73$, prime.

For $P = 4369 = 17 \cdot 257$ we have: $257 - 17 + 1 = 241$, prime.

For $P = 4681 = 31 \cdot 151$ we have: $151 - 31 + 1 = 11^2$.

Conjecture 6:

There is an infinity of 2-Poulet numbers P , $P = p_1 \cdot p_2$, for which $p_2 - p_1 - 1 = q$, where q is prime or a square of prime.

Examples:

For $P = 4033 = 37 \cdot 109$ we have: $109 - 37 - 1 = 71$, prime.

For $P = 8321 = 53 \cdot 157$ we have: $157 - 53 - 1 = 103$, prime.

Conjecture 7:

There is an infinity of 2-Poulet numbers P , $P = p_1 \cdot p_2$, for which $p_2 - p_1 - 1$ and $p_2 - p_1 + 1$ are both primes.

Conjecture 8:

For any pair of twin primes (q_1, q_2) there exist at least a pair of primes (p_1, p_2) such that $q_1 = p_2 - p_1 - 1$ and $q_2 = p_2 - p_1 + 1$ are both primes and $P = p_1 \cdot p_2$ is a 2-Poulet number.

Conjecture 9:

There is an infinity of pairs of primes, not necessarily consecutive, of the form $(p, p + 84)$.

Conjecture 10:

There is an infinity of 2-Poulet numbers P , $P = p_1 \cdot p_2$, for which $p_2 - p_1 = 84$.

Examples:

For $P = 3277 = 29 \cdot 113$ we have: $113 - 29 = 84$.

For $P = 5461 = 43 \cdot 127$ we have: $127 - 43 = 84$.

Reference:

See the sequence A214305 posted by us in OEIS for a list of Fermat pseudoprimes to base two (Poulet numbers) with two prime factors (2-Poulet numbers).

28. Two conjectures that relates any Poulet number by a type of triplets respectively of duplets of primes

Abstract. In one of my previous papers I conjectured that there exist an infinity of Poulet numbers which can be written as the sum of three primes of the same form from the following eight ones: $30k+1$, $30k+7$, $30k+11$, $30k+13$, $30k+17$, $30k+19$, $30k+23$, $30k+29$. In this paper I conjecture that any Poulet number not divisible by 5 can be written as a sum of three primes of the same form from the following four ones: $30k+1$, $30k+3$, $30k+7$ or $30k+9$ respectively as a sum from a prime and the double of the another one, both primes having the same form from the four ones mentioned above. Finally, I yet made any other two related conjectures about two types of squares of primes.

Conjecture 1:

Any Poulet number P not divisible by 5 can be written at least in one way as $P = m + n + q$, where m , n , q are primes, not necessarily all three distinct, of the same form from the following four ones: $30 \cdot k+1$, $30 \cdot k+3$, $30 \cdot k+7$ or $30 \cdot k+9$.

Note that the primes m , n , q can't be all three equal, because there is no a Poulet number divisible by 3 with only two prime factors.

Verifying the conjecture:

(for the first Poulet number from each of the forms $10 \cdot k + 1$, $10 \cdot k + 3$, $10 \cdot k + 7$ respectively $10 \cdot k + 9$)

For $P = 341$, we have:

$$: 341 = 7 + 17 + 307 = 7 + 107 + 227 = 7 + 137 + 197 = 7 + 167 + 167 = 17 + 17 + 307 = 17 + 47 + 277 = 17 + 67 + 257 = 17 + 97 + 227 = 17 + 127 + 197 (...).$$

For $P = 1387$, we have:

$$: 19 + 79 + 1289 = 19 + 89 + 1279 = 19 + 109 + 1259 = 19 + 139 + 1229 = 19 + 229 + 1129 (...).$$

For $P = 1729$, we have:

$$: 13 + 23 + 1693 = 13 + 53 + 1663 = 13 + 103 + 1613 = 13 + 163 + 1553 \quad 13 + 173 + 1543 (...).$$

For $P = 4033$, we have:

$$: 11 + 331 + 3691 = 11 + 661 + 3391 = 11 + 691 + 3331 (...).$$

Conjecture 2:

Any Poulet number P not divisible by 5 can be written at least in one way as $P = 2*m + n$, where m and n are distinct primes of the same form from the following four ones: $30*k+1$, $30*k+3$, $30*k+7$ or $30*k+9$.

Verifying the conjecture:

(for the first eight Poulet numbers not divisible by 5)

For $P = 341$, we have:

$$: 341 = 2*167 + 7 = 2*17 + 307 = 2*107 + 127 (...).$$

For $P = 561$, we have:

$$: 561 = 2*7 + 547 = 2*37 + 487 = 2*47 + 467 (...).$$

For $P = 1387$, we have:

$$: 1387 = 2*79 + 1229 = 2*139 + 1109 (...).$$

For $P = 1729$, we have:

$$: 1729 = 2*73 + 1583 = 2*103 + 1523 (...).$$

For $P = 2047$, we have:

$$: 2047 = 2*79 + 1889 = 2*379 + 1289 (...).$$

For $P = 2701$, we have:

$$: 2701 = 2*79 + 1889 = 2*379 + 1289 (...).$$

For $P = 2821$, we have:

$$: 2821 = 2*79 + 1889 = 2*379 + 1289 (...).$$

For $P = 3277$, we have:

$$: 3277 = 2*79 + 1889 = 2*379 + 1289 (...).$$

Comment:

Because sometimes seems to me that Poulet numbers behaves like squares of primes (though the only squares of primes that are Poulet numbers are the two known Wieferich primes) I also make two conjectures about primes.

Conjecture 1:

Any square of a prime of the form $p^2 = 10*k + 1$ can be written as $p^2 = m + n + q$, where m , n , q are primes, not necessarily all three distinct, of the form $10*k + 7$.

Examples:

$$: 11^2 = 121 = 37 + 37 + 47; 19^2 = 361 = 7 + 37 + 317.$$

Conjecture 2:

Any square of a prime of the form $p^2 = 10*k + 9$ can be written as $p^2 = m + n + q$, where m , n , q are primes, not necessarily all three distinct, of the form $10*k + 3$.

Examples:

$$: 7^2 = 49 = 13 + 13 + 23; 13^2 = 169 = 13 + 43 + 113.$$

29. A type of primes that seem to lead to sequences of infinite Poulet numbers in a recurrent formula

Abstract. Though I discovered a lot of sequences of Poulet numbers based on different types of formulas, I never succeeded to find a recurrent formula able to produce a subset of Poulet numbers...until now, when I incidentally noticed an interesting relation between two Poulet numbers divisible by 73. Extrapolating the result I obtained a recurrent formula based on primes of the form $30k+13$ that seem to lead often to possible infinite sequences of Poulet numbers.

Conjecture:

Any Poulet number P that has a prime divisor d of the form $60*k + 13$ is the starting term in the following recurrent formula which produce a sequence containing an infinity of Poulet numbers P_i : $P_i = ((P*2 - d)*2 - d)*2 - d...$

Examples:

For $P = 1387 = 19*73$, we have:

$$\begin{aligned} &: (1387*2 - 73) = 2701, \text{ a Poulet number;} \\ &: (((1387*2 - 73)*2 - 73))*2 - 73 = 10585, \text{ a Poulet number.} \end{aligned}$$

For $P = 1729 = 7*13*19$, we have:

$$: ((1729*2 - 13)*2 - 13) = 13741, \text{ a Poulet number.}$$

For $P = 7957 = 73*109$, we have:

$$\begin{aligned} &: (7957*2 - 73) = 15841, \text{ a Poulet number;} \\ &: ((1387*2 - 73)*2 - 73) = 31609, \text{ a Poulet number.} \\ &: (((1387*2 - 73)*2 - 73))*2 - 73 = 126217, \text{ a Poulet number.} \end{aligned}$$

For $P = 31609 = 73*433$, we have:

$$: (((31609*2 - 433)*2 - 433)*2 - 433) = 249841, \text{ a Poulet number.}$$

Note (from above) that 31609 is also a term in other recurrence relation, based on the divisor 73.

For $P = 49141 = 157*313$, we have:

$$: (((49141*2 - 313)*2 - 313)*2 - 313) = 390937, \text{ a Poulet number.}$$

For $P = 65281 = 97*673$, we have:

$$: (65281*2 - 673) = 129889, \text{ a Poulet number.}$$

30. A very interesting formula of a subset of Poulet numbers involving consecutive powers of a power of two

Abstract. I studied Poulet numbers for some time but I'm still amazed by the wealth of the patterns that this set of numbers offers; it's like everything that the prime numbers, in their stubbornly to let themselves understood and disciplined, refuse us, these exceptions of the Fermat's "little theorem" allow us. This paper states a conjecture about a new subset of Poulet numbers that I discovered by chance.

Conjecture:

There is an infinity of Poulet numbers P of the form $P = ((2^n)^k) * ((2^n)^{(k+1)} + 2^n + 1) + 1$, where k is a non-null positive integer, for any n non-null positive integer.

Examples:

For $n = 1$ the formula becomes $P = 2^k * (2^{(k+1)} + 3) + 1$ and we obtained:

$$\begin{aligned} &: 2^4 * (2^5 + 3) + 1 = 561 = 3 * 11 * 17, \text{ a Poulet number;} \\ &: 2^7 * (2^8 + 3) + 1 = 33153 = 3 * 43 * 257, \text{ a Poulet number.} \end{aligned}$$

For $n = 2$ the formula becomes $P = 4^k * (4^{(k+1)} + 5) + 1$ and we obtained:

$$\begin{aligned} &: 4^2 * (4^3 + 5) + 1 = 1105 = 5 * 13 * 17, \text{ a Poulet number;} \\ &: 4^3 * (4^4 + 5) + 1 = 16705 = 5 * 13 * 257, \text{ a Poulet number.} \end{aligned}$$

For $n = 4$ the formula becomes $P = 16^k * (16^{(k+1)} + 17) + 1$ and we obtained:

$$\begin{aligned} &: 16^1 * (16^2 + 17) + 1 = 4369 = 17 * 257, \text{ a Poulet number;} \\ &: 16^2 * (16^3 + 17) + 1 = 1052929 = 17 * 241 * 257, \text{ a Poulet number;} \\ &: 16^3 * (16^4 + 17) + 1 = 268505089 = 17 * 241 * 65537, \text{ a Poulet number.} \end{aligned}$$

For $n = 6$ the formula becomes $P = 64^k * (64^{(k+1)} + 65) + 1$ and we obtained:

$$: 64^1 * (64^2 + 65) + 1 = 266305 = 5 * 13 * 17 * 241, \text{ a Poulet number.}$$

For $n = 7$ the formula becomes $P = 128^k * (128^{(k+1)} + 129) + 1$ and we obtained:

$$: 128^1 * (128^2 + 129) + 1 = 2113665 = 3 * 5 * 29 * 43 * 113, \text{ a Poulet number.}$$

For $n = 8$ the formula becomes $P = 256^k * (256^{(k+1)} + 257) + 1$ and we obtained:

$$: 256^1 * (256^2 + 257) + 1 = 16843009 = 257 * 65537, \text{ a Poulet number.}$$

Part three. Thirty sequences of Fermat pseudoprimes

1. Poulet numbers with two prime factors

First 38 terms of the sequence (A214305 in OEIS): 341, 1387, 2047, 2701, 3277, 4033, 4369, 4681, 5461, 7957, 8321, 10261, 13747, 14491, 15709, 18721, 19951, 23377, 31417, 31609, 31621, 35333, 42799, 49141, 49981, 60701, 60787, 65077, 65281, 80581, 83333, 85489, 88357, 90751, 104653, 123251, 129889, 130561.

Conjecture 1:

For any biggest prime factor of a Poulet number P_1 with two prime factors exists a series with infinite many Poulet numbers P_2 formed this way: $P_2 \bmod (P_1 - d) = d$, where d is the biggest prime factor of P_1 .

Note: It can be seen that the Poulet numbers divisible by 73 bigger than 2701 (7957, 10585, 15841, 31609 etc.) can be written as $1314*n + 73$ as well as $2628*m + 73$.

Conjecture 2:

Any Poulet number P_2 divisible by d can be written as $(P_1 - d)*n + d$, where n is natural, if exists a smaller Poulet number P_1 with two prime factors divisible by d .

Note: This conjecture can't be extrapolated for Poulet numbers P_1 with more than two prime factors; for instance, if is taken $561 = 3*11*17$ as p_1 , are indeed found bigger Poulet numbers divisible by 17 as 1105 and 4369 that can be written as $544*n + 17$ but exists also such numbers that can't be written this way, like 2465. But can be extrapolated the first conjecture.

Conjecture 3:

For any biggest prime factor of a Poulet number P_1 exists a series with infinite many Poulet numbers p_2 formed this way: $P_2 \bmod (P_1 - d) = d$, where d is the biggest prime factor of P_1 .

Examples:

For $P_1 = 341 = 11*31$ were obtained the following Poulet numbers P_2 for which $P_2 \bmod 310 = 31$: 2821, 4371, 4681, 10261 etc.

For $P_1 = 1387 = 19*73$ were obtained the following Poulet numbers P_2 for which $P_2 \bmod 1314 = 73$: 2701, 7957, 10585, 15841 etc.

For $P_1 = 2047 = 23*89$ were obtained the following Poulet numbers P_2 for which $P_2 \bmod 1958 = 89$: 31417, 35333, 60787, 62745 etc.

For $p_1 = 2701 = 37*73$ were obtained the following Poulet numbers P_2 for which $P_2 \bmod 2628 = 73$: 7957, 10585, 15841 etc.

2. Poulet numbers with three prime factors

First 37 terms of the sequence (A215672 in OEIS): 561, 645, 1105, 1729, 1905, 2465, 2821, 4371, 6601, 8481, 8911, 10585, 12801, 13741, 13981, 15841, 16705, 25761, 29341, 30121, 30889, 33153, 34945, 41665, 52633, 57421, 68101, 74665, 83665, 87249, 88561, 91001, 93961, 113201, 115921, 121465, 137149.

Comments:

The most of the terms shown can be written in one of the following two ways:

$$(1) \quad p*((n+1)^p - n^p)*((m+1)^p - m^p);$$

$$(2) \quad p*((n^p - (n+1)^p)*(m^p - (m+1)^p),$$

where p is the smallest of the three prime factors and n, m natural numbers.

Exempli gratia for Poulet numbers from first category:

$$10585 = 5*29*73 = 5*(5*7 - 6)*(5*18 - 17);$$

$$13741 = 7*13*151 = 7*(7*2 - 1)*(7*25 - 24);$$

$$13981 = 11*31*41 = 11*(11*3 - 2)*(11*4 - 3);$$

$$29341 = 13*37*61 = 13*(13*3 - 2)*(13*5 - 4);$$

$$137149 = 23*67*89 = 23*(23*3 - 2)*(23*4 - 3).$$

Exempli gratia for Poulet numbers from second category:

$$6601 = 7*23*41 = 7*(7*4 - 5)*(7*7 - 8).$$

Note: From the numbers from the sequence above, just the numbers 30889, 88561 and 91001 can't be written in one of the two ways. What these three numbers have in common: they all have a prime divisor q of the form $30*k + 23$ (*i.e.* 23, 53, 83) and can be written as $q*((r+1)^q - r)$, where r is a natural number.

Conjecture:

Any Poulet number P with three or more prime divisors has at least one prime divisor q for that can be written as $P = q*((r+1)^q - r)$, where r is a natural number.

Note: It can be proved that a Carmichael number can be written this way for any of its prime divisors – see the sequence A213812 in OEIS.

Note: There are also a lot of Poulet numbers with two prime divisors that can be written this way, but here are few exceptions: 7957, 23377, 42799, 49981, 60787.

3. Poulet numbers with three prime factors divisible by a smaller Poulet number

First 30 terms of the sequence (A215944 in OEIS): 13981, 137149, 158369, 176149, 276013, 285541, 294409, 348161, 387731, 423793, 488881, 493697, 617093, 625921, 847261, 1052503, 1052929, 1104349, 1128121, 1152271, 1398101, 1461241, 1472353, 1507561, 1534541, 1549411, 1746289, 1840357, 1857241, 2299081.

Comments:

Almost all the numbers from the sequence above can be written as $p*((m + 1)*p - m)*((n + 1)*p - n)$, where m, n, p are natural numbers (in the brackets is written the Poulet number which every one of them is divisible by):

- (1) $n*(2*n - 1)*(3*n - 2)$: the number 294409 (2701);
- (2) $n*(2*n - 1)*(5*n - 4)$: the numbers 285541 (4681), 488881 (2701);
- (3) $n*(2*n - 1)*(11*n - 10)$: the number 625921 (10261);
- (4) $n*(2*n - 1)*(15*n - 14)$: the number 1461241 (2701);
- (5) $n*(3*n - 2)*(4*n - 3)$: the numbers 13981 (341), 137149 (2047);
- (6) $n*(3*n - 2)*(5*n - 4)$: the number 1152271 (5461);
- (7) $n*(3*n - 2)*(8*n - 7)$: the number 1840357 (5461);
- (8) $n*(3*n - 2)*(10*n - 9)$: the number 2299081 (5461);
- (9) $n*(3*n - 2)*(12*n - 11)$: the number 1746289 (4033);
- (10) $n*(3*n - 2)*(31*n - 30)$: the number 1052503 (15709);
- (11) $n*(3*n - 2)*(102*n - 101)$: the number 348161 (341);
- (12) $n*(3*n - 2)*(442*n - 441)$: the number 1507561 (341);
- (13) $n*(4*n - 3)*(7*n - 6)$: the number 176149 (1387);
- (14) $n*(4*n - 3)*(11*n - 10)$: the number 276013 (1387);
- (15) $n*(4*n - 3)*(12*n - 11)$: the number 1104349 (3277);
- (16) $n*(4*n - 3)*(31*n - 30)$: the number 1398101 (15709);
- (17) $n*(5*n - 4)*(6*n - 5)$: the number 847261 (4681);
- (18) $n*(5*n - 4)*(8*n - 7)$: the number 1128121 (4681);
- (19) $n*(5*n - 4)*(11*n - 10)$: the number 1549411 (4681);
- (20) $n*(6*n - 5)*(11*n - 10)$: the number 1857241 (10261);
- (21) $n*(6*n - 5)*(16*n - 15)$: the number 423793 (4369);
- (22) $n*(7*n - 6)*(16*n - 15)$: the number 493697 (4369);
- (23) $n*(15*n - 14)*(16*n - 15)$: the number 1052929 (4369);
- (24) $n*(16*n - 15)*(21*n - 20)$: the number 1472353 (4369).

Note: The only few numbers from the sequence above that can't be written this way are multiples of the Poulet number 5461 and can be, instead, written as $5461*(42*k - 13)$: $158369 = 5461*29$, $387731 = 5461*71$, $617093 = 5461*113$ and $1534541 = 5461*281$.

Conjecture:

The only Fermat pseudoprimes to base 2 divisible by a smaller Fermat pseudoprime to base 2 that can't be written as $p*((m + 1)*p - m)*((n + 1)*p - n)$, where m, n, p are natural numbers, are multiples of 5461 and can be written as $5461*(42*k - 13)$.

Note: Conjecture is checked for the numbers from the sequence above and for the first 15 Poulet numbers with four prime factors.

Note: There are Fermat pseudoprimes to base 2 divisible with 5461 that can be written as $p*((m + 1)*p - m)*((n + 1)*p - n)$; these ones can be written as $5461*(42*k + 43)$. Numbers from this category are: $1152271 = 5461*211$, $1840357 = 5461*337$, $2299081 = 5461*421$.

4. Poulet numbers of the form $(6*k + 1)*(6*k*n + 1)$, where k, n are integers different from 0

First 30 terms of the sequence (A214607 in OEIS): 1105, 1387, 1729, 2701, 2821, 4033, 4681, 5461, 6601, 8911, 10261, 10585, 11305, 13741, 13981, 14491, 15841, 16705, 18721, 29341, 30121, 30889, 31609, 31621, 39865, 41041, 41665, 46657, 49141, 52633, 57421, 63973, 65281, 68101, 75361.

Comments:

A few examples of how the formula looks like for k and n from 1 to 4:

For $k = 1$ the formula becomes $42*n + 7$.

For $k = 2$ the formula becomes $156*n + 13$.

For $k = 3$ the formula becomes $342*n + 19$.

For $k = 4$ the formula becomes $600*n + 25$.

For $n = 1$ the formula generates a perfect square.

For $n = 2$ the formula becomes $(6*k + 1)*(12*k + 1)$ and were found the following Poulet numbers: 2701, 8911, 10585, 18721, 49141 etc.

For $n = 3$ the formula becomes $(6*k + 1)*(18*k + 1)$ and were found the following Poulet numbers: 2821, 4033, 5461, 15841, 31621, 68101 etc.

For $n = 4$ the formula becomes $(6*k + 1)*(24*k + 1)$. See the sequence A182123 in OEIS.

Note: The formula is equivalent to Poulet numbers of the form $p*(n*p - n + 1)$, where p is of the form $6*k + 1$. From the first 68 Poulet numbers just 7 of them (7957, 23377, 33153, 35333, 42799, 49981, 60787) can't be written as $p*(n*p - n + 1)$, where p is of the form $6*k \pm 1$ and k, n are integers different from 0.

5. Poulet numbers of the form $(6*k - 1)*((6*k - 2)*n + 1)$, where k, n are integers different from 0

First 37 terms of the sequence (A210993 in OEIS): 341, 561, 645, 1105, 1905, 2047, 2465, 3277, 4369, 4371, 6601, 8321, 8481, 10585, 11305, 12801, 13747, 13981, 15709, 16705, 18705, 19951, 23001, 30889, 31417, 34945, 39865, 41041, 41665, 55245, 60701, 62745, 65077, 68101, 72885, 74665, 75361.

Comments:

A Poulet number can be written in more than one way in this form: e.g. $561 = (6*2 - 1)*((6*2 - 2)*5 + 1) = (6*3 - 1)*((6*3 - 2)*2 + 1)$.

Few examples of how the formula looks like for k and n from 1 to 4:

For $k = 1$ the formula becomes $20*n + 5$ and generates all the Poulet numbers divisible by 5 from the sequence above (beside 645, all of them have another solutions beside $k = 1$).

For $k = 2$ the formula becomes $110*n + 11$ and generates the Poulet numbers: 341, 561 etc.

For $k = 3$ the formula becomes $272*n + 17$ and generates the Poulet numbers: 561, 1105, 2465, 4369 etc.

For $k = 4$ the formula becomes $506*n + 23$ and generates the Poulet numbers: 2047, 6601 etc.

For $n = 1$ the formula generates a perfect square.

For $n = 2$ the formula becomes $3*(6*k - 1)*(4*k - 1)$ and were found the following Poulet numbers: 561 etc.

For $n = 3$ the formula becomes $(6*k - 1)*(18*k - 5)$ and were found the following Poulet numbers: 341, 2465, 8321, 83333 etc.

For $n = 4$ the formula becomes $(6*k - 1)*(24*k - 7)$ and were found the following Poulet numbers: 1105, 2047, 3277, 6601, 13747, 16705, 19951, 31417, 74665, 88357 etc.

Note: The formula is equivalent to Poulet numbers of the form $p*(n*p - n + 1)$, where p is of the form $6*k - 1$. From the first 68 Poulet numbers just 26 of them (1387, 2701, 2821, 4033, 4681, 5461, 7957, 8911, 10261, 13741, 14491, 18721, 23377, 29341, 31609, 31621, 33153, 35333, 42799, 46657, 49141, 49981, 57421, 60787, 63973, 65281) can't be written as $p*(n*p - n + 1)$, where p is of the form $6*k - 1$ and k, n are integers different from 0.

6. Poulet numbers of the form $7200*n^2 + 8820*n + 2701$

First 29 terms of the sequence (A214016 in OEIS): 2701, 18721, 49141, 93961, 226801, 314821, 534061, 665281, 1537381, 1755001, 1987021, 2233441, 3059101, 3363121, 4014361, 5489641, 6313681, 8134561, 9131401, 10185841, 13073941, 13694761, 18443701, 21474181, 27331921, 30058381, 30996001, 32914441, 34890481.

Comments:

Poulet numbers were obtained for the following values of n : 0, 1, 2, 3, 5, 6, 8, 9, 14, 15, 16, 17, 20, 21, 23, 27, 29, 33, 35, 37, 42, 43, 50, 54, 61, 64, 65, 67, 69.

Conjecture:

There are infinite many Poulet numbers of the form $7200*n^2 + 8820*n + 2701$.

7. Poulet numbers of the form $144*n^2 + 222*n + 85$

First 29 terms of the sequence (A214017 in OEIS): 1105, 2047, 3277, 6601, 13747, 16705, 19951, 31417, 74665, 88357, 275887, 514447, 604117, 642001, 741751, 916327, 1293337, 1433407, 1520905, 2205967, 2387797, 2976487, 2976487, 3316951, 3539101, 4005001, 4101637, 4863127, 5575501, 8209657.

Comments:

Poulet numbers were obtained for the following values of n : 2, 3, 4, 6, 9, 10, 11, 14, 22, 43, 59, 64, 66, 71, 79, 94, 99, 102, 123, 128, 143, 151, 156, 166, 168, 183, 196, 238.

Conjecture:

There are infinite many Poulet numbers of the form $144*n^2 + 222*n + 85$.

8. Poulet numbers of the form $8*p*n + p^2$, where p is prime

First 29 terms of the sequence (A218483 in OEIS): 561, 1105, 1729, 1905, 2465, 4033, 4369, 4681, 6601, 8321, 8481, 10585, 11305, 12801, 15841, 16705, 18705, 18721, 23001, 23377, 25761, 30121, 30889, 31417, 31609, 33153, 34945, 35333, 39865, 41041, 41665, 46657, 52633, 62745, 65281, 74665, 75361, 83665, 85489.

Comments:

For $p = 5$ the formula becomes $40*n + 25$. From the first 15 pseudoprimes divisible by 5, 12 are of the form $40*n + 25$ (beside 3 of them which are of the form $40*n + 5$).
Conjecture: there are no pseudoprimes to base 2 of the form $40*n + 15$.

Note: it can be seen that a pseudoprime can be written in this formula in more than one way:
e.g. $561 = 8*3*23 + 3^2 = 8*11*5 + 11^2 = 8*17*2 + 17^2$ or $1905 = 8*3*79 + 3^2 = 8*5*47 + 5^2$.

Conjecture 1:

If a Fermat pseudoprime to base 2 can be written as $8*p*n + p^2$, where n is an integer number and p one of it's prime factors, than can be written this way for any of it's prime factors. Checked for all pseudoprimes from the sequence above.

Conjecture 2:

If a Fermat pseudoprime to base 2 with two prime factors can be written as $8*p_1*n + p_1^2$, where n is a natural number and p_1 one of it's two prime factors, than can be written too as $8*p_2*(-n) + p_2^2$, where p_2 is the other prime factor. Checked for $4033 = 37*109(n = 9)$, $4369 = 17*257(n = 30)$, $4681 = 31*151(n = 15)$, $8321 = 53*157(n = 13)$, $18721 = 97*193(n = 12)$, $23377 = 97*241(n = 18)$, $31417 = 89*353(n = 33)$, $31609 = 73*433(n = 45)$, $65281 = 97*673(n = 72)$, $85489 = 53*1613(n = 195)$.

Conjecture 3:

If a Fermat pseudoprime to base 2 can't be written as $8*p*n + p^2$, where n is an integer number and p one of it's prime factors, than can't be written this way for any of it's prime factors. Checked for the following pseudoprimes: 341, 645, 1387, 2047, 2701, 2821, 3277, 4371, 5461, 7957, 10261, 13741, 13747, 13981, 14491, 15709, 19951, 29341, 31621, 42799, 49141, 49981, 55245, 60701, 60787, 63973, 65077, 68101, 72885, 80581, 83333.

Note: from the first 72 pseudoprimes, 39 can be written this way.

9. Poulet numbers of the form $(n^2 + 2*n)/3$

First 33 terms of the sequence (A216170 in OEIS): 341, 645, 2465, 2821, 4033, 5461, 8321, 15841, 25761, 31621, 68101, 83333, 162401, 219781, 282133, 348161, 530881, 587861, 653333, 710533, 722261, 997633, 1053761, 1082401, 1193221, 1246785, 1333333, 1357441, 1398101, 1489665, 1584133, 1690501, 1735841.

Comments:

The corresponding values of n : 31, 43, 85, 91, 109, 127, 157, 217, 277, 307, 451, 499, 697, 811, 919, 1021, 1261, 1327, 1399, 1459, 1471, 1729, 1777, 1801, 1891, 1933, 1999, 2017, 2047, 2113, 2177, 2251.

The formula can be generalised this way: Fermat pseudoprimes to base 2 of the form $(n^m + m*n)/(m + 1)$.

For $m = 3$ the formula becomes $(n^3 + 3*n)/4$ and were obtained the following Poulet numbers: 341, 1729, 188461, 228241, 1082809 (for $n = 11, 19, 91, 97, 163$).

Conjecture:

For any m natural, $m > 1$, there exist a series with infinite many Fermat pseudoprimes to base 2, P , formed this way: $P = (n^m + m*n)/(m+1)$.

10. Poulet numbers that can be written as $2*p^2 - p$, where p is also a Poulet number

First 22 terms of the sequence (A215343 in OEIS): 831405, 5977153, 15913261, 21474181, 38171953, 126619741, 210565981, 224073865, 327718401, 377616421, 390922741, 558097345, 699735345, 1327232481, 1999743661, 4996150741, 8523152641, 11358485281, 13999580785, 15613830541, 17657245081, 20442723301.

Comments:

The correspondent p for the numbers from the sequence above: 645, 1729, 2821, 3277, 4369, 7957, 10261, 10585, 12801, 13741, 13981, 16705, 18705, 25761, 31621, 49981, 65281, 75361, 83665, 88357, 93961, 101101.

Note that for 22 from the first 80 Poulet numbers we obtained thru this formula another Poulet numbers!

The formula could be generalised this way: Poulet numbers that can be written as $(n + 1)*p^2 - n*p$, where n is natural, $n > 0$, and p is another Poulet number.

For $n = 1$ that formula becomes the formula set out for the sequence above.

For $n = 2$ that formula becomes $3*p^2 - 2*p$ and were obtained the following Poulet numbers: 348161, 1246785 (for $p = 341, 645$) etc.

For $n = 3$ that formula becomes $4*p^2 - 3*p$ and were obtained the following Poulet numbers: 119273701 (for $p = 5461$) etc.

For $n = 4$ that formula becomes $5*p^2 - 4*p$ and were obtained the following Poulet numbers: 2077545, 9613297 (for $p = 645, 1387$) etc.

Conjecture:

There are infinite many Poulet numbers that can be written as $(n + 1)*p^2 - n*p$, where n is natural, $n > 0$, and p is another Poulet number.

Note: Finally, considering, *e.g.*, that for the Poulet number 645 were obtained Poulet numbers for $n = 1, 2, 4$ (i.e. 831405, 1246785, 2077545), yet another conjecture.

Conjecture: For any Poulet number p there are infinite many Poulet numbers that can be written as $(n + 1)*p^2 - n*p$, where n is natural, $n > 0$.

11. Poulet numbers of the form $m*n^2 + (11*m - 23)*n + 19*m - 49$

First 37 terms of the sequence (A215326 in OEIS): 341, 645, 1105, 1387, 2047, 2465, 2821, 3277, 4033, 5461, 6601, 7957, 8321, 11305, 13747, 15841, 16705, 19951, 23001, 25761, 30889, 31417, 31621, 39865, 41665, 49981, 65077, 68101, 74665, 83333, 83665, 85489, 88357, 90751, 107185, 137149, 158369.

Comments:

The solutions (m,n) for the Poulet numbers from the sequence above are: (3, 9); (3, 13); (4, 14); (4, 16); (9, 11) and (4, 20); (3, 27); (3, 29); (4, 26); (3, 35); (290, 0) and (3, 41); (350, 0) and (4, 38); (259, 1); (3, 51); (367, 1); (4, 56); (94, 8) and (3, 71); (4, 62); (329, 3) and (4, 68); (379, 3); (3, 91); (182, 8); (319, 5) and (4, 86); (3, 101); (888, 2); (928, 2) and (66, 20); (43, 29); (659, 5); (3, 149); (438, 8) and (4, 134); (3, 165); (4406, 0) and (4, 142); (4502, 0); (4, 146); (4, 148); (2384, 2) and (38, 48); (1387, 5); (5111, 1).

Few examples of how the formula looks like for m from 3 to 4.

For $m = 3$ the formula becomes $3*n^2 + 10*n + 8$ and were found the following Poulet numbers: 341, 645, 2465, 2821, 4033, 5461, 8321, 15841, 25761, 31621, 68101, 83333 etc. (12 from the first 100 Poulet numbers can be written this way!).

For $m = 4$ the formula becomes $4*n^2 + 21*n + 27$ and were found the following Poulet numbers: 1105, 1387, 2047, 3277, 6601, 13747, 16705, 19951, 31417, 83665, 88357, 90751 etc. (12 from the first 100 Poulet numbers can be written this way!).

Note: For $n = -2$ the formula becomes $(m - 3)$ and for $n = -9$ becomes $(m + 158)$ so all the Poulet numbers have at least these integer solutions to this formula.

Note: For $n = -1$ the formula becomes $(9*m - 26)$ and 37 from the first 100 Poulet numbers can be written this way! That means that for more than a third from Poulet numbers P checked is true that $(P + 8)$ is divisible by 9 (for comparison, this relation is true for just 14 from the first 100 primes).

12. Poulet numbers that can be written as $(p^2 + 2p)/3$, where p is also a Poulet number

First 22 terms of the sequence (A216276 in OEIS): 997633, 1398101, 3581761, 26474581, 37354465, 63002501, 70006021, 82268033, 93030145, 561481921, 804978721, 1231726981, 2602378721, 12817618945, 15516020833, 16627811905, 22016333333, 25862624705, 53707855201, 67220090785, 95074073281, 144278347201.

Comments:

The corresponding values of the Fermat pseudoprime p : 1729, 2047, 3277, 8911, 10585, 13747, 14491, 15709, 16705, 41041, 49141, 60787, 88357, 196093, 215749, 223345, 256999, 278545, 401401, 449065, 657901.

Note that for 22 from the first 200 Fermat pseudoprimes to base 2 were obtained also Fermat pseudoprimes to base 2 through this formula!

Conjecture 1:

For any Fermat pseudoprime to base 2, p_1 , there exist infinite many Fermat pseudoprimes to base 2, p_2 , formed this way: $p_2 = (p_1^n + n \cdot p_1)/(n + 1)$, where n natural, $n > 1$.

Conjecture 2:

For any Carmichael number, c_1 , there exist infinite many Carmichael numbers, c_2 , formed this way: $c_2 = (c_1^n + n \cdot c_1)/(n + 1)$, where n natural, $n > 1$. Note that, in the sequence above, from Fermat pseudoprimes to base 2 that are also Carmichael numbers (1729, 8911, 10585, 41041, 278545, 449065) were obtained too Carmichael numbers.

13. Poulet numbers that can be written as $p^{2n} - p^n + p$, where p is also a Poulet number

First 22 terms of the sequence (A217835 in OEIS): 348161, 831405, 1246785, 1275681, 2077545, 2513841, 5977153, 9613297, 13333441, 13823601, 18137505, 19523505, 21474181, 21880801, 37695505, 38171953, 44521301, 47734141, 54448153, 72887585, 75151441, 95423329.

Comments:

The numbers from sequence are the all numbers of this type up to 10^8 .

The corresponding (p, n) : (341, 3), (645, 2), (645, 3), (341, 11), (645, 5), (561, 8), (1729, 2), (1387, 5), (341, 120), (561, 44), (1905, 5), (645, 47), (3277, 2), (2701, 3), (2047, 9), (4369, 2), (341, 384), (2821, 6), (2047, 13), (2465, 12), (3277, 7), (4369, 5).

Conjecture 1:

For any Fermat pseudoprime p to base 2 there are infinitely many Fermat pseudoprimes to base 2 equal to $p^{2^n} - p^n + p$, where n is natural.

Note: See the sequence A215343: the generalised formula from there is $p^{2^n} - p^n + p^2$, which suggests an extrapolated formula for obtaining some Fermat pseudoprime to base 2 from other: $p^{2^n} - p^n + p^k$.

Conjecture 2:

For any Fermat pseudoprime p to base 2 and any k natural, $k > 0$, there are infinitely many Fermat pseudoprimes to base 2 equal to $p^{2^n} - p^n + p^k$, where n is natural.

14. Primes of the form $(24*p + 1)/5$, where p is a Poulet number

First 22 terms of the sequence (A218010 in OEIS): 1637, 2693, 20981, 22469, 40709, 42773, 49253, 65957, 69557, 123653, 140837, 235877, 451013, 623621, 626693, 716549, 1095557, 1370597, 1634693, 2108597, 2459813, 2548229, 2554421, 2563493, 2869781, 3534197, 3669557, 3755237, 4093637, 4337429, 4567109.

Comments:

The corresponding values of p : 341, 561, 4371, 4681, 8481, 8911, 10261, 13741, 14491, 25761, 29341, 49141, 93961, 129921, 130561, 149281, 228241, 285541, 340561, 439291, 512461, 530881, 532171, 534061, 597871, 736291, 764491, 782341, 852841, 903631, 951481.

It is notable that, from the first 128 natural solutions of this equation $((24*p + 1)/5)$, where p is Fermat pseudoprime to base 2), 31 are primes (the ones from the sequence above), 51 are products (not necessary squarefree) of two prime factors and 41 are products of three prime factors; only 5 of them are products of four prime factors.

It is notable yet another relation between numbers of the form $(24*n + 1)/5$, where n natural, and Fermat pseudoprimes:

Conjecture:

There is no absolute Fermat pseudoprime m for which $n = (5*m - 1)/24$ is a natural number (checked for the first 300 Carmichael numbers; if true, then the formula is a criterion to separate pseudoprimes at least from a subset of primes, because there are 37 primes m from the first 300 primes for which $n = (5*m - 1)/24$ is a natural number).

15. The smallest m for which the n -th Carmichael number can be written as $p^{2*(m + 1)} - p*m$

First 60 terms of the sequence (A213812 in OEIS): 1, 3, 4, 2, 2, 3, 1, 1, 2, 7, 24, 4, 4, 7, 47, 80, 9, 1, 23, 2, 46, 15, 24, 21, 24, 1, 1, 76, 8, 21, 16, 14, 6, 2, 150, 16, 8, 16, 3, 156, 36, 232, 2, 13, 10, 788, 40, 25, 2, 4, 123, 12, 44, 16, 8, 207, 226, 462, 92, 6.

Comments:

The corresponding values of p are (we write the Carmichael number in brackets): 17(561), 17(1105), 19(1729), 29(2465), 31(2821), 41(6601), 67(8911), 73(10585), 73(15841), 61(29341), 41(41041), 97(46657), 103(52633), 89(62745), 37(63973), 31(75361), 101(101101), 241(115921), 73(126217), 233(162401), 61(172081), 109(188461), 101(252601), 113(278545), 109(294409), 397(314821), 409(334153), 67(340561), 211(399001), 137(410041), 163(449065), 181(488881), 271(512461), 421(530881), 61(552721), 197(656601), 271(658801), 199(670033), 433(748657), 73(825265), 151(838201), 61(852841), 577(997633), 271(1024651), 307(1033669), 37(1050985), 163(1082809), 211(1152271), 631(1193221), 541(1461241), 113(1569457), 353(1615681), 199(1773289), 331(1857241), 461(1909001), 101(2100901), 97(2113921), 73(2433601), 163(2455921), 599(2508013).

Note: Any Carmichael number C can be written as $C = p^{2*(n+1)} - p^n$, where p is any prime divisor of C (it can be seen that the smallest n is obtained for the biggest prime divisor). The formula $C = p^{2*(n+1)} - p^n$ is equivalent to $C = p^{2*m} - p^{m-1} = p^{2*m} - p^m + p$, equivalent to $p^2 - p$ divides $C - p$, which is a direct consequence of Korselt's criterion. It can be shown from $p - 1$ divides $C - 1$ not that just $p^2 - p$ divides $C - p$ but even that $p^2 - p$ divides $C - p^k$ (if $C > p^k$) or $p^k - C$ (if $p^k > C$) which leads to the generic formula for a Carmichael number: $C = p^k + n*p^2 - n*p$ (if $C > p^k$) or $C = p^k - n*p^2 + n*p$ (if $p^k > C$) for any p prime divisor of C and any k natural number.

Note: The formulas generated giving values of k seems to be very useful in the study of Fermat pseudoprimes; also, the composite numbers C for which the equation $C = p^k - n*p^2 + n*p$ gives, over the integers, as solutions, all their prime divisors, for a certain k , deserve further study.

16. Carmichael numbers of the form $(30*k + 7)*(60*k + 13)*(150*k + 31)$

First 18 terms of the sequence (A182085 in OEIS): 2821, 488881, 288120421, 492559141, 776176261, 1632785701, 3835537861, 6735266161, 9030158341, 21796387201, 167098039921, 288374745541, 351768558961, 381558955141, 505121232001, 582561482161, 915245066821, 2199733160881.

Conjecture:

The number $C = (30*k + 7)*(60*k + 13)*(150*k + 31)$ is a Carmichael number if (but not only if) $30*k + 7$, $60*k + 13$ and $150*k + 31$ are all three prime numbers.

Note: We got Carmichael numbers with three prime divisors for $k = 0, 1, 10, 12, 18, 24, 32, 43, 85, 102, 123, 129, 150, 201, 207, 256$.

We got Carmichael numbers with more than three prime divisors for $n = 14, 29, 109, 112$.

Note: All these numbers can be written as well as $N = (n + 1)*(2*n + 1)*(5*n + 1)$, where $n = 30*k + 6$.

17. Carmichael numbers of the form $C = (30*n - 7)*(90*n - 23)*(300*n - 79)$

First 16 terms of the sequence (A182132 in OEIS): 340561, 4335241, 153927961, 542497201, 1678569121, 2598933481, 25923026641, 63280622521, 88183003921, 155999871721, 209850699601, 240893092441, 274855097881, 380692027321, 733547013841, 1688729866321.

Conjecture:

The number $C = (30*n - 7)*(90*n - 23)*(300*n - 79)$ is a Carmichael number if (but not only if) $30*n - 7$, $90*n - 23$ and $300*n - 79$ are all three prime numbers.

Note: We got Carmichael numbers with three prime divisors for $n = 2, 9, 15, 32, 43, 48, 58, 64, 67, 78, 97, 128$.

We got Carmichael numbers with more than three prime divisors for $n = 1, 6, 13, 70$.

18. Carmichael numbers of the form $C = (30*n - 17)*(90*n - 53)*(150*n - 89)$

First 17 terms of the sequence (A182133 in OEIS): 29341, 1152271, 34901461, 64377991, 775368901, 1213619761, 4562359201, 8346731851, 9293756581, 48874811311, 68926289491, 72725349421, 147523256371, 235081952731, 672508205281, 707161856941, 779999961061.

Conjecture:

The number $C = (30*n + 13)*(90*n + 37)*(150*n + 61)$ is a Carmichael number if (but not only if) $30*n + 13$, $90*n + 37$ and $150*n + 61$ are all three prime numbers.

Note: We got Carmichael numbers with three prime divisors for $n = 0, 1, 5, 12, 14, 12, 27, 28, 49, 55, 56, 71, 83, 121, 125$.

We got Carmichael numbers with more than three prime divisors for $n = 4$ and $n = 119$.

19. Carmichael numbers $C = (60*k + 13)*(180*k + 37)*(300*k + 61)$

First 16 terms of the sequence (A182416 in OEIS): 29341, 34901461, 775368901, 1213619761, 4562359201, 9293756581, 72725349421, 672508205281, 707161856941, 779999961061, 983598759361, 1671885346141, 1800095194261, 3459443867461, 6513448976101, 8369282635561.

Conjecture:

$N = (60*k + 13)*(180*k + 37)*(300*k + 61)$ is a Carmichael number if (but not only if) $60*k + 13$, $180*k + 37$ and $300*k + 61$ are all three prime numbers.

Note: We got Carmichael numbers with three prime divisors for $k = 0, 6, 7, 11, 14, 28, 60, 62, 80, 102, 126, 137, 139, 157, 171$.

We got Carmichael numbers with more than three prime divisors for $k = 2, 59, 67, 82$.

Note: We can see that $13 = 7*2 - 1$, $37 = 7*6 - 5$ and $61 = 7*10 - 9$, while $60 = 30*2$, $180 = 30*6$ and $300 = 30*10$; we also have Carmichael numbers that can be written as $(30*n - 11)*(60*n - 23)*(150*n - 59)$, for instance 63973, or as $(30*n - 7)*(90*n - 23)*(300*n - 79)$, for instance 340561; we can see that, this time, $23 = 11*2 + 1$, $59 = 11*5 + 4$, $23 = 7*3 + 2$ and $79 = 7*10 + 9$, while $60 = 30*2$, $150 = 30*5$, $90 = 30*3$ and $300 = 30*10$.

Observation:

Many Carmichael numbers, not only with three prime divisors, can be written in one of the following two forms: $C = ((30*a*m - (a*p + a - 1))*((30*b*m - (b*p + b - 1))*((30*c*m - (c*p + c - 1)))$ or $C = ((30*a*m + (a*p - a + 1))*((30*b*m + (b*p - b + 1))*((30*c*m + (c*p - c + 1)))$, where $p, a*p + a - 1, b*p + b - 1$ and $c*p + c - 1$ are all (four or three, if $a = 1$) primes (in the first case) or $p, a*p - a + 1, b*p - b + 1$ and $c*p - c + 1$ are all primes (in the second case).

20. Carmichael numbers $C = (30*n - 29)*(60*n - 59)*(90*n - 89)*(180*n - 179)$

First 9 terms of the sequence (A182088 in OEIS): 31146661, 2414829781, 192739365541, 197531244744661, 741700610203861, 973694665856161, 2001111155103061, 2278278996452641, 4271903575869601.

Conjecture:

The number $C = (30*n - 29)*(60*n - 59)*(90*n - 89)*(180*n - 179)$ is a Carmichael number if (but not only if) $30*n - 29, 60*n - 59, 90*n - 89$ and $180*n - 179$ are all four prime numbers.

Note: We got Carmichael numbers with three prime divisors for $n = 10, 52, 77, 143$.

We got Carmichael numbers with more than three prime divisors for $n = 2, 4, 72, 92, 95, 111$.

21. Carmichael numbers $C = (330*k + 7)*(660*k + 13)*(990*k + 19)*(1980*k + 37)$

First 11 terms of the sequence (A182089 in OEIS): 63973, 461574735553, 7103999557333, 35498632881313, 111463190499493, 271061745643873, 560604728986453, 1036648928639233, 1765997490154213, 2825699916523393, 4303052068178773.

Conjecture:

The number $C = (330k + 7)(660k + 13)(990k + 19)(1980k + 37)$ is a Carmichael number if $330k + 7$, $660k + 13$, $990k + 19$ and $1980k + 37$ are all four prime numbers.

22. Carmichael numbers of the form $C = (30*n - p)(60*n - (2*p + 1))(90*n - (3*p + 2))$, where p , $2*p + 1$, $3*p + 2$ are all three primes

First 17 terms of the sequence (A182087 in OEIS): 1729, 172081, 294409, 1773289, 4463641, 56052361, 118901521, 172947529, 216821881, 228842209, 295643089, 798770161, 1150270849, 1299963601, 1504651681, 1976295241, 2301745249.

Comments:

These numbers can be reduced to only two possible forms: $C = (30*n - 23)(60*n - 47)(90*n - 71)$ or $C = (30*n - 29)(60*n - 59)(90*n - 89)$. In the first form, for the particular case when $30*n - 23$, $60*n - 47$ and $90*n - 71$ are all three prime numbers, we obtain the Chernick numbers of the form $10*m + 1$ (for $k = 5*n - 4$ we have $C = (6*k + 1)(12*k + 1)(18*k + 1)$). In the second form, for the particular case when $30*n - 29$, $60*n - 59$ and $90*n - 89$ are all three prime numbers, we obtain the Chernick numbers of the form $10*m + 9$ (for $k = 5*n - 5$ we have $C = (6*k + 1)(12*k + 1)(18*k + 1)$). So the Chernick numbers can be divided into two categories: Chernick numbers of the form $(30*n + 7)(60*n + 13)(90*n + 19)$ and Chernick numbers of the form $(30*n + 1)(60*n + 1)(90*n + 1)$.

23. Carmichael numbers of the form $C = p(2*p - 1)(3*p - 2)(6*p - 5)$, where p is prime

First 15 terms of the sequence (A182518 in OEIS): 63973, 31146661, 703995733, 21595159873, 192739365541, 461574735553, 3976486324993, 10028704049893, 84154807001953, 197531244744661, 741700610203861, 973694665856161, 2001111155103061, 3060522900274753, 3183276534603733.

Comments:

We get Carmichael numbers with four prime divisors for $p = 7, 271, 337, 727, 1237, 1531, 2281, 3037, 3067$.

We get Carmichael numbers with more than four prime divisors for $p = 31, 67, 157, 577, 2131, 2731, 3301$.

Note: We can see that p , $2*p - 1$, $3*p - 2$ and $6*p - 5$ can all four be primes only for $p = 6*k + 1$ (for $p = 6*k + 5$, we get $2*p - 1$ divisible by 3), so in that case the formula is equivalent to $C = (6*k + 1)(12*k + 1)(18*k + 1)(36*k + 1)$.

**24. Carmichael numbers of the form $C = p \cdot (2^p - 1) \cdot (n \cdot (2^p - 2) + p)$,
where p and $2^p - 1$ are primes**

First 29 terms of the sequence (A182207 in OEIS): 1729, 2821, 41041, 63973, 101101, 126217, 172081, 188461, 294409, 399001, 488881, 512461, 670033, 748657, 838201, 852841, 997633, 1033669, 1050985, 1082809, 1461241, 2100901, 2113921, 2628073, 4463641, 4909177, 7995169, 8341201, 8719309.

Conjecture:

Any Carmichael number C divisible by p and $2^p - 1$ (where p and $2^p - 1$ are prime numbers) can be written as $C = p \cdot (2^p - 1) \cdot (n \cdot (2^p - 2) + p)$.

Checked for the first 30 Carmichael numbers divisible by p and $2^p - 1$.

Note: We can see how easy is to obtain Carmichael numbers with this formula:

For $n = 1$ we get $p \cdot (2^p - 1) \cdot (3^p - 2)$ and Carmichael numbers 1729, 172081, 294409 etc.

For $n = 2$ we get $p \cdot (2^p - 1) \cdot (5^p - 4)$ and Carmichael numbers 2821, 63973, 488881 etc.

For $n = 3$ we get $p \cdot (2^p - 1) \cdot (7^p - 6)$ and Carmichael numbers 399001, 53711113 etc.

25. Carmichael numbers of the form $n \cdot (2^n - 1) \cdot (p^n - p + 1) \cdot (2^p \cdot n - 2^p + 1)$, where p is odd

First 17 terms of the sequence (A212882 in OEIS): 63973, 172081, 31146661, 167979421, 277241401, 703995733, 1504651681, 2414829781, 117765525241, 192739365541, 461574735553, 881936608681, 2732745608209, 3145699746793, 3307287048121, 3976486324993, 7066238244481.

Comments:

The following Carmichael numbers are of the form $n \cdot (2^n - 1) \cdot (3^n - 2) \cdot (6^n - 5)$: 63973, 31146661, 703995733, 2414829781, 192739365541, 461574735553, 3976486324993.

The following Carmichael numbers are of the form $n \cdot (2^n - 1) \cdot (5^n - 4) \cdot (10^n - 9)$: 172081, 881936608681, 3307287048121, 8916642713161.

The following Carmichael number is of the form $n \cdot (2^n - 1) \cdot (7^n - 6) \cdot (14^n - 13)$: 167979421.

The following Carmichael number is of the form $n \cdot (2^n - 1) \cdot (9^n - 8) \cdot (18^n - 17)$: 277241401.

The following Carmichael number is of the form $n \cdot (2^n - 1) \cdot (11^n - 10) \cdot (22^n - 21)$: 9924090391909.

The following Carmichael number is of the form $n \cdot (2^n - 1) \cdot (15^n - 14) \cdot (30^n - 29)$: 7932245192461.

The following Carmichael number is of the form $n \cdot (2^n - 1) \cdot (17^n - 16) \cdot (34^n - 33)$: 3145699746793.

The following Carmichael numbers are of the form $n*(2^n - 1)*(21^n - 20)*(42^n - 41)$: 1504651681, 117765525241, 2732745608209.

The following Carmichael number is of the form $n*(2^n - 1)*(23^n - 22)*(46^n - 45)$: 7066238244481.

For $p = 13$ and $p = 19$, there is no Carmichael number up to 10^{13} .

There is not any other Carmichael number of this form, for p from 3 to 23, up to 10^{13} .

Conjecture:

For any odd number p we have an infinite number of Carmichael numbers of the form $n*(2^n - 1)*(p^n - p + 1)*(2^p n - 2^p + 1)$.

Note: Many numbers of the form $n*(2^n - 1)*(p^n - p + 1)*(2^p n - 2^p + 1)$, not divisible by 2, 3 or 5, where p is odd or even, are squarefree and respects the Korselt's criterion for many of their prime divisors or are not squarefree but respects the Korselt's criterion sometimes even for all their divisors (but we didn't find Carmichael numbers when p is even).

26. Carmichael numbers of the form $3^n*(9^n + 2)*(18^n - 1)$, where n is odd

First 29 terms of the sequence (A213071 in OEIS): 561, 13833, 62745, 170625, 360801, 656601, 1081353, 1658385, 2411025, 3362601, 4536441, 5955873, 7644225, 9624825, 11921001, 14556081, 17553393, 20936265, 24728025, 28952001, 33631521, 38789913, 44450505, 50636625, 57371601, 64678761, 72581433, 81102945, 90266625.

Comments:

Carmichael numbers (561, 62745, 656601, 11921001, 174352641) were obtained for the following values of n : 1, 5, 11, 29, 71.

Note: The sequence can be generalized this way: $C = p^n*(3^p n + 2)*(6^p n - 1)$, where p is prime.

Few examples for p from 5 to 23:

For $p = 5$ the formula becomes $5^n*(15^n + 2)*(30^n - 1)$ and were obtained the following Carmichael numbers: 2465, 62745, 11119105, 3249390145 (for $n = 1, 3, 17, 113$);

For $p = 7$ the formula becomes $7^n*(21^n + 2)*(42^n - 1)$ and were obtained the following Carmichael numbers: 6601 (for $n = 1$);

For $p = 11$ the formula becomes $11^n*(33^n + 2)*(66^n - 1)$ and were obtained the following Carmichael numbers: 656601 (for $n = 3$);

For $p = 13$ the formula becomes $13^n*(39^n + 2)*(78^n - 1)$ and were obtained the following Carmichael numbers: 41041, 271794601 (for $n = 1, 21$);

For $p = 17$ the formula becomes $17^n*(51^n + 2)*(102^n - 1)$ and were obtained the following Carmichael numbers: 11119105, 2159003281 (for $n = 5$);

For $p = 19$ the formula becomes $19^n*(57^n + 2)*(114^n - 1)$ and were obtained the following Carmichael numbers: 271794601 (for $n = 13$);

For $p = 23$ the formula becomes $23^n \cdot (69n + 2) \cdot (138n - 1)$ and were obtained the following Carmichael numbers: 5345340001 (for $n = 29$).

27. Carmichael numbers that have only prime divisors of the form $10k+1$

First 28 terms of the sequence (A212843 in OEIS): 252601, 399001, 512461, 852841, 1193221, 1857241, 1909001, 2100901, 3828001, 5049001, 5148001, 5481451, 6189121, 7519441, 8341201, 9439201, 10024561, 10837321, 14676481, 15247621, 17236801, 27062101, 29111881, 31405501, 33302401, 34657141, 40430401, 42490801.

Conjecture:

All Carmichael numbers C (not only with three prime divisors) of the form $10^n + 1$ that have only prime divisors of the form $10k+1$ can be written as $C = (30^a + 1) \cdot (30^b + 1) \cdot (30^c + 1)$, $C = (30^a + 11) \cdot (30^b + 11) \cdot (30^c + 11)$, or $C = (30^a + 1) \cdot (30^b + 11) \cdot (30^c + 11)$. In other words, there are no numbers of the form $C = (30^a + 1) \cdot (30^b + 1) \cdot (30^c + 11)$.

28. Carmichael numbers divisible by a smaller Carmichael number

First 29 terms of the sequence (A214758 in OEIS): 63973, 126217, 172081, 188461, 278545, 748657, 997633, 1773289, 5310721, 8719921, 8830801, 9890881, 15888313, 18162001, 26474581, 26921089, 31146661, 36121345, 37354465, 41471521, 93614521, 93869665, 101957401, 120981601, 151813201.

Comments:

Carmichael numbers by which the numbers from sequence are divisible: 1729, 1729, 2821, 1729, 2465, 1729, 1729, 8911, 29341, 6601, 8911, 41041, 8911, 75361, 8911, 46657, 2821 and 172081, 1105, 10585, 2821 and 172081, 41041, 2465, 1729 and 188461, 46657, 252601.

Note: A Carmichael number can be divisible by more than one Carmichael number: e.g. 31146661, 41471521, 101957401.

A subsequence of this sequence contains the numbers C_1 (and another subsequence the numbers C_3) that can be written as $C_1 = (C_2 + C_3)/2$, where C_1 , C_2 and C_3 are Carmichael numbers and C_1 and C_3 are both divisible by C_2 (e.g. $63973 = (1729 + 126217)/2$; $93614521 = (41041 + 187188001)/2$).

Conjecture:

A Carmichael number C_1 can be written as $C_1 = (C_2 + C_3)/2$, where C_2 and C_3 are also Carmichael numbers, only if both C_1 and C_3 are divisible by C_2 .

29. Carmichael numbers divisible by 1729

First 29 terms of the sequence (A212920 in OEIS): 1729, 63973, 126217, 188461, 748657, 997633, 101957401, 509033161, 705101761, 1150270849, 1854001513, 2833846561, 7103660473, 8039934721, 9164559313, 10298458261, 14530739041, 23597511301, 41420147041, 49923611101, 50621055121, 55677010753, 65039877721.

Conjecture:

If $m \cdot 126 + n = 1729$, $m \cdot 126 > n$, then exists a series with infinite many Carmichael terms of the form $C \bmod m \cdot 234 = n$.

Verifying the conjecture:

- (1) For $m < 7$ we have $m \cdot 126 < n$;
- (2) For $m = 7$ the formula becomes $C \bmod 882 = 847$ and were obtained the Carmichael numbers: 1729, 15841, 1033669 etc.;
- (3) For $m = 8$ the formula becomes $C \bmod 1008 = 721$ and were obtained the Carmichael numbers: 1729, 15841, 41041, 172081, 670033, 748657, 825265, 997633 etc.;
- (4) For $m = 9$ the formula becomes $C \bmod 1134 = 595$ and were obtained the Carmichael numbers: 1729, 1033669 etc.;
- (5) For $m = 10$ the formula becomes $C \bmod 1260 = 469$ and were obtained the Carmichael numbers: 1729, 1033669 etc.;
- (6) For $m = 11$ the formula becomes $C \bmod 1386 = 343$ and were obtained the Carmichael numbers: 1729, 1082809 etc.;
- (7) For $m = 12$ the formula becomes $C \bmod 1512 = 217$ and were obtained the Carmichael numbers: 1729, 41041 etc.;
- (8) For $m = 13$ the formula becomes $C \bmod 1638 = 91$ and were obtained the Carmichael numbers: 1729, 41041, 63973, 670033, 997633 etc.

Conjecture:

If $m \cdot 234 + n = 1729$, $m \cdot 234 > n$, then exists a series with infinite many Carmichael terms of the form $C \bmod m \cdot 234 = n$.

Verifying the conjecture:

- (1) For $m < 4$ we have $m \cdot 234 < n$;
- (2) For $m = 4$ the formula becomes $C \bmod 936 = 793$ and were obtained the Carmichael numbers: 1729, 41041, 46657, 126217, 748657, 4909177, 65037817, 193910977, 311388337, 633639097 etc.;
- (3) For $m = 5$ the formula becomes $C \bmod 1170 = 559$ and were obtained the Carmichael numbers: 1729, 1033669, 1082809, 7995169, 26921089 etc.;
- (4) For $m = 6$ the formula becomes $C \bmod 1404 = 325$ and were obtained the Carmichael numbers: 1729, 41041, 46657, 188461, 314821 etc.;
- (5) For $m = 7$ the formula becomes $C \bmod 1638 = 91$ and the case is similar with one from precedent conjecture.

Conjecture:

If $m^{342} + n = 1729$, $m^{342} > n$, then exists a series with infinite many Carmichael terms of the form $C \bmod m^{342} = n$.

Verifying the conjecture:

- (1) For $m < 2$ we have $m^{342} < n$;
- (2) For $m = 3$ the formula becomes $C \bmod 1026 = 703$ and were obtained the Carmichael numbers: 1729, 8911 etc.;
- (3) For $m = 4$ the formula becomes $C \bmod 1368 = 361$ and were obtained the Carmichael numbers: 1729, 126217 etc.;
- (4) For $m = 5$ the formula becomes $C \bmod 1710 = 19$ and were obtained the Carmichael numbers: 1729, 1773289 etc.

Conclusion:

We can see that $126 = 18 \cdot 7$, $234 = 18 \cdot 13$ and $342 = 18 \cdot 19$ and 7, 13, and 19 are the prime factors of 1729, so the three conjectures could be expressed all in one. Even more than that, taking randomly another Carmichael number, $8911 = 7 \cdot 19 \cdot 67$, taking randomly $m = 7$ in the formula $m^{18 \cdot 67}$, we obtain the formula $C \bmod 8442 = 469$, which, indeed, leads to a series of Carmichael numbers: 8911, 1773289, 8830801 etc., which means that the conjecture could be generalised:

Conjecture:

For any prime factor of a Carmichael number $C1$ exists a series with infinite many Carmichael terms $C2$ formed this way: $C2 \bmod m^{18 \cdot d} = n$, where $m^{18 \cdot d} + n = C1$, where d is the prime factor of $C1$ and m, n are natural numbers, $m^{18 \cdot d} < n$.

Note: Finally, if we have a Carmichael number divisible by 1729 (i.e. 63973, see the sequence above), we can see that the formula $C \bmod 62244 = 1729$ (it can see that $62244 + 1729 = 63973$) leads too to a series of Carmichael numbers: 126217 etc. which means that 1729 can be treated like a prime factor. This can be probably generalised to the Carmichael numbers that are divisible with other Carmichael numbers or probably even for a randomly chosen product of prime factors.

30. Fermat pseudoprimes n to base 3 of the form $n = (3^{(4 \cdot k + 2)} - 1)/8$

First 9 terms of the sequence (A217853 in OEIS): 91, 7381, 597871, 48427561, 3922632451, 317733228541, 25736391511831, 2084647712458321, 168856464709124011.

Comments:

These numbers were obtained for values of k from 1 to 20, with the following exceptions: $k = 10, 12, 13, 16, 17, 19$, for which were obtained $3^n \bmod n = 3^7, 3^{31}, 3^{37}, 3^{25}, 3^{31}, 3^{13}$.

Conjecture:

There are infinitely many Fermat pseudoprimes to base 3 of the form $(3^{(4 \cdot k + 2)} - 1)/8$, where k is a natural number.

Part four. One hundred and fifty open problems regarding Fermat pseudoprimes

1. Study the sequence of Poulet numbers of the form $(6^k - 1)(18^k - 5)$: 341, 8321, 83333, 162401, 348161, 587861, 653333, 722261, 1053761, 1398101, 1735841 (...), obtained for $k = 2, 9, 28, 39, 57, 74, 78, 82, 99, 114, 127$ (...).
2. Study the sequence of Carmichael numbers of the form $(6^k + 1)(12^k + 1)$: 2821, 15841, 530881, 997633, 1193221, 3581761, 4767841 (...).
3. Study the sequence of Carmichael numbers of the form $(6^k + 1)(18^k + 1)$: 8911, 10585, 115921, 314821, 334153, 6313681, 8134561 (...).
4. Study the sequence of Carmichael numbers of the form $7 \cdot 13(18^k + 1)$: 1729, 63973, 126217, 188461, 670033, 748657, 997633, 1033669, 1082809, 4463641, 4909177, 7995169, 18900973, 27336673, 31146661, 41298985 (...), obtained for $k = 1, 39, 77, 105, 115, 409, 457, 609, 631, 661, 2725, 2997, 4881, 11539, 16689, 19015, 25213$ (...).

Conjecture: Any Carmichael number C divisible by 7 and 13 can be written as $C = 7 \cdot 13(18^k + 1)$ or as $C = 7 \cdot 13(18^k + 13)$, where k is natural; checked for the first 17 Carmichael numbers divisible by 7 and 13.

5. Study the sequence of Carmichael numbers of the form $19 \cdot 37(18^k + 1)$: 63973, 8719309, 12490201, 69331969, 105117481, 136625941, 144218341, 242641153, 247095361, 292244833 (...), obtained for $k = 5, 689, 987, 5479, 8307, 10797, 11397, 19175, 19527, 23095$ (...).

Conjecture: Any Carmichael number C divisible by 19 and 37 can be written as $C = 19 \cdot 37(18^k + 1)$, where k is natural; checked for the first 10 Carmichael numbers divisible by 19 and 37.

6. Study the following possible generic form of the Carmichael numbers which are not divisible by 3, not necessarily with only three prime factors: $C = [a \cdot 10^m - (a \cdot p + a - 1)] \cdot [b \cdot 10^m - (b \cdot p + b \pm 1)] \cdot [c \cdot 10^m - (c \cdot p + c \pm 1)]$, where m, a, b, c are natural numbers and p is a prime number, $5 < p < 29$. The formula is interesting because it highlights the potential series of Carmichael numbers.

Examples:

: $2821 = 7 \cdot 13 \cdot 31 = [1 \cdot 30 - (1 \cdot 23 + 1 - 1)] \cdot [2 \cdot 30 - (2 \cdot 23 + 2 - 1)] \cdot [5 \cdot 30 - (5 \cdot 23 + 5 - 1)]$; 2821 is the first term in the sequence of Carmichael numbers of the form $(30^n - 23)(60^n - 47)(150^n - 119)$;

: $340561 = 13 \cdot 17 \cdot 23 \cdot 67 = 23 \cdot 67 \cdot 221 = [1 \cdot 30 - (1 \cdot 7 + 1 - 1)] \cdot [3 \cdot 30 - (3 \cdot 7 + 3 - 1)] \cdot [10 \cdot 30 - (10 \cdot 7 + 10 - 1)]$; 340561 is the first term in the sequence of Carmichael numbers of the form $(30^n - 7)(90^n - 23)(300^n - 79)$.

7. Study the sequence of Carmichael numbers of the form $20 \cdot n^2 + 12 \cdot n + 1$: 561, 1729, 41041, 46657, 52633 (...), obtained for $k = 5, 9, 45, 48, 51$ (...).

8. Study the sequence of Carmichael numbers of the form $n*(2^n - 1)*(5^n - 4)$: 2821, 63973, 488881, 288120421 (...), obtained for $k = 7, 37, 19, 307$ (...).

Conjecture: Any Carmichael number C can be written as a product of three numbers in one of two ways:

- I. $C = n*(a^n - a + 1)*(b^n - b + 1)$, where n is (the smallest prime) divisor of C and a, b natural numbers;
 II. $C = n*(a^n - a - 1)*(b^n - b - 1)$, where n is (the smallest prime) divisor of C and a, b natural numbers.

Examples:

Carmichael numbers of form I: $8911 = 7*(3^7 - 2)*(11^7 - 10)$.

Carmichael numbers of form II: $6601 = 7*(4^7 - 5)*(7^7 - 8)$.

9. Study the sequence of Carmichael numbers of the form $n*(2^n - 1)*(7^n - 6)*(14^n - 13)$, where $n, 2^n - 1, 7^n - 6$ and $14^n - 13$ are all four prime numbers. We found, to the numbers of this form which are not Carmichael, an interesting property: Korselt's criterion applies to smaller two prime divisors and for the other two we obtained, for the rational number resulted from fraction $(C - 1)/(p - 1)$, the following periods: 428571 (= $3^4*11*13*37$), 714285 (= $5*3^3*11*13*37$), and 857142 (= $2*3^4*11*13*37$). We found the following such numbers N : 7415394013, 71970231133, 2336497750621.

10. Study the sequence of Carmichael numbers of the form $(6^n - 5)*(24^n - 5)$.

11. Study the sequence of Carmichael numbers of the form $C = n*(2^n - 1)*(6^n - 5)*(12^n - 11)$, C not divisible by 2, 3 or 5; the numbers with only four prime divisors that we got (245791, 129736725, 52889761459, 86242504141) respects a "larger" form of Korselt's criterion [n divides $(C - 1)$, $(2^n - 1)$ and $(6^n - 5)$ divides $2*(C - 1)$ and $(12^n - 11)$ divides $4*(C - 1)$]. Also, for many other squarefree (not Carmichael numbers) of this form, $(p - 1)$ divides $2^r*(C - 1)$ for every prime divisor p .

12. Study the following generic form for Poulet numbers: $P = ((d - 1)*(d + m)/m) + 1$, where d is a prime factor of P .

Conjecture: Any Poulet number P , except for a set of definable exceptions, can be written as $P = ((d - 1)*(d + m)/m) + 1$, where d is any prime factor of P (the first exception is the 15-th Poulet number, the number 4371).

13. Study the following conjecture:

Conjecture: For any Poulet number P , except for a set of definable exceptions, is true one of the following two relations: $(P/d) - 1$ is divisible by $d - 1$ or $d - 1$ is divisible by $(P/d) - 1$ for any d prime factor of P (the first exception is the 15-th Poulet number, the number 4371).

14. Study the following conjecture:

Conjecture: For any 2-Poulet number P , except for a set of definable exceptions, is true the following relation: $P = ((d - 1)*(d + m)/m) + 1$, where m is natural and d is any of the two prime factors of P (the first three exceptions are the 19-th, the 35-th and the 38-th Poulet numbers, *i.e.* the numbers $7957 = 73*109$, $18721 = 97*193$ and $23377 = 97*241$).

15. Study the following function: $f = 2^C \pmod{560}$, where C is a Carmichael number (for $C = 1105, 1729, 2821, 6601, 10585, 15841, 29341, 41041, 46657, 52633, 63973, 75361, 101101, 115921, 126217$ we have $f = 352 = 2^5 \cdot 11$).
16. Study the following conjecture:
Conjecture: Any Carmichael number C can be written in one of the following three forms: $C = 10 \cdot k + 1$; $C = 12 \cdot k + 1$ or $C = 14 \cdot k + 1$, where k natural (checked for the first 300 Carmichael numbers).
17. Study the numbers obtained by concatenation of a Carmichael number with its reversal; interesting palindromic primes are obtained: for instance, $1584114851 = 11 \cdot 144010441$.
18. Study the following relation: $p \cdot (C_1 - 1) - C_2 = C_3$, where p is prime and C_1, C_2 and C_3 are Carmichael numbers. Are there any other such triplets of Carmichael numbers beside $(C_1, C_2, C_3) = (561, 1729, 8911)$?
19. Study the following relation: $C - 1440 = p^2$, where C is Carmichael number and p is prime; are there any other Carmichael numbers beside 1729 ($1729 - 1440 = 17^2$) and 41041 ($41041 - 1440 = 199^2$) that satisfy this relation?
20. Study the Carmichael numbers equal to $a + b$, where b is equal to the concatenation of the numbers a and 0 (e.g. $561 = 51 + 510$) or is equal to the concatenation of the numbers a and 1 (e.g. $15841 = 1440 + 14401$).
21. Study the Carmichael numbers equal to $23^2 + n^2$ (e.g. $1105 = 23^2 + 24^2$; $2465 = 23^2 + 44^2$).
22. Study the following relation: $C = 176 \cdot n + 1$, where C is Carmichael number and n is natural; are there any other Carmichael numbers beside 2465 ($2465 = 176 \cdot 14 + 1$) and 15841 ($15841 = 176 \cdot 90 + 1$) that satisfy this relation?
23. Study the following relation: $n \cdot (C_1 - 1) + p^2 = C_2$, where p is prime, n natural and C_1 and C_2 are Carmichael numbers. Are there any other such duplets of Carmichael numbers beside $(C_1, C_2) = (1729, 15841)$?
24. Study the Carmichael numbers which satisfy the relation $\text{abs}\{C - 13^4\} = 2^a \cdot 3^b \cdot 5^c \cdot 7^d \cdot 11^e \cdot 13^f \cdot p$, where p is prime (the first 40 Carmichael numbers satisfy this relation). Also the first 16 absolute Euler pseudoprimes satisfy the relation $\text{abs}\{E - 13^4\} = 2^a \cdot 3^b \cdot 7^c \cdot 13^d \cdot p$, where p is prime.
25. Study the Carmichael numbers of the form $C = 420 \cdot n + 7^4$ (for $n = 1$ and $n = 10$ we obtain $C = 2821$ and $C = 6601$). Also Carmichael numbers of the forms $8 \cdot n + 11^2$, $560 \cdot n + 161$, $1728 \cdot n + 11 \cdot m$ or $1728 \cdot n + 17 \cdot m$.
26. Study the following two conjectures:
Conjecture 1: For any Poulet number P there exist n such that $P - 80 \cdot n$ is prime.
Conjecture 2: For any Poulet number P there exist n such that $P - 150 \cdot n$ is prime.

27. Study the following conjecture:

Conjecture: For any 3-Carmichael number $C = d_1*d_2*d_3$, except for a set of definable exceptions, is true that $d_3 > d_1^2 + d_2^2$ (the first exception is the Carmichael number $116682721 = 281*617*673$).

28. Study the Carmichael numbers of the form $C = (2^m + 1)^2 + 2^n*p$, where p is prime and m, n are naturals [for $(m, n, p) = (4, 4, 17)$ we obtain $C = 561$ and for $(m, n, p) = (4, 7, 17)$ we obtain $C = 6601$].

29. Study the Carmichael numbers which satisfy the relation $(C^2 + 1)/2 = 37*p$, where p is prime (such numbers are 561 and 8911); also Carmichael numbers that satisfy the relation $(C^2 + 1)/2 = 181*p$, where p is prime (such numbers are 1105 and 29341).

30. Study the Carmichael numbers which satisfy the relation $C - 180 = p*q$, where p and q are primes and $q - p$ is divisible by 23 (such numbers are 29341 and 41041).

31. Study the operation modulo reiterated on Carmichael numbers in the following way: $2821 \bmod (2821 \bmod 90) = 0$ (which appears to be true for many Carmichael numbers).

32. Study the following function: $f = 2^C \bmod (30*s)$, where C is a Carmichael number and s is the sum of its digits; for the first 19 Carmichael numbers we obtain $f = 2^n$, where n non-null positive integer, or $f = 2^n*p$, where n non-null positive integer and p prime.

33. Study the following function: $f = C \bmod (60*d)$, where C is a Carmichael number and d is one of its prime factors; interesting results might be obtain: for instance, we obtain $f = 7*19$ for all the first seven Carmichael numbers of the form $10*k + 3$ divisible by 7 (*i.e.* 52633, 63973, 334153, 670033, 997633, 2628073, 18900973). Also the function $f = C \bmod (90*d)$.

34. Study the following function: $f = C \bmod (90*s)$, where C is a Carmichael number and s is its digital root.

35. Study the following conjecture:

Conjecture: Any Carmichael number C , except for a set of definable exceptions, can be written as $C = p*q + p - q$, where p and q are primes; such duplets of primes are: $(p, q) = (41, 13)$ for $C = 561$, $(p, q) = (73, 23)$ for $C = 1729$ etc.

36. Study the Carmichael numbers C which satisfy the relation $C = p*q^2 + 30*q$, where p and q are primes (*e.g.* $1729 = 31*7^2 + 30*7$).

37. Study the Poulet numbers which satisfy the relation $P = 1800*n^2 + 840*n + 1$ (for $n = 3$ we obtain the Poulet number 18721) or $P = 1800*n^2 + 840*n + 31$ (for $n = 2$ we obtain the Poulet number 8911) or $P = 1800*n^2 + 840*n + 61$ (for $n = 1$ we obtain the Poulet number 2701).

38. Study the following conjecture:

Conjecture: Any Carmichael number C of the form $10*k + 1$, not divisible by 3, can be written at least in one of the following two forms: $C = 1800*n^2 + 840*n + p$ or $C = 1800*n^2 - 840*n - q$, where p and q are primes (p of the form $30*h + 1$ and q of the form $30*h + 29$).

39. Study the following relation: $2*C_1 + C_2 - 30 = C_3$, where C_1, C_2 and C_3 are Carmichael numbers. Are there any other such triplets of Carmichael numbers beside $(C_1, C_2, C_3) = (561, 1729, 2821)$?
40. Study the following relation: $C^2 - 2*C - 30*k = p$, where C is Carmichael number not divisible with 3 or 5 and p is prime. Is there for any such C a natural number k for which p is prime? For $C = 1729$ we obtain, for $k = 1$, $p = 2985953$ prime, for $k = 3$, $p = 2985893$ prime; for $C = 2821$ we obtain, for $k = 1$, $p = 7952369$ prime, for $k = 3$, $p = 7952309$ prime; for $C = 6601$ we obtain, for $k = 2$, $p = 43559939$ prime etc. The same problem for the following relation: $C_1^2 - 2*C_2 - 30*k = p$, where C_1 and C_2 are distinct Carmichael numbers. For $(C_1, C_2) = (1729, 561)$ we obtain, for $k = 1$, $p = 2988289$ prime, for $k = 2$, $p = 2988259$ prime. For $(C_1, C_2) = (2821, 561)$ we obtain, for $k = 1$, $p = 7956889$ prime, for $k = 2$, $p = 7956859$ prime etc.).
41. Study the following operations on 2-Poulet numbers P : $P \bmod (n*d_1 - n)$ and $P \bmod (n*d_2 - n)$, where n is non-null positive integer and d_1 and d_2 are the two prime factors of P . Interesting results are obtained. For instance, for $P = 23377 = 97*241$, we obtain $P \bmod (4*97 - 4) = P \bmod (4*241 - 4) = 337$; note that the Poulet number 23377 it is the first exception for many generic forms that are suitable for the all smaller Poulet numbers, like for instance $P = d_1*d_2 = d_1*(d_1*n - n + 1)$.

Conjecture: For any 2-Poulet number $P = d_1*d_2$, except for a set of definable exceptions, there exist the non-null positive integers m, n such that $P \bmod (n*d_2 - n) = (m + 1)*d_1 - m$. For $P = 341 = 11*31$ we obtain $P \bmod (3*31 - 3) = 7*11 - 6$ and $P \bmod (4*31 - 4) = 10*11 - 9$. For $P = 1387 = 19*73$ we obtain $P \bmod (4*73 - 4) = 4*19 - 3$. For $P = 2047 = 23*89$ we obtain $P \bmod (4*89 - 4) = 13*23 - 12$. For $P = 2701 = 37*73$ we obtain $P \bmod (4*73 - 4) = 3*37 - 2$. For $P = 3277 = 29*113$ we obtain $P \bmod (3*113 - 3) = 9*29 - 8$, $P \bmod (4*113 - 4) = 5*29 - 4$, $P \bmod (5*113 - 5) = 17*29 - 16$, $P \bmod (6*113 - 6) = 21*29 - 20$.

42. Study the numbers N , where $N = p*(p + 2) + C$, where p and $p + 2$ are twin primes and C a Carmichael number; for $C = 1729$ and the first twenty pairs of twin primes, $N = 2^m*3^n*p$, where m and n are non-null positive integers and p is prime. This formula might be also a method for obtaining big primes; we have, for instance, for $C = 1729$, for the pair of primes $(982449569, 982449571)$, $N = 2^2*3^2*p$, where $p = 26811309933144073$ is prime; for the pair of primes $(982450457, 982450459)$, $N = 2^2*3^3*p$, where $p = 8937119466892699$ is prime; for the pair of primes $(982451579, 982451581)$, $N = 2^4*3^2*p$, where $p = 6702854910031237$ is prime.

Conjecture: The number $N = p*(p + 2) + 1729$, where p and $p + 2$ are twin primes, is always equal to a product of powers of the numbers 2 and 3 and of prime factors of the form $30*k + 1, 30*k + 7, 30*k + 13$ and $30*k + 19$.

43. Study the functions f and g , where $f(C) = C - s(C) + 1$ and $g(C) = s(C)*(C - s(C)) + 1$, where C is a Carmichael number and $s(C)$ is the sum of the digits of C . For instance, $f(6601) = 6601 - 13 + 1 = g(561) = 12*(561 - 12) + 1 = 6589$.
44. Study the Carmichael numbers C that satisfy the relation $C = 561 + 272*n$. Are there any other Carmichael numbers that satisfy this relation beside $C = 1105$ (corresponding to $n = 2$) and $C = 2465$ (corresponding to $n = 7$)?
45. Study the Carmichael numbers C that satisfy the relation $C = n*(C - 1) - p*(p + 2)$, where n is non-null natural integer and p and $p + 2$ are twin primes. Are there any other Carmichael numbers that satisfy this relation beside $C = 1729$ (corresponding to $n = 4$ and $p = 71$)?

46. Study the following conjecture:

Conjecture: Any Poulet number P of the form $10*k + 1$, $10*k + 5$ or $10*k + 9$ (not of the form $10*k + 3$ or $10*k + 7$) can be written as $P = (a^2 - b^2)/8$, where a and b are non-null positive integers.

47. Study the following conjecture:

Conjecture: Let C be a Carmichael number and $s(C)$ be its digital root. If C is not a Harshad number and $s(C)$ is not equal to 1, then any prime factor d of C , $d \neq 3$, can be written as $d = n*s(C) - n \pm 1$, where n is a non-null positive integer.

Examples:

For $C = 561 = 3*11*17$, $s(561) = 3$ and $d_1 = 11 = 5*3 - 5 + 1$ and $d_2 = 17 = 8*3 - 8 + 1$.

For $C = 1105 = 5*13*17$, $s(1105) = 7$ and $d_1 = 5 = 1*7 - 1 - 1$, $d_2 = 13 = 2*7 - 2 + 1$ and $d_3 = 17 = 3*7 - 3 - 1$.

For $C = 6601 = 7*23*41$, $s(6601) = 4$ and $d_1 = 7 = 2*4 - 2 + 1$, $d_2 = 23 = 7*4 - 6 + 1$ and $d_3 = 41 = 14*4 - 14 - 1$.

48. Study the following conjecture:

Conjecture: Let C be a Carmichael number and $s(C)$ be the sum of its digits. Any C except for a set of definable exceptions can be written as $C = n*s(C) - n + 1$, where n is a non-null positive integer.

Examples:

For $C = 1105$ we have $s(C) = 7$ and $1105 = 184*7 - 184 + 1$.

For $C = 1729$ we have $s(C) = 19$ and $1729 = 96*19 - 96 + 1$.

For $C = 2465$ we have $s(C) = 17$ and $2465 = 154*17 - 154 + 1$.

For $C = 6601$ we have $s(C) = 13$ and $6601 = 550*13 - 550 + 1$.

49. Study the Carmichael numbers $C_1 = d_1*d_2*d_3$ and C_2 that satisfy the relation $C_2 = 2*(d_1 + d_2 + d_3 - 6)*(C_1 - 2) - 1$. Are there any other Carmichael numbers that satisfy this relation beside the pair $(C_1, C_2) = (1105, 63973)$? It can be seen that, for $1105 = 5*13*17$ we obtain $2*(5 + 13 + 17 - 6)*(1105 - 2) - 1 = 7*13*19*37 = 63973$. The function $f(C) = 2*(d_1 + d_2 + \dots + d_n - 2*n)*(C - 2) - 1$, where C is a Carmichael number and n is the number of the prime factors d_1, d_2, \dots, d_n of C might also be interesting to be studied.

Conjecture 1: There exist an infinity of 3-Carmichael numbers $C = d_1 * d_2 * d_3$ for which the number $N = 2 * (d_1 + d_2 + d_3 - 6) * (C - 2) - 1$ is also a Carmichael number.

Conjecture 2: There exist an infinity of n-Carmichael numbers $C = d_1 * d_2 * \dots * d_n$ for which the number $N = 2 * (d_1 + d_2 + \dots + d_n - 2 * n) * (C - 2) - 1$ is also a Carmichael number.

50. Study the following conjecture:

Conjecture: For any Carmichael number C , except for a set of definable exceptions, is true at least one of the two following statements:

(1) $C - 1$ is divisible by $s(C) - 1$;

(2) $C - 1$ is divisible by $s(C) + 1$, where $s(C)$ is the sum of digits of C .

Note: The first exception is the first Carmichael number, 561, and the second exception is the 15-th Carmichael number, 63973; the majority of Carmichael numbers satisfy the relation (1), which is also the case of primes: there are 23 primes p for which the relation $p - 1$ is divisible by $s(p) - 1$ is true (*i.e.* the primes 11, 13, 19, 31, 37, 41, 43, 61, 71, 73, 101, 103, 109, 113, 127, 151, 157, 163, 181, 191, 193, 199, 211) smaller than the first prime for which the relation $p - 1$ is divisible by $s(p) + 1$ is true, *i.e.* the prime 241, for which is also true the relation $p - 1$ is divisible by $s(p) - 1$.

Observation: For $C = 46657$ we obtain $(46657 - 1) / (28 - 1) = 1728 = 1729 - 1$; are there any Carmichael numbers C_1 and C_2 for which $(C_1 - 1) / (s(C_1) - 1) = C_2 - 1$?

51. Study the following conjectures:

Conjecture 1: Any Carmichael number C can be written as $C = (n * (n + p)) / 8$, where n is non-null positive integer and p is prime or power of prime.

Examples:

$561 = (51 * (51 + 37)) / 8$; $1105 = (85 * (85 + 19)) / 8$; $1729 = (104 * (104 + 29)) / 8$; $2465 = (136 * (136 + 3^2)) / 8$; $2821 = (104 * (104 + 113)) / 8$.

Conjecture 2: There is an infinity of Carmichael numbers of the form $(104 * (104 + p)) / 8$, where p is prime (for $p = 29$ and 113 are obtained Carmichael numbers 1729 and 2821).

Conjecture 3: There is an infinity of primes p of the form $p = 7 * q - 104$, where q is a prime of the form $6 * k + 1$.

Examples: for $q = 19, 31, 43, 79$ we obtain $p = 29, 113, 197, 449$

Conjecture 4: There is an infinity of primes p of the form $p = q_1 * q_2 - 8 * (q_1 + 6)$, where q_1 and $q_1 + 6$ are primes of the form $6 * k + 1$ and q is also prime of the form $6 * k + 1$.

Examples: for $(q_1, q_2) = (13, 31)$, we obtain $p = 251$; for $(q_1, q_2) = (31, 19)$, we obtain $p = 293$.

52. Study the following conjecture:

Conjecture: Any Carmichael number C can be written as $C = (n * (n + a)) / (a + 1)$, where n and a are non-null positive integers.

Examples:

$561 = (33 * (33 + 1)) / 2$; $1105 = (65 * (65 + 3)) / 4$; $1729 = (91 * (91 + 4)) / 5$; $2465 = (145 * (145 + 8)) / 9$; $2821 = (91 * (91 + 2)) / 3$.

- 53.** Study the Poulet numbers P_1 and P_2 that satisfy the relation $P_1 + p*(p - 1) = P_2$ and the ones that satisfy the relation $P_1 + 2*p*(p - 1) = P_2$, where p is prime. Are there any pairs of Poulet numbers beside $(P_1, P_2) = (1387, 1729)$ that satisfy the first relation and beside $(341, 561)$ that satisfy the second relation?
It can be seen that $1729 = 1387 + 18*19$ and $561 = 341 + 2*10*11$.

Conjecture 1: For any Poulet number P_1 there is an infinity of primes p such that $P_1 + p*(p - 1) = P_2$, where P_2 is also a Poulet number.

Conjecture 2: For any Poulet number P_1 there is an infinity of primes p such that $P_1 + 2*p*(p - 1) = P_2$, where P_2 is also a Poulet number.

- 54.** Study the Carmichael numbers of the form $2821 + 26520*n$; for $n = 0, 1$ and 7 are obtained the Carmichael numbers $2821, 29341$ and 188461 .

Conjecture: There is an infinity of Carmichael numbers C of the form $C = 2821 + 26520*n$, where n is non-negative integer.

- 55.** Study the Carmichael numbers of the form $p^4 + 60*p$, where p is prime; for $p = 7$ and 13 are obtained the Carmichael numbers 2821 and 29341 .

Conjecture: There is an infinity of Carmichael numbers C of the form $C = p^4 + 60*p$, where p is prime.

- 56.** Study the following conjecture:

Conjecture: Any Carmichael number C can be written as $C = 2^n*p + q$, where p and q are primes and $\text{abs}\{p - q\} = 10*k$, where k is non-negative integer.

Examples:

$$C = 561 = 2*277 + 7 = 2*257 + 47 = 2^3*59 + 89;$$

$$C = 1105 = 2^2*269 + 29 = 2^2*263 + 53 = 2^6*17 + 17;$$

$$C = 1729 = 2*853 + 23 = 2^3*211 + 41 = 2^4*107 + 17.$$

- 57.** Study the Carmichael numbers of the form $n*(n + 1) + p^2$, where n is non-null positive integer and p is prime; for $(n, p) = (23, 3)$ is obtained 561 and for $(n, p) = (32, 7)$ is obtained 1105 .

Conjecture: There is an infinity of Carmichael numbers C of the form $C = n*(n + 1) + p^2$, where n is non-null positive integer and p is prime.

- 58.** Study the Carmichael numbers C which can be written as $C = 2^m + p^n$, where p is prime and m, n are non-null positive integers.

Examples: $561 = 2^9 + 7^2 = 2^5 + 23^2$; $2465 = 2^6 + 7^4$.

Conjecture: There is an infinity of Carmichael numbers C which can be written as $C = 2^m + 7^n$, where m and n are non-null positive integers.

- 59.** Study the Carmichael numbers C which can be written as $C = (p^2 + n)/(n + 1)$, where p is prime and n non-null positive integer.

Examples: $561 = (41^2 + 2)/3$; $1105 = (47^2 + 1)/2$.

Conjecture: There is an infinity of Carmichael numbers C which can be written as $C = (p^2 + n)/(n + 1)$, where p is prime and n non-null positive integer.

60. Study the Carmichael numbers C which can be written as $C = (p^2 * q^2 + n)/(n + 1)$, where p and q are primes and n non-null positive integer.
Example: $1729 = (7^2 * 23^2 + 14)/15$.

Conjecture: There is an infinity of Carmichael numbers C which can be written as $C = (p^2 * q^2 + n)/(n + 1)$, where p and q are primes and n non-null positive integer.

61. Study the Carmichael numbers C which can be written as $C = (\sqrt{4 * p - 1})/3 + 2$, where p is prime. For $p = 59527711$ is obtained $C = 8911$, for $p = 83999917$ is obtained $C = 10585$, for $p = 188155441$ is obtained 15841 .

Conjecture: There is an infinity of Carmichael numbers C which can be written as $C = (\sqrt{4 * p - 1})/3 + 2$, where p is prime.

62. Study the following triplets of primes: $(p, p + C_1 - 1, p + C_2 - 1)$, where C_1 and C_2 are Carmichael numbers. For $p = 59$, $C_1 = 561$ and $C_2 = 1729$ we have the triplet of primes $(59, 619, 1787)$ and the relation $59 + 619 + 1787 = 2465$, also a Carmichael number. Are there any other numbers p, C_1, C_2, C_3 that satisfy this relation?

63. Study the numbers of the form $N = \text{abs}\{C_3 - (C_1 + C_2 - 2)\}$, where C_1, C_2 and C_3 are Carmichael numbers; appears that N is often prime of product of very few prime factors; for C_1 and C_2 equal to 1105 and 1729 , we obtain N prime for C_3 equal to $561, 1729, 2465, 2821, 6601, 8911, 10585, 15841$, *i.e.* for the first 8 from 9 Carmichael numbers.

64. Study the following conjecture:

Conjecture: Any Carmichael number C can be written as $C = p * q + q - p$, where p and q are primes (appears that often $q - p$ is of the form $10 * k$).

Examples: $C = 561 = 19 * 29 + 29 - 19$; $C = 1729 = 23 * 73 + 73 - 23$.

65. Study the following conjectures:

Conjecture: Any 3-Carmichael number C divisible by 5 can be written as $C = p * (q - 2^n)$, where p is one of its prime factors beside 5, q is a prime and n is non-null positive integer.

Examples:

$$C = 1105 = 13 * (89 - 2^2) = 17 * (67 - 2);$$

$$C = 2465 = 17 * (149 - 2^2) = 29 * (89 - 2^2);$$

$$C = 10585 = 29 * (367 - 2) = 73 * (149 - 2^2).$$

Conjecture: For any p odd prime, $p \neq 5$, there exist an infinity of numbers n such that $q = 5 * p + 2^n$ is prime.

Example: for $p = 3$, q is prime for $n = 1, 2, 3, 4, 5, 6$.

66. Study the following conjectures:

Conjecture 1: For any Poulet number P , P not divisible by 3 and 5, there exist non-null positive integers m, n such that the number $(P + 5^m)/(6^n)$ is prime or power of prime.

Examples:

For $P = 341$ we have $(341 + 5^2)/(6^1) = 61$, prime.

For $P = 1387$ we have $(1387 + 5^3)/(6^3) = 7$, prime.

For $P = 1729$ we have $(1729 + 5^1)/(6^1) = 17^2$, square of prime.

Conjecture 2: For any Poulet number P , P divisible by 5 and not divisible by 3, there exist a non-null positive integer m such that the number $(P + 5^m)/30$ is prime.

Examples:

For $P = 1105$ we have $(1105 + 5^1)/30 = 37$, prime.

For $P = 2465$ we have $(2465 + 5^2)/30 = 83$, prime.

For $P = 10585$ we have $(2465 + 5^1)/30 = 353$, prime.

Conjecture 3: For any Poulet number P , P divisible by 3 and 5, there exist a non-null positive integer m such that the number $(P + 5^m)/10$ is prime.

Examples:

For $P = 1905$ we have $(1905 + 5^1)/10 = 191$, prime.

For $P = 18705$ we have $(18705 + 5^1)/10 = 1871$, prime.

For $P = 55245$ we have $(55245 + 5^2)/10 = 5527$, prime.

Conjecture 4: For any n non-null positive integer, n not divisible by 5, except for a set of definable exceptions, there exist a non-null positive integer m such that the number $6^n - 5^m$ is prime or is equal to 1.

Note: the first three exceptions are the numbers 134, 341 and 431.

67. Study the Poulet numbers P_1 and P_2 with the property that $P_1 + P_2 + 1$ is also a Poulet number. Are there any other such Poulet numbers beside $P_1 = 341$ and $P_2 = 1387$ for which $P_1 + P_2 + 1 = 1729$?
68. Study the Poulet numbers P_1 and P_2 with the property that $P_1 + P_2 - 1$ is also a Poulet number. Are there any other such Poulet numbers beside $P_1 = 561$ and $P_2 = 1905$ for which $P_1 + P_2 - 1 = 2465$?
69. Study the Poulet numbers P_1 and P_2 with the property that $P_1 + r(P_2) - 2$ is also a Poulet number, where $r(P_2)$ is the reversal of P_2 . Are there any other such Poulet numbers beside $P_1 = 561$ and $P_2 = 645$ for which $P_1 + r(P_2) - 2 = 561 + 546 - 2 = 1105$?
70. Study the following conjectures:

Conjecture 1: For any Poulet numbers P_1 and P_2 there exist an infinity of primes p such that $P_1 + P_2 + p$ is also a Poulet number.

Conjecture 2: For any Poulet numbers P_1, P_2, \dots, P_n there exist an infinity of primes p such that $P_1 + P_2 + \dots + P_n + p$ is also a Poulet number.

71. Study the formula $p = (15 \cdot C + d)/(2^m \cdot d^n)$, where C is a Carmichael number and d one of its prime factors, formula which appears to generate often primes.
Examples:

For $(C, d, m, n) = (2465, 29, 2, 2)$ we have $p = 11$, prime; for $(C, d, m, n) = (4954039956700380001, 17, 6, 1)$ we have $p = 68300183226567739$, prime.

72. Study the formula $p = (15*d + C)/(2^m*d^n)$, where C is a Carmichael number and d one of its prime factors, formula which appears to generate often primes.

Examples:

For $(C, d, m, n) = (4954039956700380001, 17, 5, 1)$ we have $p = 9106691096875699$, prime; for $(C, d, m, n) = (3589102252820237401, 101, 2, 2)$ we have $p = 87959568983929$, prime.

73. Study the Carmichael numbers C_1 and C_2 which satisfy the formula $C_2 = C_1*(2^m*3^n + 1)$, where m, n non-null positive integers.

Conjecture: There exist an infinity of Carmichael numbers C which can be written as $C = 1729*(2^m*2^n + 1)$.

Examples: such numbers are 63973, 126217, 188461, obtained for $(m, n) = (2, 2), (3, 2), (2, 3)$.

74. Study the Carmichael numbers C_1 and C_2 which satisfy the formula $C_2 - C_1 = C_1 - d = n$, where d is a prime factor of C_1 .

Examples:

: $1105 - 561 = 561 - 17 = 544$ (where $561 = 3*11*17$);

: $1024651 - 512461 = 512461 - 271 = 512190$ (where $512461 = 31*61*271$).

Conjecture: There exist an infinity of Carmichael numbers C_1 for which exist a Carmichael number C_2 such that $C_2 = 2*C_1 - d$, where d is a prime factor of C_1 .

75. Study the Carmichael numbers C_1 and C_2 which satisfy the formula $C_2^2 - C_1^2 = n^2$, where n is non-null positive integer.

Example: $1105^2 - 561^2 = 952^2$.

Also study the function $f(C) = C^2 \bmod 952$, where C is Carmichael number; interesting results are obtained; for instance, $f(1729) = f(6601) = 161$

Conjecture: There exist an infinity of pairs of Carmichael numbers C_1 and C_2 such that $C_2^2 - C_1^2 = n^2$, where n is non-null positive integer.

76. Study the Carmichael numbers C_1, C_2 and C_3 which satisfy the formula $2*C_2 - C_1 = C_3$.

Example: $2*93614521 - 41041 = 187188001$.

Conjecture: There exist an infinity of distinct Carmichael numbers C_1, C_2 and C_3 which satisfy the formula $2*C_2 - C_1 = C_3$.

77. Study the Poulet numbers P which can be written as $P = 126*n + 1$ (nine from the first hundred of Poulet numbers can be written this way: 1397, 3277, 4033, 14491, 49141, 83665, 115921, 121465, 176149. Also study the Poulet numbers of the form $P = 126*n + 37$ (16705, 18721, 74665, 104653 etc.).

78. Study the following conjecture:

Conjecture: For any Poulet number $P = n + d$, where d is a prime factor of P , there exist an infinity of primes p that can be written as $p = (n + d^k)/d$, where k positive integer.

Examples:

For $P = 1729 = 1722 + 7$ (where $1729 = 7 \cdot 13 \cdot 19$), q is prime for $k = 5, 6, 8$ etc.

For $P = 1729 = 1710 + 19$ (where $1729 = 7 \cdot 13 \cdot 19$), q is prime for $k = 2, 4, 5$ etc.

79. Study the following conjecture:

Conjecture: There exist an infinity of Poulet numbers P of the form $P = 3 \cdot n^2 + 6 \cdot n + 4$ (such numbers are, *e.g.*, 1729 and 2701 obtained for n equal to 23 and 29).

80. Study the following conjectures:

Conjecture 1: There exist an infinity of primes p which can be written as $p = (P + n)/(n + 1)$, for any Poulet number P , where n is non-null positive integer.

Examples:

For $P = 561$ we have $p = 71 = (561 + 7)/(7 + 1)$; also $p = 41 = (561 + 13)/(13 + 1)$ etc.

Conjecture 2: There exist an infinity of primes p which can be written as $p = \sqrt{n \cdot P - n + 1}$, for any Poulet number P , where n is non-null positive integer.

Examples:

For $P = 561$ we have $p = 41 = \sqrt{3 \cdot 561 - 3 + 1}$; also $p = 71 = \sqrt{9 \cdot 561 - 9 + 1}$.

Conjecture 3: There exist an infinity of primes p which can be written as $p = (q^2 + n)/(n + 1)$, for any prime number q , where n is non-null positive integer.

Examples:

For $q = 41$ we have $p = 337 = (41^2 + 4)/5$; also $p = 211 = (41^2 + 7)/8$.

Conjecture 4: There exist an infinity of primes p which can be written as $p = \sqrt{(q^2 + n)/(n + 1)}$, for any prime number q , where n is non-null positive integer.

Examples:

For $q = 41$ we have $p = 29 = \sqrt{(41^2 + 1)/2}$; also $p = 11 = \sqrt{(41^2 + 13)/14}$.

81. Study the following conjectures:

Conjecture 1: There exist an infinity of 3-Poulet numbers $P = d_1 \cdot d_2 \cdot d_3$, where $d_1 < d_2 < d_3$, for which the numbers $2 \cdot d_1 + 1$, $3 \cdot d_2 + 2$ and $3 \cdot d_3 + 2$ are all three primes.

Examples:

For $P = 645 = 3 \cdot 5 \cdot 43$, we have $2 \cdot 3 + 1 = 7$, $5 \cdot 3 + 2 = 17$, $43 \cdot 3 + 2 = 131$, where (11, 41, 53) is a triplet of primes.

For $P = 1105 = 5 \cdot 13 \cdot 17$, we have $2 \cdot 5 + 1 = 11$, $13 \cdot 3 + 2 = 41$, $17 \cdot 3 + 2 = 53$, where (11, 41, 53) is a triplet of primes.

For $P = 2465 = 5 \cdot 17 \cdot 29$, we have $2 \cdot 5 + 1 = 11$, $17 \cdot 3 + 2 = 53$, $29 \cdot 3 + 2 = 89$, where (11, 53, 89) is a triplet of primes.

Conjecture 2: There exist, for any Poulet number $P = d_1 \cdot d_2 \cdot d_3$, where $d_1 < d_2 < d_3$, an infinity of positive integers n such that the numbers $n \cdot d_1 + n - 1$, $(n + 1) \cdot d_2 + n$ and $(n + 1) \cdot d_3 + n$ are all three primes.

Example: For $P = 1729 = 7 \cdot 13 \cdot 19$, we have $9 \cdot 7 + 8 = 71$, $10 \cdot 13 + 9 = 139$, $10 \cdot 19 + 9 = 199$, where (71, 139, 199) is a triplet of primes.

82. Study the following conjectures:

Conjecture 1: There exist an infinity of 3-Poulet numbers $P = d_1 * d_2 * d_3$, where $d_1 < d_2 < d_3$, for which the number $p = 60 * d_1 + 30 * d_2 + d_3$ is prime or square of prime.

Examples:

For $P = 1905 = 3 * 5 * 127$, $p = 60 * 3 + 30 * 5 + 127 = 457$, prime.

For $P = 1729 = 7 * 13 * 19$, $p = 60 * 7 + 30 * 13 + 19 = 829$, prime.

For $P = 2821 = 7 * 13 * 31$, $p = 60 * 7 + 30 * 13 + 31 = 841 = 29^2$, square of prime.

Conjecture 2: There exist an infinity of 4-Poulet numbers $P = d_1 * d_2 * d_3 * d_4$, where $d_1 < d_2 < d_3 < d_4$, for which the number $p = 90 * d_1 + 60 * d_2 + 30 * d_3 + d_4$ is prime or square of prime.

Examples:

For $P = 41041 = 7 * 11 * 13 * 41$, $p = 90 * 7 + 60 * 11 + 30 * 13 + 41 = 1721$, prime.

For $P = 62745 = 3 * 5 * 47 * 89$, $p = 90 * 3 + 60 * 5 + 30 * 47 + 89 = 2069$, prime.

Conjecture 3: There exist an infinity of n-Poulet numbers $P = d_1 * d_2 * d_3 * ... * d_n$, where $d_1 < d_2 < d_3 < ... < d_n$, for which the number $p = (n - 1) * 30 * d_1 + ... + 2 * 30 * d_2 + 30 * d_3 + d_4$ is prime or square of prime.

Example:

For $P = 999607982113 = 13 * 19 * 37 * 97 * 307 * 3673$, $p = 150 * 13 + 120 * 19 + 90 * 37 + 60 * 97 + 30 * 307 + 3673 = 1721$, prime.

83. Study the following sequences of Carmichael numbers:

- (1) of the form $1729 * n * (m * n - m + 1)$: 101957401, 705101761, 1150270849, 7103660473, obtained for $(m, n) = (5, 109), (11, 193), (2, 577), (349, 109)$;
- (2) of the form $1729 * 37 * (m * 37 - m + 1) * (n * 37 - n + 1)$: 509033161, 1854001513, 8039934721, obtained for $(m, n) = (2, 3), (2, 11), (3, 32)$;
- (3) of the form $1729 * 271 * n * (13 * m + 5)$: 10298458261, 41420147041, obtained for $(m, n) = (2, 709), (8, 811)$.

Questions: All Carmichael numbers divisible by 1729 can be written only in one of the three ways from above (indeed, all of them until the number 65039877721 can be written in one of the three ways from above)? Note that all of the three forms can be reduced to the form $1729 * (36 * n + 1)$.

84. Study if the following emirical observations about the number 1729 can be extended somehow to other Fermat pseudoprimes and can say something about (a class of) these numbers:

- (1) $15^{49} \bmod 1729 = 15^{121} \bmod 1729 = 15^{13} \bmod 1729 = 561$ (it seems that $15^{(36 * n - 23)} \equiv 561 \pmod{1729}$);
 - (2) $13^{17} \bmod 1729 = 1105$;
 - (3) $1729 + 6601 - 7^4 = 7^2 * 11^2$;
 - (4) $2 * 1729 - (561 + 1105) = 2^8 * 7$;
 - (5) $2 * 1729 - 1105 + 1 = 561 + 2^6$;
 - (6) $1728 + 11^6 = 1773289$;
 - (7) $15^8 \bmod 2821 = 15^8 \bmod 1729 = 841 = 29^2$,
- where 561, 1105, 1729, 2821, 6601, 1773289 are Carmichael numbers.

85. Study the following conjectures:

Conjecture 1: For any 3-Poulet number P one of the two following statements it's true:

- (1) $36*(P - d)$ is divisible by $(d^2 - d)$;
- (2) $36*(d^2 - d)$ is divisible by $(P - d)$, where d is any prime factor of P .

Conjecture 2: For any 2-Poulet number P , squarefree, one of the two following statements it's true:

- (1) $36*(P - d)$ is divisible by $(d^2 - d)$;
- (2) $36*(d^2 - d)$ is divisible by $(P - d)$, where d is any prime factor of P .

Conjecture 3: For any 2-Poulet number P , squarefree, the following statement it's true:
 $360*(P - d_1)/(d_1^2 - d_1) = 360*(d_2^2 - d_2)/(P - d_2) = m$, where d_1 and d_2 are the two prime factors and m is a natural number.

Conjecture 4: For any Fermat pseudoprime to base n with two prime factors, P , squarefree, the following statement it's true:

$360*(P - d_1)/(d_1^2 - d_1) = 360*(d_2^2 - d_2)/(P - d_2) = m$, where d_1 and d_2 are the two prime factors and m is a natural number.

Conjecture 5: For any 3-Poulet number P one of the two following statements it's true:

- (1) $12*d_1*(P - d_2)$ is divisible by $(d_2^2 - d_2)$;
- (2) $12*d_1*(d_3^2 - d_3)$ is divisible by $(P - d_3)$, where $d_1 < d_2 < d_3$ are the three prime factors of P .

86. Study the Poulet numbers (which are not also Carmichael numbers) P that can be written as $P = n^3 - m*n^2 + m*n$, where n, m are natural numbers and $n \neq P$: {341, 645, 1387, 1905, 2047, 2701, 3277, 4033, 4369, 4371, 4681, 5461, 8321, 8481, 10261, 12801, 13747, 13981, 14491, 16705, 18705, 18721, 19951, 25761, 30889, 31417, 31609, 31621, 33153, 39865, 41665, 49141, 57421, 60701, 65077, 65281, 68101, 74665, 80581...}. Poulet numbers that can't be written this way are: {7957, 11305, 15709, 23377, 30121, 34945, 42799, 49981, 55245, 60787, 72885...}.

Note: We didn't consider Carmichael numbers because all of them can be written this way.

Observation: From the first 51 Poulet numbers (that are not Carmichael), only 11 can't be written in this form; 8 have the solution $m = n - 2$ which means that can be written as $3*n^2 - 2*n$ as well as $3*m^2 + 10*m + 8$, other 8 have the solution $m = n - 3$ which means that can be written as $4*n^2 - 3*n$ as well as $4*m^2 + 21*m + 27$, while other 5 have the solution $m = n - 1$ which means that can be written as $2*n^2 - n$ as well as $2*m^2 + 3*m + 1$. It can also be said, from $P = n^3 - m*n^2 + m*n$, taking $m = n - q$, that very many Poulet numbers (so far we checked, around eighty percent) can be written as $P = n^3 - m*n^2 + m*n$ as well as $P = n^2*(q + 1) - n*q$. But even more Poulet numbers satisfy only the second formula. *E.g.*, from Poulet numbers that don't have natural solutions for $P = n^3 - m*n^2 + m*n$, 7957, 23377, 42799, 49981, 60787 have no natural solutions for $P = n^2*(q + 1) - n*q$ too, while 11305, 15709, 30121, 34945, 55245, 72885 do have, which means that around ninety percent from Poulet numbers we checked can be written as $P = n^2*(q + 1) - n*q$, where n, q natural numbers.

- 87.** Study the following sequences of 3-Poulet numbers:
- (1) of the form $n*(2*n - 1)*(3*n - 2)$: $18705 = 15*29*43$, $172081 = 31*61*91$, $294409 = 37*73*109$, $464185 = 43*85*127$ etc.;
 - (2) of the form $n*(2*n - 1)*(5*n - 4)$: $63973 = 19*37*91$, $285541 = 31*61*151$, $488881 = 37*73*181$, $7428421 = 91*181*451$ etc.;
 - (3) of the form $n*(3*n - 2)*(4*n - 3)$: $13981 = 11*31*41$, $137149 = 23*67*89$ etc.

- 88.** Study the numbers of the form $C = 7*11*13*41*(30*n + 1)$. The first few such numbers are: {41041, 1272271, 2503501, 3734731, 4965961, 6197191, 7428421, 8659651, 9890881, 11122111, 12353341, 13584571, 14815801, 16047031, 17278261, 18509491, 19740721, 20971951, 22203181, 23434411, 24665641, 25896871, 27128101, 28359331, 29590561, 30821791, 32053021, 33284251...}. Are obtained Carmichael numbers for the following values of n : 0, 6, 72, 76, 152, 456, 684, 18720, 49896, 82352, 208052, 314640, 496080, 614952, 622440.

Note: It is interesting that for many numbers of this form, which are not Carmichael numbers but are still squarefree, the Korselt's criterion ($C - 1$ is divisible by $p - 1$, where p is a prime divisor of C) is valid for many from their prime divisors, and a "larger" form of this criterion, *id est* taken $2*(C - 1)$, $4*(C - 1)$ or $5*(C - 1)$ beside $C - 1$, applies to even more prime divisors.

Conjecture: There exist an infinity of Carmichael numbers of the form $C = 7*11*13*41*(30*n + 1)$.

- 89.** Study the Carmichael numbers divisible by Carmichael number 8911: {8911, 1773289, 8830801, 15888313, 26474581, 703995733, 1949646601, 10501586767, 43721295751, 45685048465, 84954809611, 87433760701, 113267783377, 139041817201, 174858690601, 260896819393, 342629865865, 343709665201, 504197488081, 574278582241, 576480525985...}.

Conjecture: Any Carmichael number C divisible by Carmichael number 8911 can be written as $C = 8911*(198*n + 1)$.

Note: The Carmichael number 8911 itself can be written as $198*45 + 1$, so the formula of numbers from the sequence above becomes $C = (198*45 + 1)*(198*n + 1)$.

- 90.** Study the Carmichael numbers C that can be written as $C = 952*n + 561$: {561, 2465, 278545, 552721...}. Note that $952 = \sqrt{1105^2 - 561^2}$, where 1105 and 561 are the first two Carmichael numbers.

Questions: Are there any Carmichael numbers C_1 and C_2 such that $C_2^2 - C_1^2 = n^2$, where n positive integer? Are there any other sequences of Carmichael numbers of the form $C_3 = n*\sqrt{C_2^2 - C_1^2} + C_1$?

Conjecture: There exist an infinity of Carmichael numbers of the form $C = 952*n + 561$.

- 91.** Study the Carmichael numbers C of the form $C = ((p^2 + 1)/2)*p$, where p is prime: {1105, 2465...}.

Conjecture: There exist an infinity of Carmichael numbers of the form $C = ((p^2 + 1)/2)*p$, where p is prime.

92. Study the Carmichael numbers of the form $10*k + 9$: {1729, 294409, 1033669, 1082809, 1773289, 5444489, 7995169, 8719309, 17098369, 19384289, 23382529, 26921089, 37964809, 43620409, 45890209, 50201089, 69331969, 84311569, 105309289, 114910489, 146843929, 168659569, 172947529, 180115489, 188516329, 194120389, 214852609...}. Note that very few of them are of the form $30*k + 29$ (from the numbers above just two: 5444489 and 19384289), most of them having the form $30*k + 19$.

93. Study the Carmichael numbers C of the form $C = 6*n^3 + 11*n^2 + 6*n + 1$: {1729, 172081, 294409, 1773289, 4463641, 56052361...}. Note that the formula becomes $(30n + 1)*(60*n + 1)*(90*n + 1)$ for $n = 10*n$ and $(30n + 7)*(60*n + 13)*(90*n + 19)$ for $n = 10*n + 6$. The formula is larger than (and includes all the terms of) the Chernick series.

94. Study the Poulet numbers P of the form $P = 4*n^2 + 172*n + 529$: {6601, 8481, 15841...}.

Conjecture: There exist an infinity of Poulet numbers of the form $P = 4*n^2 + 172*n + 529$, where n is positive integer.

95. Study the Poulet numbers P of the form $P = 20*n^2 + 364*n + 177$: {561, 4369, 6601...}.

Conjecture: There exist an infinity of Poulet numbers of the form $P = 20*n^2 + 364*n + 177$, where n is positive integer.

96. Study the Carmichael numbers C of the form $C = p^2 + n^2 - 1$, where p is prime and n is non-null positive integer; for $n = 60$, the formula becomes $C = p^2 + 360$ and were obtained the Carmichael numbers 1729 and 6601 for $p = 37$ and $p = 79$.

Conjecture: There exist an infinity of Carmichael numbers of the form $C = p^2 + n^2 - 1$, where p is prime and n is non-null positive integer.

97. Study the Poulet numbers P of the form $P = (60*m*n - 29)*(60*m*n - (60*m + 29))$, where m, n are non-null positive integers.

Notes:

For $m = 1$ the formula becomes $(60*n - 29)*(60*n - 89)$ and were found the following Poulet numbers: 2821, 13741...;

For $m = 2$ the formula becomes $(120*n - 29)*(120*n - 149)$ and were found the following Poulet numbers: 149281...;

For $m = 3$ the formula becomes $(180*n - 29)*(180*n - 209)$ and were found the following Poulet numbers: 49981...

Conjecture: For any m non-null positive integer there exist a series with infinite many Poulet numbers P formed this way: $P = (60*m*n - 29)*(60*m*n - (60*m + 29))$, where n non-null positive integer.

98. Study the Poulet numbers P of the form $P = (40*n - 10*m - 9)*(40*n - 10*m - 129)$, where m is natural and n is non-null positive integer.

Notes:

For $m = 0$ the formula becomes $(40*n - 9)*(40*n - 129)$ and were obtained the following Poulet numbers: 4681...;

For $m = 1$ the formula becomes $(40*n - 19)*(40*n - 139)$ and were obtained the following Poulet numbers: 75361...;

For $m = 2$ the formula becomes $(40*n - 29)*(40*n - 149)$ and were obtained the following Poulet numbers: 149281...;

For $m = 3$ the formula becomes $(40*n - 39)*(40*n - 159)$ and were obtained the following Poulet numbers: 6601...

Observation: It can be seen that for m of the form $4*k$ the formula becomes $(40*r - 9)*(40*r - 129)$, for m of the form $4*k + 1$ the formula becomes $(40*r - 19)*(40*r - 139)$, for m of the form $4*k + 2$ the formula becomes $(40*r - 29)*(40*r - 149)$ and for p of the form $4*k + 3$ the formula becomes $(40*r - 39)*(40*r - 159)$.

Conjecture: For any m natural number there exist a series with infinite many Poulet numbers P formed this way: $P = (40*n - 10*m - 9)*(40*n - 10*m - 129)$, where n is non-null positive integer.

99. Study the following conjecture:

Conjecture: There exist an infinity of triplets of Carmichael numbers (C_1, C_2, C_3) such that $C_1 = 2*C_2 + C_3 - 30$.

Example: $2821 = 2*561 + 1729 - 30$.

100. Study the Carmichael numbers C with the property that the number $p = (C - 1)/(s - 1) + 1$, where s is the sum of the digits of C , is prime.

Examples: $(1729 - 1)/(19 - 1) + 1 = 97$, prime; $(994788345601 - 1)/(64 - 1) + 1 = 15790291201$, prime.

101. Study the Carmichael numbers C of the form $C = (40*k + 1)*(60*k + 1)*(100*k + 1)$: {252601, 82929001, 241242001...}.

Conjecture: There exist an infinity of Carmichael numbers C of the form $C = (40*k + 1)*(60*k + 1)*(100*k + 1)$.

102. Can be the formula $(1729^2 + 1729*2)/(2 + 1) = 997633$, where the numbers 1729 and 997633 are Poulet numbers, extended to other Poulet numbers, in one of the following two ways:

(1) There exist a set of Poulet numbers of the form $P = (1729^n + 1729*n)/(n + 1) = 997633$;

(2) There exist a set of Poulet numbers of the form $P_2 = (P_1^2 + P_1*2)/3$?

103. Study the Poulet numbers P of the form $P = (42*m + 1)*(42*n + 1)$: {5461, 14491, 42799...}.

Conjecture: There exist an infinity of Poulet numbers P of the form $P = (42*m + 1)*(42*n + 1)$.

104. Study the Poulet numbers P for which $\text{sqr}(8*n + 1)$ is a natural number: {561, 2701, 4371, 8911, 10585, 18721, 33153, 41041, 49141, 93961, 104653, 115921, 157641, 226801, 289941, 314821, 334153, 534061, 665281, 721801, 831405, 873181, 915981, 1004653, 1373653, 1537381, 1755001, 1815465, 1987021, 2035153, 2233441, 2284453...}. Note the amount of Fermat pseudoprimes to base 2 for which that relation is true.

105. Study the Poulet numbers P of the form $P = n^2 + 81*n + 1599$: {2821, 6601, 10261, 15709...}.

Conjecture: There exist an infinity of Poulet numbers P of the form $P = n^2 + 81*n + 1599$.

106. Study the Poulet numbers P of the form $P = n^2 + 81*n + 159$: {4371, 4681...}.

Conjecture: There exist an infinity of Poulet numbers P of the form $P = n^2 + 81*n + 159$.

107. Study the Poulet numbers P of the form $P = n^2 + 81*n + 1629$: {4681, 29371...}.

Conjecture: There exist an infinity of Poulet numbers P of the form $P = n^2 + 81*n + 1629$.

108. Study the Poulet numbers P of the form $P = n^2 + 81*n + 479$: {561, 645...}.

Conjecture: There exist an infinity of Poulet numbers P of the form $P = n^2 + 81*n + 479$.

109. Study the Poulet numbers P of the form $P = n^2 + 81*n + 39$: {561, 2821...}.

Conjecture: There exist an infinity of Poulet numbers P of the form $P = n^2 + 81*n + 39$.

110. Study the Poulet numbers P of the form $P = n^2 + 10*n + 10$: {561, 5461...}.

Conjecture: There exist an infinity of Poulet numbers P of the form $P = n^2 + 10*n + 10$.

111. Study the following conjecture:

Conjecture: There exist an infinity of Poulet number P_2 of the form $21*q - (P_1 + 1)$, where q is prime and P is also a Poulet number.

Examples:

: $21*43 - (561 + 1) = 341$;
: $21*47 - (645 + 1) = 341$.

112. Study the following conjectures:

Conjecture 1: For any Poulet number P_1 of the form $3^k + 1$ there exist another Poulet number P_2 of the form $3^h + 2$ such that $P_1 + P_2 + 1 = P_3$, where P_3 is also a Poulet number (obviously of the form $3^g + 1$).

Example: $341 + 1387 + 1 = 1729$.

Conjecture 2: For any Poulet number P_1 of the form $3^k + 2$ there exist another Poulet number P_2 of the form $3^h + 1$ such that $P_1 + P_2 + 1 = P_3$, where P_3 is also a Poulet number (obviously of the form $3^g + 1$).

113. Study the following conjecture:

Conjecture: There exist an infinity of Poulet number P_2 of the form $(P_1 + 30) \cdot 9 + 10$, where P_1 is also a Poulet number.

Example: $(1729 + 30) \cdot 9 + 10 = 15841$.

114. Study the relation $C \bmod 936$, where C is a Carmichael number: many interesting sequences are obtained, *e.g.* many Carmichael numbers C of the form $C = 936 \cdot n + 793$: {1729, 41041, 46657, 115921, 126217, 172081, 340561, 658801, 670033, 748657, 997633...}. Study the relation $C \bmod (72 \cdot p)$, where p is prime.

115. Study the following two conjectures:

Conjecture 1: All Carmichael numbers which are divisible both with 37 and 73 are of the form $10^k + 1$.

Conjecture 2: All Carmichael numbers which are divisible both with 37 and 73 have a prime divisor of the form $30 \cdot h + 23$.

116. Study the following conjecture:

Conjecture: There exist an infinity of Carmichael numbers C such that $C \cdot s(C) - 90 = p$, where $s(C)$ is the sum of the digits of C and p is prime or power of prime.

Examples:

: $1729 \cdot 19 - 90 = 181^2$, square of prime;

: $2821 \cdot 13 - 90 = 36583$, prime.

117. Study the Carmichael numbers C divisible by 3 but not divisible by 5: for most of them is true the relation $C + 27$ is divisible by $84 = 7 \cdot 12$ (for the first eight such numbers: 561, 656601, 11921001, 26719701, 45318561, 174352641, 230996949, 662086041; for the 9-th such number, 684106401, $C + 27$ is divisible by 12 but not by 7).

118. Study the Carmichael numbers C divisible by 5 but not divisible by 3: interesting results are obtained for $N = (C + 180)/5$.

119. Study the numbers $N = (560 + 7 \cdot p)/7$, where p is a prime of the form $24 \cdot k + 17$. N is often a prime (for $p = 17, 113, 353 \dots$) or a square of prime (for $p = 41, 89, 281, 449 \dots$). Note that $561 = 3 \cdot 11 \cdot 17$ is the first Carmichael number; can be this prime generating relation extrapolated for other Carmichael numbers?
120. Study the numbers $N = p \cdot q \cdot r + C - 1$, where C is a Carmichael number having a prime factor d and p, q, r are prime numbers of the form $30 \cdot k + d + 10$. Interesting results are obtained for N : for Carmichael numbers 1729, 8911, 63973 (all three divisible by 19) and the primes $(p, q, r) = (359, 389, 419)$, all three of the form $30 \cdot k + 29$, we have, *exempli gratia*: $359 \cdot 389 \cdot 419 + 1728 = 19 \cdot 3079763$; $359 \cdot 389 \cdot 419 + 8910 = 19 \cdot 3080141$; $359 \cdot 389 \cdot 419 + 63972 = 19 \cdot 3083039$, where 3079763, 3080141 and 3083039 are primes.
121. Study the numbers N which can be written as $N = 3 \cdot p^2 + 2 \cdot p \cdot (p + 10)$, where p and $p + 10$ are both primes. Note that the second Carmichael number, the number 1105, is the first integer which can be written like this in two distinct ways: $1105 = 3 \cdot 13^2 + 2 \cdot 13 \cdot 23 = 3 \cdot 17^2 + 2 \cdot 7 \cdot 17$.
122. Study the Carmichael numbers which can be written as $p + q + r$, where p, q, r are of the same form from the following eight possible forms: $30 \cdot k + 1, 30 \cdot k + 7, 30 \cdot k + 11, 30 \cdot k + 13, 30 \cdot k + 17, 30 \cdot k + 19, 30 \cdot k + 23, 30 \cdot k + 29$. For instance $561 = 47 + 167 + 347$, where 47, 167 and 347 are primes of the form $30 \cdot k + 17$. Are there any other Carmichael numbers or Poulet numbers which can be written this way?
123. Study the numbers $N = C - (1 + d_1 + d_2 + \dots + d_n)$, where C is a Carmichael number and d_1, d_2, \dots, d_n are its prime factors. Is just a hazard that N is a square of prime (23^2) for $C = 561 = 3 \cdot 11 \cdot 17$?
124. Study the numbers $N = C - s(d)$, where C is a 3-Carmichael number and $s(d)$ is the sum of its three prime factors. For most of them is true that N is divisible by 10 (just for three from the first thirteen 3-Carmichael numbers, *i.e.* the numbers $2465 = 5 \cdot 17 \cdot 29$, $8911 = 7 \cdot 19 \cdot 67$ and $10585 = 5 \cdot 29 \cdot 73$, N is not divisible by 10).
125. Study the Carmichael numbers C of the form $C = n^k + 10^h$, where n, k, h are non-null positive integers.

Conjecture: There exist an infinity of Carmichael numbers C of the form $n^k + 10^h$, where n is positive odd integer greater than or equal to 3 and k, h non-null positive integers.

Examples:

- : $1729 = 729 + 1000 = 3^6 + 10^3$;
- : $29341 = 29241 + 100 = 171^2 + 10^2$.

126. Study the Poulet numbers P of the form $P = (d^2 + d)/2$, where d is a prime factor of P . Are there any other such Poulet numbers beside $2701 = (73^2 + 73)/2$?

Conjecture: There exist an infinity of Poulet numbers P of the form $P = (d^2 + d)/2$, where d is a prime factor of P .

127. Study the Poulet numbers P_2 of the form $P_2 = 2 \cdot P_1 - 1$, where P_1 is also a Poulet number. Are there any other such Poulet numbers beside $130561 = 2 \cdot 65281 - 1$?

Conjecture: There exist an infinity of Poulet numbers P_2 of the form $P_2 = 2 \cdot P_1 - 1$, where P_1 is also a Poulet number.

128. Study relations between a Carmichael number C and the fourth power of its lesser prime factor (d^4); Carmichael numbers can be classified in two classes by the criterion if $C < d^4$ (the case of $C = 1729 < 7^4 = 2401$) or $C > d^4$ (the case of $C = 29341 > 13^4 = 28561$). Also $N = \text{abs} \{C - d^4\}$ has interesting values: $2821 - 7^4 = 7 \cdot 60$ and $29341 - 13^4 = 13 \cdot 60$.

Conjecture: There exist an infinity of Carmichael numbers C of the form $C = p^4 + 60 \cdot p$, where p is prime.

129. Study the following two conjectures:

Conjecture 1: There are no Carmichael numbers of the following forms: $30 \cdot k + 3$, $30 \cdot k + 17$, $30 \cdot k + 23$, $30 \cdot k + 27$.

Conjecture 2: There exist an infinity of Carmichael numbers for each of the following forms: $30 \cdot k + 1$, $30 \cdot k + 5$, $30 \cdot k + 7$, $30 \cdot k + 9$, $30 \cdot k + 11$, $30 \cdot k + 13$, $30 \cdot k + 15$, $30 \cdot k + 19$, $30 \cdot k + 21$, $30 \cdot k + 25$, $30 \cdot k + 29$.

130. Study the numbers N of the form $N = (2 \cdot p \cdot (p + 100 \cdot k) + 3 \cdot p^2) / 5$, where p and $p + 100 \cdot k$ are both primes. For $p = 7$ and $k = 6$ is obtained the Poulet number 1729. Are there any other such Poulet numbers beside 1729?

Conjecture: There exist an infinity of Poulet numbers of the form $(2 \cdot p \cdot (p + 100 \cdot k) + 3 \cdot p^2) / 5$, where k is non-null positive integer and p and $p + 100$ are both primes.

131. Study the numbers N of the form $N = 12 \cdot C^2 + 1$, where C is a Carmichael number; seems that N is often a prime or a semiprime (this is the case for the first 10 Carmichael numbers, 561, 1105, 1729, 2465, 2821, 6601, 8911, 10585, 15841, 29341) and generally is a product of very few primes. The formula is useful for finding big primes; taking a greater Carmichael numbers, e.g. $C = 999607982113$, with 12 digits, we obtain $N = 11990593414848284735337227$, a prime with 26 digits.

132. Study the following conjecture:

Conjecture: Without a set of definable exceptions, is true the following statement for Poulet numbers: if a Poulet number can be written as $2^k \cdot p^n + p^2$, where k, n are integers, $k \geq 0$, and p one of its prime factors, that can be written as $2^k \cdot q^m + q^2$, where m is also integer, for any of its other prime factors q . Thus, if a Poulet number can't be written as $2^k \cdot p^n + p^2$, where k, n are integers, $k \geq 0$, and p one of its prime factors, that can't be written as $2^k \cdot q^m + q^2$, where m is also integer, for any of its other prime factors q .

Note that any 2-Poulet number can be obviously written as $2 \cdot p \cdot n + p^2$, where p is one of its two prime factors and n integer (n is different by 0 if the two prime factors are distinct and is equal to 0 if the Poulet number is a square of a Wieferich prime).

But, it can also be seen that, if a 2-Poulet number can be written as $2^k \cdot p_1 \cdot n + p_1^2$, where k, n are integers, $k \geq 0$, and p_1 one of its two prime factors, than can be written too as $2^k \cdot p_2 \cdot (-n) + p_2^2$, where p_2 is the other prime factor. Let's take, *exempli gratia*, the 2-Poulet number $233017 = 43 \cdot 5419$; we have the following relations:

$$\begin{aligned} 233017 &= 2^0 \cdot 43 \cdot 5376 + 43^2 = 2^1 \cdot 5419 \cdot (-5376) + 5419^2. \\ 233017 &= 2^1 \cdot 43 \cdot 2688 + 43^2 = 2^1 \cdot 5419 \cdot (-2688) + 5419^2. \\ 233017 &= 2^2 \cdot 43 \cdot 1344 + 43^2 = 2^2 \cdot 5419 \cdot (-1344) + 5419^2. \\ &\dots\dots\dots \\ 233017 &= 2^8 \cdot 43 \cdot 21 + 43^2 = 2^8 \cdot 5419 \cdot (-21) + 5419^2. \end{aligned}$$

The conjecture above extrapolates this relation for n -Poulet numbers; let's take a n -Poulet number, *e.g.* $13741 = 7 \cdot 13 \cdot 151$; we have the following relations:

$$\begin{aligned} 13741 &= 2^0 \cdot 7 \cdot 1956 + 7^2 = 2^0 \cdot 13 \cdot 1044 + 13^2 = 2^0 \cdot 151 \cdot (-60) + 151^2. \\ 13741 &= 2^1 \cdot 7 \cdot 978 + 7^2 = 2^1 \cdot 13 \cdot 522 + 13^2 = 13741 = 2^1 \cdot 151 \cdot (-30) + 151^2. \\ 13741 &= 2^2 \cdot 7 \cdot 489 + 7^2 = 2^2 \cdot 13 \cdot 261 + 13^2 = 2^2 \cdot 151 \cdot (-15) + 151^2. \end{aligned}$$

133. Study the following conjecture:

Conjecture: For any Poulet number P is true that $P - 5$ is equal to a sum of distinct powers of the number two.

Examples:

$$\begin{aligned} : 341 - 5 &= 2^8 + 2^6 + 2^4. \\ : 645 - 5 &= 2^9 + 2^7. \\ : 561 - 5 &= 2^9 + 2^5 + 2^3 + 2^2. \\ : 1105 - 5 &= 2^{10} + 2^6 + 2^3 + 2^2. \\ : 1729 - 5 &= 2^{10} + 2^9 + 2^7 + 2^5 + 2^4 + 2^3 + 2^2. \end{aligned}$$

134. Study the numbers N of the form $N = P + 8$, where P is a Poulet number of the form $3^k + 2$ (it is known that most of the Poulet numbers have the form $3^k + 1$); seems that N is often a prime or a semiprime (this is the case for the first 8 such Poulet numbers, 341, 2465, 8321, 35333, 8333, 91001, 113201, 150851) and generally seems to be a product of very few primes.

135. Study the numbers N of the form $N = m^2 + m - C$, where C is a Carmichael number and m is the integer part of the square root of C ($m = \text{isqrt } C$); interesting results are obtained: for instance, $N = 53^2 + 53 - 2821 = 81^2 + 81 - 6601 = 41$ for $C = 2821$ and $C = 6601$.

136. Study the following two formulas for a Carmichael number C :

- (1) $C - 1 = p^2 - q^2$, where p and q are primes ; the relation is true for the triplet $(C, p, q) = (1729, 43, 11)$; are there other such triplets?
- (1) $C = p^2 - 60^2$, where p is prime; the relation is true for the duplets $(1729, 73)$ and $(6601, 101)$: are there other such duplets?

Conjecture 1: There is an infinity of Carmichael numbers C of the form $C = p^2 - q^2 + 1$, where p and q are primes.

Conjecture 2: There is an infinity of Carmichael numbers C of the form $C = p^2 - 60^2$, where p is prime.

- 137.** Study the numbers N of the form $N = 41^2 + 41 + p$, where p is a prime of the form $30*k + 7$ (note that $41^2 + 41 + 7 = 1729$, the Hardy-Ramanujan number); seems that N is often a product of very few primes.
- 138.** Study the numbers N of the form $N = p*q - C^2$, where C is a Carmichael number and p, q are primes with the property that $q - p = C - 1$; interesting results are obtained: for instance, $N = 449*1009 - 561^2 = 2^4*5*1729$, where 449 and 1009 are primes with the property that $1009 - 449 = 561 - 1$ and 561 and 1729 are Carmichael numbers.
- 139.** Study the following recurrent sequence: the first term of the sequence, $a(0)$, is equal to $2*(C_1 - 1)$, where C_1 is the first Carmichael number, 561, then $a(n + 1) = a(n) + C_{n+1}$, where C_{n+1} is the $(n+1)$ th Carmichael number. The sequence is: {561, 1120, 2225, 3954, 6419, 9240, 15841...}; it can be seen that the term $a(6) = 15841$ is also a Carmichael number.

Conjecture: There is an infinity of Carmichael numbers in every recurrent sequence formed in the following way: $a(0) = 2*(C_m - 1)$, where C_m is the m -th Carmichael number, then $a(n + 1) = a(n) + C_{m+n+1}$, where C_{m+n+1} is the $(m+n+1)$ th Carmichael number.

- 140.** Study the numbers $N_i = (C - d_i^2 + d_i)/d_i$, where C is a Carmichael number and the numbers d_i are its prime factors. Many Carmichael numbers have prime factors d_i for which N_i is prime but for some Carmichael numbers (*e.g.* for $C = 1729$, for which d_1, d_2 and d_3 are equal to 7, 13, 19 and N_1, N_2, N_3 are equal to 247, $11^2, 73$) all the numbers N_i are primes.

Question: Do these Carmichael numbers form a set based on some special properties (beside, of course, the one that I just enunciated it)?

- 141.** Study the numbers $N = (P - 1)*15 + 1$, where P is a Poulet number divisible by 15. The numbers N seem to be often primes, semiprimes or products of very few primes.

Examples:

: $(999016383345 - 1)*15 + 1 = 14985245750161$, prime;

: $(993004965345 - 1)*15 + 1 = 14895074480161$, prime;

: $(990890597985 - 1)*15 + 1 = 1069*13903984069$, semiprime.

Note:

A very interesting result is obtained for $P = 42842904105$; in this case is obtained $N = 642643561561$, a number from which are obtained, by deconcatenation, two consecutive numbers (642 and 643) and a Poulet number repeated two times (561).

To highlight the interest in the result above, is worth mentioning that, for $P = 18705$, is obtained $N = 280561$, a prime number from which are obtained, by deconcatenation, the number 561 and the number $280 = (561 - 1)/2$.

- 142.** Study the numbers N_i formed by concatenation from any Poulet number P_i in the following way: $N = 170341$ for $C = 341$, $N = 280561$ for $C = 561$ etc.; it can be seen that we concatenated C to the left with the number $(C - 1)/2$. The numbers N_i seem to be often primes (this is the case for 170341 and 280561), but not always: *e.g.*, for $P = 2821$ we have $N = 1063 \cdot 13267$.

Question: Do the Poulet numbers for which N is prime form a set based on some special properties?

- 143.** Study the following conjecture:

Conjecture: There is an infinity of Poulet numbers P such that the number N obtained by concatenating to the left P with the number 28 is (equal to) a fourth power of a prime.

Examples:

: For $P = 561$ we have $N = 28561 = 13^4$;

: For $P = 25761$ we have $N = 2825761 = 41^4$.

- 144.** Study the following conjecture:

Conjecture: There is an infinity of duplets of numbers (q, n) , where q is prime and n is non-null positive integer, such that the number $P = (q^4 - 1)/3 \cdot n + 1$ is a Poulet number.

Example: For the duplets $(13, 5)$, $(13, 17)$ and $(13, 28)$ are obtained the Poulet numbers 1905, 561 and 341.

- 145.** Study the following conjecture:

Conjecture: Any Carmichael number C can be written as $C = p \cdot q + q - p$, where p and q are primes.

Example:

: $1729 = 3 \cdot 433 + 433 - 3 = 17 \cdot 97 + 97 - 17 = 23 \cdot 73 + 73 - 23$.

- 146.** Study the following conjecture:

Conjecture: Any Carmichael number C can be written as $C = p^2 + p + d$, where p is prime and d is a prime factor of C .

Example:

: $1729 = 41^2 + 41 + 7$.

- 147.** Study the following conjecture:

Conjecture: Any Carmichael number C can be written as $C = a^2 + b^2 + c^2 - 2$, where a, b, c are primes.

Example:

: $561 = 15^2 + 17^2 + 7^2 - 2$.

148. Study the following conjecture:

Conjecture: Any Carmichael number C can be written as $C = p \cdot 2^n + 2^n - p$, where p is prime and n is non-null positive integer.

Example: $561 = 79 \cdot 2^3 + 2^3 - 79$.

149. Study the following conjectures:

Conjecture: There exist infinite many Poulet numbers of the form $P = 1029600 \cdot p + 1$.

Examples: $5148001 = 1029600 \cdot 5 + 1$; $7207201 = 1029600 \cdot 7 + 1$.

Conjecture: There exist infinite many Poulet numbers of the form $P_2 = 9 \cdot (P_1 - 1) + q^2$, where q is prime and P_1 is also a Poulet number.

Example: $15841 = 9 \cdot (1729 - 1) + 17^2$.

150. Study the following conjecture:

Conjecture: For any two Poulet numbers P_1 and P_2 that share at least one prime divisor, the diophantine equation $x \cdot (P_1 - 1) = y \cdot (P_2 - 1)$ has solutions $[x, y]$ on integers, other than the trivial one $[x = (P_2 - 1), y = (P_1 - 1)]$.

Verifying the conjecture for few pairs of Poulet numbers that have the number 3 as a common prime divisor:

: for $P_1 = 341$ and $P_2 = 645$ we have $(x, y) = (161, 85)$;

: for $P_1 = 341$ and $P_2 = 1905$ we have $(x, y) = (28, 5)$;

: for $P_1 = 341$ and $P_2 = 4371$ we have $(x, y) = (437, 34)$;

: for $P_1 = 341$ and $P_2 = 8481$ we have $(x, y) = (424, 17)$.

Prime numbers have always fascinated mankind.

For mathematicians, they are a kind of “black sheep” of the family of integers by their constant refusal to let themselves to be disciplined, ordered and understood.

But we have at hand a powerful tool, insufficiently investigated yet, which can help us in understanding them: Fermat pseudoprimes.

Exceptions to Fermat’s “little” Theorem, these numbers seem to be more malleable than primes, more willing to let themselves to be ordered than them, and their depth study will surely shed light on many properties of the primes.

I titled the book this way to show how many new and exciting things one can say more about this class of numbers, but, beside the two hundred conjectures (listed in the Part one of this book) and the one hundred and fifty open problems (listed in the Part four of this book), promised in the title, there are many other observations about Fermat pseudoprimes and generic formulas of subsets of Carmichael numbers and Poulet numbers contained in this book.

All the articles in this book of collected papers use well-known notions of number theory, with only two exceptions, namely one article in which I defined a new concept, id est “*a set of Smarandache - Coman divisors of order k of a composite integer n with m prime factors*” and one article in which I have showed several applications of this concept in the study of Fermat pseudoprimes.